The Secret Algebraic Lives of Graphs

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Clemson University

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ASU Mathematical and Statistical Sciences Colloquium

Acknowledgements

(Land Acknowledgement)

I acknowledge that the land where I live and work in northern Virginia is the ancestral home of numerous Native American tribes from the Doeg to the Piscataway. It is also the site of enslavement of many African Americans. Black lives matter.

(Trans Awareness Week)

Transgender Awareness Week, observed November 13–19, is a celebration to educate about transgender and gender non-conforming people and the issues associated with their transition or identity. It ends on Transgender Day of Remembrance on November 20 to memorialize those who were murdered due to anti-transgender hatred or prejudice.

Collaborators

(COURAGE)

COURAGE REU Summer 2020: Clemson Online Undergraduate Research on Algebra and Graphs Expanded Devyn Adams, Vi Anh, Caroline Daw, James Gossell, Aayahna Herbert, John Lim, Yifan Qian, KSW, Matt Schaller, Zoe Zhou, Yyuyang Zhuo - in preparation.

(JHKSW)

Jacob Honecutt, KSW - *La Matematica*, 2022.

Outline

How It Started

(PhD: Homological Commutative Algebra)

Intersection multiplicities and symbolic powers: requires

- Graduate Algebra 1–2,
- Commutative Algebra,
- Homological Algebra, and
- Algebraic Geometry

(Postdoc and Beyond: Derived Commutative Algebra)

- Derived categories, derived functors: requires even more
- Hard to work with grad students, let alone undergrads
- Hard to conceive realistic elevator talks

How It's Going

- I'm still interested in these things
- Take-Home Point (THP) 1: Now I use connections with graph theory and other areas as entry points for students
- THP2: We can see the algebra
- Elevator talks are much easier

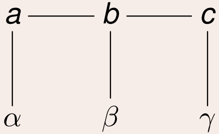
Setup

(Assumption: for the remainder G is a (finite simple) graph)

- vertex set $V = V(G) = \{x_2, \ldots, x_d\}$
- edge set $E = E(G)$.
- “Simple” means no loops, directed edges, nor multi-edges.

(Example: $P_2 =$ a —— b —— c)

- vertices a, b, c and edges ab, bc
- b is “adjacent to” a and c
- “Whiskering” of P_2: $\Sigma P_2 =$



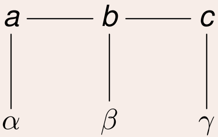
Vertex Covers: How to Monitor Networks

(Definition: A vertex cover of G is a subset $V' \subseteq V$ such that)

- slogan: every edge of G is covered by V'
- i.e., every $x_i x_j \in E$ has $x_i \in V'$ or $x_j \in V'$

(Example: $P_2 =$ a —— b —— c)

- $V' = \{b\} \quad \{a, b\} \quad \{a, c\} \quad \{b, c\} \quad \{a, b, c\}$
- For $\Sigma P_2 =$



$V' =$

$\{a, b, c\} \quad \{a, b, \gamma\} \quad \{a, \beta, c\} \quad \{\alpha, b, c\} \quad \{\alpha, b, \gamma\} \quad \ldots$

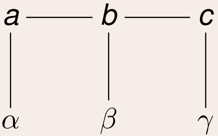
Minimal Vertex Covers: Monitor Networks Efficiently

(Definition: A vertex cover of G is minimal if)

it does not properly contain another vertex cover of G

(Example: $P_2 =$ a —— b —— c)

- $V' = \quad \underline{\{b\}} \quad \{a, b\} \quad \underline{\{a, c\}} \quad \{b, c\} \quad \{a, b, c\}$
- For $\Sigma P_2 =$



$V' =$

$\underline{\{a, b, c\}} \quad \underline{\{a, b, \gamma\}} \quad \underline{\{a, \beta, c\}} \quad \underline{\{\alpha, b, c\}} \quad \underline{\{\alpha, b, \gamma\}} \quad \ldots$

Well Covered Graphs

(Problem: Efficient versus Inexpensive)

The naive algorithm for producing minimal vertex covers doesn't necessarily product one of smallest size/cardinality

(Definition: *G* is <u>well covered</u> if)

every minimal vertex cover of *G* has the same size/cardinality

(Example: P_2 is not well covered)

- $V' = \quad \underline{\{b\}} \quad \{a, b\} \quad \underline{\{a, c\}} \quad \{b, c\} \quad \{a, b, c\}$
- ΣP_2 is well covered: $\quad V' =$

$$\underline{\{a, b, c\}} \quad \underline{\{a, b, \gamma\}} \quad \underline{\{a, \beta, c\}} \quad \underline{\{\alpha, b, c\}} \quad \underline{\{\alpha, b, \gamma\}} \quad \ldots$$

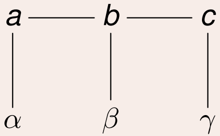
Well Covered Trees

(Theorem: Villarreal 1990?)

Let T be a tree with at least two vertices.

- T is well covered if and only if it is whiskered, i.e.,
- T is well covered if and only if $T = \Sigma U$ for some tree U

(Example: $P_2 =$ a —— b —— c)

- P_2 is not whiskered and not well covered
- $\Sigma P_2 =$ 

is whiskered and well covered

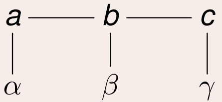
Dominating Sets: Another Way to Monitor Networks

(Definition: A dominating set for G is a subset $V' \subseteq V$ s.t.)

- slogan: every *vertex* of G is dominated by V', i.e.,
- every $x_i \in V$ has $V' \cap \overline{N}_G(x_i) \neq \emptyset$, i.e.,
- every $x_i \in V$ has either $x_i \in V'$ or $x_j \in V'$ for some $x_i x_j \in E$

(Example: $P_2 = a$ —— b —— c)

- $V' = \quad \{b\} \quad \{a, b\} \quad \{a, c\} \quad \{b, c\} \quad \{a, b, c\}$
- For $\Sigma P_2 =$



$$V' = \begin{matrix} \{a, b, c\} & \{a, b, \gamma\} & \{a, \beta, c\} & \{\alpha, b, c\} & \\ \{a, \beta, \gamma\} & \{\alpha, b, \gamma\} & \{\alpha, \beta, c\} & \{\alpha, \beta, \gamma\} & \dots \end{matrix}$$

Minimal Vertex Covers: Monitor Networks Efficiently

(Definition: A dominating set for G is minimal if)

it does not properly contain another dominating set for G

(Definition: G is well dominated if)

every minimal dominating set for G has the same size

(Example)

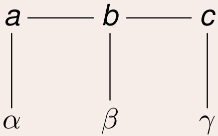
- P_2 is not well dominated
- ΣP_2 is well dominated

Well Dominated Trees

(Theorem: JHKSW)

A tree with at least two vertices is well dominated iff whiskered

(Example: $P_2 = \quad a \text{ —— } b \text{ —— } c$)

- P_2 is not whiskered and not well dominated
- $\Sigma P_2 =$ 

is whiskered & well dominated

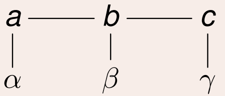
Total Dominating Sets: Even More Network Monitoring

(Definition: A total dominating set for G is a subset $V' \subseteq V$ s.t.)

- slogan: every *vertex* of G is totally dominated by V', i.e.,
- every $x_i \in V$ has $V' \cap N_G^{\circ}(x_i) \neq \emptyset$, i.e.,
- every $x_i \in V$ has $x_j \in V'$ for some $x_i x_j \in E$

A vertex cannot totally dominate itself

(Example: $P_2 = \quad a —— b —— c$)

- $V' = \quad \{a, b\} \quad \{b, c\} \quad \{a, b, c\}$
- For $\Sigma P_2 =$  $V' = \quad \{a, b, c\} \quad \ldots$

Totally Well Dominated Trees

(Definition: G is totally well dominated if)

every minimal total dominating set for G has the same size

(Theorem: COURAGE)

We characterize the totally well dominated trees.

(Fact)

Whiskered implies totally well dominated, but not conversely

(Example)

- P_2 is not whiskered but is totally well dominated
- ΣP_2 is whiskered and totally well dominated

Edge Ideals: Villarreal 1990

(Assumptions: for the rest of the talk:)

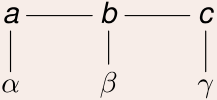
k is a field, and $R = k[V] = k[x_1, \ldots, x_d]$ is a polynomial ring

(Def'n: The edge ideal of G is generated by the edges of G)

$$\mathcal{E}(G) = \langle x_i x_j \in E(G) \rangle \subseteq R$$

(Example: $P_2 =$ a —— b —— c)

- $\mathcal{E}(P_2) = \langle ab, bc \rangle \subseteq k[a, b, c]$.
- For $\Sigma P_2 =$



$$\mathcal{E}(\Sigma P_2) = \langle ab, bc, a\alpha, b\beta, c\gamma \rangle \subseteq k[a, b, c, \alpha, \beta, \gamma]$$

Decompositions of Edge Ideals

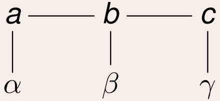
(Theorem: Villarreal 1990)

$$\mathcal{E}(G) = \bigcap_{V' \text{ vc of } G} \langle V' \rangle = \bigcap_{V' \text{ mvc of } G} \langle V' \rangle$$

and the second decomposition is irredundant/minimal.
In particular, $\mathcal{E}(G)$ is unmixed if and only if G is well covered.

(Example: $P_2 =$ a —— b —— c)

- $\mathcal{E}(P_2) = \langle ab, bc \rangle = \langle b \rangle \cap \langle a, c \rangle$.
- For $\Sigma P_2 =$



$\mathcal{E}(\Sigma P_2) = \langle ab, bc, a\alpha, b\beta, c\gamma \rangle =$
$\langle a, b, c \rangle \cap \langle a, b, \gamma \rangle \cap \langle a, \beta, c \rangle \cap \langle \alpha, b, c \rangle \cap \langle \alpha, b, \gamma \rangle$

Closed Neighborhood Ideals

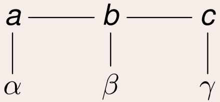
(Definition: Sharifan and Moradi 2020)

The closed neighborhood ideal of G is generated by the closed neighborhoods of G:

$$\overline{\mathcal{N}}(G) = \langle \overline{N}_G(x_1), \ldots, \overline{N}_G(x_d) \rangle \subseteq R$$

(Example: $P_2 = \quad a \text{ —— } b \text{ —— } c$)

- $\overline{\mathcal{N}}(P_2) = \langle ab, abc, bc \rangle = \langle ab, bc \rangle \subseteq k[a, b, c]$.
- For $\Sigma P_2 =$



$$\overline{\mathcal{N}}(\Sigma P_2) = \langle ab\alpha, abc\beta, bc\gamma, a\alpha, b\beta, c\gamma \rangle = \langle a\alpha, b\beta, c\gamma \rangle$$

Decompositions of Closed Neighborhood Ideals

(Theorem: JHKSW)

$$\overline{\mathcal{N}}(G) = \bigcap_{V' \text{ ds for } G} \langle V' \rangle = \bigcap_{V' \text{ mds for } G} \langle V' \rangle$$

and the second decomposition is irredundant/minimal.
In particular, $\overline{\mathcal{N}}(G)$ is unmixed if and only if G is well dominated.

(Example: $P_2 = \quad a \text{ ------ } b \text{ ------ } c$)

- $\overline{\mathcal{N}}(P_2) = \langle ab, bc \rangle = \langle b \rangle \cap \langle a, c \rangle$.
- $\overline{\mathcal{N}}(\Sigma P_2) = \langle a\alpha, b\beta, c\gamma \rangle$
  $$= \langle a, b, c \rangle \cap \langle a, b, \gamma \rangle \cap \langle a, \beta, c \rangle \cap \langle \alpha, b, c \rangle$$
  $$\cap \langle a, \beta, \gamma \rangle \cap \langle \alpha, b, \gamma \rangle \cap \langle \alpha, \beta, c \rangle \cap \langle \alpha, \beta, \gamma \rangle$$

Open Neighborhood Ideals

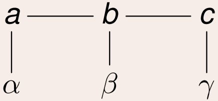
(Definition: COURAGE)

The open neighborhood ideal of G is generated by the open neighborhoods of G:

$$\mathcal{N}^\circ(G) = \langle N_G^\circ(x_1), \ldots, N_G^\circ(x_d)\rangle \subseteq R$$

(Example: $P_2 =$ a —— b —— c)

- $\mathcal{N}^\circ(P_2) = \langle b, ac, b\rangle = \langle b, ac\rangle \subseteq k[a, b, c]$.
- For $\Sigma P_2 =$



$$\mathcal{N}^\circ(\Sigma P_2) = \langle b\alpha, ac\beta, b\gamma, a, b, c\rangle = \langle a, b, c\rangle$$

Decompositions of Closed Neighborhood Ideals

(Theorem: COURAGE)

$$\mathcal{N}^\circ(G) = \bigcap_{V' \text{ tds for } G} \langle V' \rangle = \bigcap_{V' \text{ mtds for } G} \langle V' \rangle$$

and the second decomposition is irredundant/minimal.
In particular, $\mathcal{N}^\circ(G)$ is unmixed iff G is totally well dominated.

(Example: $P_2 = \quad a \text{ —— } b \text{ —— } c$)

- $\mathcal{N}^\circ(P_2) = \langle b, ac \rangle = \langle a, b \rangle \cap \langle b, c \rangle$.
- $\overline{\mathcal{N}}(\Sigma P_2) = \langle a\alpha, b\beta, c\gamma \rangle$
  $= \langle a, b, c \rangle \cap \langle a, b, \gamma \rangle \cap \langle a, \beta, c \rangle \cap \langle \alpha, b, c \rangle$
  $\cap \langle a, \beta, \gamma \rangle \cap \langle \alpha, b, \gamma \rangle \cap \langle \alpha, \beta, c \rangle \cap \langle \alpha, \beta, \gamma \rangle$

I'll Tell You What I Want, What I Really, Really Want

(Cohen-Macaulayness is a niceness condition)

- The Cohen-Macaulay property is stronger than being unmixed, but weaker than being regular or smooth.
- Hochster ($\approx$): Life is truly nice in a Cohen-Maculay ideal.

(It solves a problem from Algebraic Geometry)

- A smooth variety intersected with a generic hypersurface will be unmixed.
- However, an unmixed variety intersected with a generic hypersurfece need not be unmixed.
- OTOH, a Cohen-Macaulay variety intersected with a generic hyperplane will be Cohen-Macaulay.

Unmixedness Implies Cohen-Macaulay, but not Conversely, unless...

(Assumption)

For the remainder, T is a tree.

(Theorem: Villarreal 1990)

TFAE

- T is well covered,
- T is whiskered,
- $\mathcal{E}(T)$ is unmixed,
- $\mathcal{E}(T)$ is Cohen-Macaulay.

Unmixedness Implies Cohen-Macaulay, but not Conversely, unless...

(Theorem: JHKSW)

TFAE

- T is well dominated,
- T is whiskered,
- $\overline{\mathcal{N}}(T)$ is unmixed,
- $\overline{\mathcal{N}}(T)$ is Cohen-Macaulay,
- $\overline{\mathcal{N}}(T)$ is a complete intersection.

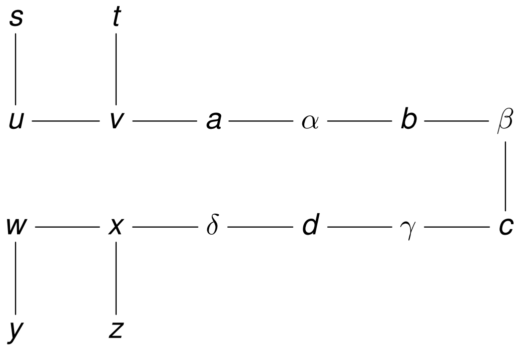
Unmixedness Implies Cohen-Macaulay, but not Conversely, unless...

(Theorem: COURAGE)

TFAE

- T is totally well dominated,
- $\overline{\mathcal{N}}(T)$ is unmixed,
- $\overline{\mathcal{N}}(T)$ is Cohen-Macaulay.

A Totally Well Dominated Tree that is CM but not CI



$$\mathcal{N}^\circ = \langle u, v, w, x, ab, bc, cd, \alpha\beta\beta\gamma, \gamma\delta \rangle$$

This is Cohen-Macaulay of type 4, so not Gorenstein and thus not a complete intersection.

Conclusion

(Take Home Points)

1. Combinatorial Commutative Algebra gives a reasonable entry point for students in commutative algebra
2. We can see the algebra
3. Unmixed graph ideals for trees tend to be Cohen-Macaulay

Thanks!