MATH 725, FALL 2012, HOMEWORK 4–5
DUE FRIDAY 02 NOVEMBER

Let R be a commutative ring with identity, and let M be an R-module.

Exercise 1. Assume that R is noetherian and M is artinian. Prove that for every $x \in M$ there is an integer $n \geqslant 1$ such that $J(R)^n x = 0$. Here $J(R)$ is the Jacobson radical of R.

Exercise 2. Consider the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$, and let n, p be integers such that $p > 1$ is prime and $n \geqslant 0$.

(a) Let $r \in \mathbb{Q}$, and let $o(r)$ denote the order of the element $r + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ in the additive abelian group $\mathbb{Q}/\mathbb{Z}$. Let t be a positive integer. Prove that $o(r)$ divides t if and only if there is an integer m such that $r = m/t$.
(b) Set $G_{p,n} = \{r + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid r \in \mathbb{Q} \text{ and } o(r) \mid p^n\}$, and prove that $G_{p,0} \subsetneq G_{p,1} \subsetneq G_{p,2} \subsetneq \cdots$ is an ascending chain of $\mathbb{Z}$-submodules of $\mathbb{Q}/\mathbb{Z}$ that does not stabilize.
(c) Prove that $G_{p,n}$ is cyclic such that $\operatorname{Ann}_{\mathbb{Z}}(G_{p,n}) = p^n\mathbb{Z}$. Conclude that $G_{p,n} \cong \mathbb{Z}/p^n\mathbb{Z}$ and $\operatorname{length}_{\mathbb{Z}}(G_{p,n}) = n$.
(d) Set $G_{p,\infty} = \cup_{i=0}^{\infty} G_{p,i}$. Prove that $G_{p,\infty}$ is not noetherian as a $\mathbb{Z}$-module.
(e) Prove or disprove: $G_{p,\infty}/G_{p,n}$ is noetherian as a $\mathbb{Z}$-module.
(f) Let $H \subsetneq G_{p,\infty}$ be a proper $\mathbb{Z}$-submodule of $G_{p,\infty}$. Prove that there is an integer i such that $H = G_{p,i}$.
(g) Prove that $G_{p,\infty}$ is artinian as a $\mathbb{Z}$-module and that $\operatorname{Ann}_{\mathbb{Z}}(G_{p,\infty}) = 0$.
(h) Prove or disprove: $G_{p,\infty}/G_{p,n}$ is artinian as a $\mathbb{Z}$-module.
(i) Prove that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module and that $\mathbb{Q}/\mathbb{Z} = \oplus_q G_{q,\infty}$, where the sum is indexed by the set of primes $q > 1$. Conclude that $G_{p,\infty}$ is an injective $\mathbb{Z}$-module. (It is a non-trivial fact that every injective $\mathbb{Z}$-module is isomorphic to a coproduct $\mathbb{Q}^{(\mu)} \coprod (\coprod_q G_{q,\infty}^{(\mu_p)})$ for some index sets $\mu, \mu_2, \mu_3, \mu_5, \ldots$.)
(j) Prove or disprove: $\mathbb{Q}/\mathbb{Z}$ is artinian as a $\mathbb{Z}$-module.
(k) Prove that $G_{p,\infty} = p^n G_{p,\infty}$, that is, that the map $G_{p,\infty} \xrightarrow{p^n} G_{p,\infty}$ is onto; prove also that if $n \geqslant 1$, then this map is not 1-1.
(l) Prove that if $q > 1$ is prime, then $G_{p,\infty} \otimes_{\mathbb{Z}} G_{q,\infty} = 0$. (Hint: The case $p = q$ uses part (k). For the case $p \neq q$, use the fact that, for all positive integers a, b we have $(p^a, q^b) = 1$.)
(m) Prove that if $q > 1$ is prime and $p \neq q$, then $\operatorname{Hom}_{\mathbb{Z}}(G_{p,\infty}, G_{q,\infty}) = 0$. (It is a non-trivial fact that $\operatorname{Hom}_{\mathbb{Z}}(G_{p,\infty}, G_{p,\infty}) \cong \widehat{\mathbb{Z}}^p$. The proof goes something like this: $\operatorname{Hom}_{\mathbb{Z}}(G_{p,\infty}, G_{p,\infty}) \cong \operatorname{Hom}_{\mathbb{Z}}(\varinjlim_{i} G_{p,i}, G_{p,\infty}) \cong \varprojlim_{i} \operatorname{Hom}_{\mathbb{Z}}(G_{p,i}, G_{p,\infty}) \cong \varprojlim_{i} \mathbb{Z}/p^n\mathbb{Z} \cong \widehat{\mathbb{Z}}^p$.)

Exercise 3. (a) Prove that if M is finitely generated as an R-module then there is an integer $n \geqslant 0$ and an R-module monomorphism $R/\operatorname{Ann}_R(M) \to M^n$.
(b) Prove that if M is noetherian as an R-module, then $R/\operatorname{Ann}_R(M)$ is a noetherian ring.
(c) Prove or disprove: if M is artinian as an R-module, then $R/\operatorname{Ann}_R(M)$ is a artinian ring.