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Semidualizing modules of 2×2 Ladder determinantal rings

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ARTICLE INFO

Article history:

Received 1 November 2018

Available online 13 August 2019

Communicated by Kazuhiko Kurano

MSC:

13C20

13B30

13C40

Keywords:

Ladder determinantal ring

Gorenstein ring

Semidualizing module

ABSTRACT

We continue our study of ladder determinantal rings over a field k from the perspective of semidualizing modules. In particular, given a ladder of variables Y , we show that the associated ladder determinantal ring $k[Y]/I_2(Y)$ admits exactly 2^n non-isomorphic semidualizing modules where n is determined from the combinatorics of the ladder Y : the number n is essentially the number of non-Gorenstein factors in a certain decomposition of Y . From this, for each n , we show explicitly how to find ladders Y such that $k[Y]/I_2(Y)$ admits exactly 2^n non-isomorphic semidualizing modules. This is in contrast to our previous work, which demonstrates that large classes of ladders have exactly 2 non-isomorphic semidualizing modules.

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² Sandra Spiroff was supported in part by Simons Foundation Collaboration Grant 245926.

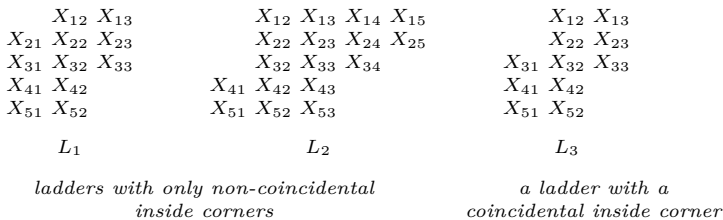
1. Introduction

Let R be a commutative noetherian ring and let k be a field. We are interested in the question of how many non-isomorphic semidualizing modules the ring R has, where a finitely generated R -module C is **semidualizing** if $\text{Hom}_R(C, C) \cong R$ and $\text{Ext}_R^i(C, C) = 0$ for all $i \geq 1$. Examples include the free R -module of rank 1 and, if R is local and Cohen-Macaulay, a canonical or dualizing module. This general class of modules was introduced by Foxby [10] in part to understand Auslander and Bridger’s G-dimension [3] and non-finitely generated versions due to Enochs and Jenda [9]. The number of non-isomorphic semidualizing R -modules is finite when R is local or a standard graded normal domain by Nasseh and Sather-Wagstaff [12,14]. In each of these cases, the number of these modules measures how far R is from being Gorenstein.

The question [14, Question 4.13] that we address in this paper relates to the cardinality of the set of semidualizing modules $\mathfrak{S}_0(R)$ of a Noetherian local ring R : Must it always be a power of 2? We provide more evidence towards an affirmative answer in the case of ladder determinantal rings. These rings generalize the classical determinantal rings in a way that is useful, e.g., for studying Young tableaux [1].

Referring back to the detailed background section in [13], we briefly recall that a **ladder** is a subset Y of an $m \times n$ matrix $X = (X_{ij})$ of indeterminates satisfying the property that if $X_{ij}, X_{pq} \in Y$ satisfy $i \leq p$ and $j \leq q$, then $X_{iq}, X_{pj} \in Y$. Then $R_2(Y) = k[Y]/I_2(Y)$ is the associated **ladder determinantal ring** of 2-minors, as $I_2(Y)$ is the ideal generated by the 2×2 minors lying entirely in Y . To avoid trivialities, we assume that X is the smallest matrix containing Y and that every variable of Y is part of a 2×2 minor. To wit, we consider only **2-connected** ladders; i.e., those ladders Y satisfying the property that there do not exist two subladders $\emptyset \neq Z_1, Z_2 \subseteq Y$ such that $Z_1 \cap Z_2 = \emptyset$, $Z_1 \cup Z_2 = Y$, and every 2-minor of Y is contained in Z_1 or Z_2 .

We provide a construction to produce ladder determinantal rings with exactly 2^n semidualizing modules for any $n \in \mathbb{N}$, and we show that $|\mathfrak{S}_0(R_2(Y))|$ is always, in fact, a power of 2. To describe our results explicitly, we use the corners of a ladder; see Definition 2.4 and the sample ladders below. If a lower and upper inside corner of a ladder Y coincide, then we say that Y has a coincidental inside corner.



Our main result says that if Y is a two-sided 2-connected ladder with w coincidental inside corners, then the set of semidualizing modules has cardinality 2^n , where $0 \leq n \leq w$. To be specific, we prove the following; see Notation 3.1 for an explanation of the symbol $Z_0 \# \dots \# Z_w$.

Main Theorem (See Theorem 3.12.). Let $Y = Z_0 \# \cdots \# Z_w$ be a 2-connected ladder, where each Z_u is a 2-connected ladder with no coincidental inside corners. Let $R = R_2(Y)$. Then $|\mathfrak{S}_0(R)| = \prod_{u=0}^w |\mathfrak{S}_0(R_2(Z_u))| = 2^{\varepsilon_0 + \cdots + \varepsilon_w}$, where $\varepsilon_u = 0$ if $R_2(Z_u)$ is Gorenstein and $\varepsilon_u = 1$ otherwise.

Moreover, since Gorenstein ladder determinantal rings determined by 2-minors are completely classified, we have:

Corollary (See Corollary 3.13.). For any $N \in \mathbb{N}$, there exist ladders Y such that $|\mathfrak{S}_0(R_2(Y))| = 2^N$. In fact, infinitely many such ladders exist.

The paper is organized as follows. We begin with a Background section which, after a brief review of the relevant terms, provides some material on Bass classes. The results on Bass classes will allow us to establish a lower bound on the number of semidualizing modules of ladder determinantal rings constructed from ladders with coincidental inside corners; see Corollary 2.13. In Section 3, we prove the Main Theorem. We begin the section by providing several base cases of ladder determinantal rings with coincidental inside corners (called corners of type 1 in [7]). This is necessary, since the ladder may take many different shapes, requiring careful consideration of each possibility. We work up to the case that $Y = Z_1 \# Z_2$, where each Z_i is a two-sided ladder with no coincidental inside corner (see Proposition 3.11). Then we are in a position to prove the Main Theorem.

2. Background-brief recap and material on Bass classes

Citing the detailed background section in [13], we provide only a brief recap of the relevant terms and facts, before proceeding to the material on Bass classes.

2.1. Brief recap of relevant terms and results

Definition 2.1. The **divisor class group** of a normal domain R , denoted $\text{Cl}(R)$, is the set of isomorphism classes of rank-1 reflexive modules, or equivalently, height-1 reflexive ideals. Denoting a module class by $[M]$, the operations $[M] + [N] = [(M \otimes_R N)^{**}]$, where $(-)^* = \text{Hom}_R(-, R)$ and $[M] - [N] = [\text{Hom}_R(N, M)]$, with additive identity $[R]$, make $\text{Cl}(R)$ into an abelian group.

Definition 2.2. A semidualizing R -module of finite injective dimension is a **dualizing R -module**. If R is Cohen-Macaulay, then a dualizing module is a **canonical module**. A ring R admits only **trivial semidualizing modules** if

$$\mathfrak{S}_0(R) = \begin{cases} \{[R], [\omega_R]\} & \text{if } R \text{ has a dualizing module } \omega_R; \\ \{[R]\} & \text{otherwise.} \end{cases}$$

When we see no danger of confusion, we write $M \in \mathfrak{S}_0(R)$ instead of $[M] \in \mathfrak{S}_0(R)$. A **semidualizing ideal** is an ideal of R that is semidualizing as an R -module.

Fact 2.3. The results below will be used repeatedly.

- (1) If $\mathfrak{a}, \mathfrak{b}$ are semidualizing ideals and $\mathfrak{a} \otimes_R \mathfrak{b}$ is semidualizing, then the multiplication map $\mu : \mathfrak{a} \otimes_R \mathfrak{b} \rightarrow \mathfrak{a}\mathfrak{b}$ is an isomorphism by [14, Proposition 3.3].
- (2) If R is a normal domain, then $\mathfrak{S}_0(R) \subseteq \text{Cl}(R)$ by [14, Proposition 3.4].
- (3) If R is Cohen-Macaulay with (semi)-dualizing modules C, ω_R , respectively, then $\text{Hom}_R(C, \omega_R)$ is semidualizing. Moreover, $\text{Hom}_R(C, \omega_R) \otimes_R C \cong \omega_R$ via evaluation, and $\text{Tor}_i^R(\text{Hom}_R(C, \omega_R), C) = 0$ for all $i \geq 1$. [6, 2.11; 4.4; 4.10].
- (4) If R is a Cohen-Macaulay normal domain with C, ω as in part (3), and $C \neq \omega_R$ are height-1 reflexive ideals, then $\text{Hom}_R(C, \omega_R)$ is naturally isomorphic to a height-1 reflexive ideal C' , and $\omega_R \xleftarrow{\cong} C \otimes_R C' \xrightarrow[\mu]{\cong} CC'$. Thus, $[C] + [C'] = [\omega_R]$. Conversely, if C' is a height-1 reflexive ideal such that $[C] + [C'] = [\omega_R]$, then $C' \cong \text{Hom}_R(C, \omega_R)$, and hence is semidualizing.

Let Y be a ladder, as described in the Introduction. The associated **ladder determinantal ring** of t -minors is $R_t(Y) = \mathbf{k}[Y]/I_t(Y)$, where $I_t(Y)$ is the ideal generated by the $t \times t$ minors of X lying entirely in Y . The ring $R_t(Y)$ is known to be Cohen-Macaulay by Herzog and Trung [11, Corollary 4.10] and a normal domain by Conca [8, Proposition 3.3]. Let x_{ij} denote the residue of $X_{ij} \in Y$ in $R_t(Y)$.

Definition 2.4. The **lower inside corners**³ of Y are the points (a, b) such that the variables $X_{ab}, X_{a-1b}, X_{ab-1} \in Y$, but $X_{a-1b-1} \in X \setminus Y$; these are denoted $X_{a_i b_i}$, or simply (a_i, b_i) , with $1 < a_1 < \dots < a_h < m$. For notational convenience, we also set $(a_0, b_0) = (1, n)$ and $(a_{h+1}, b_{h+1}) = (m, 1)$. Likewise, the **upper inside corners** of a ladder Y are the points (c, d) such that $X_{cd}, X_{c+1d}, X_{cd+1} \in Y$, but $X_{c+1d+1} \in X \setminus Y$; these are denoted $X_{c_j d_j}$, or simply (c_j, d_j) , with $1 < c_1 < \dots < c_k < m$. The ladder Y has **coincidental corners** if $(a_i, b_i) = (c_j, d_j)$ for some $i \in \{1, \dots, h\}$ and $j \in \{1, \dots, k\}$ [13, Section 1]. For notational convenience, we also set $(c_0, d_0) = (1, n)$ and $(c_{k+1}, d_{k+1}) = (m, 1)$. A ladder Y is **one-sided** if it is path-connected, and $h = 0$ or $k = 0$. A ladder Y is **two-sided** if it is path-connected and $h, k > 0$ [13, Definition 1.8].

Example 2.5. In the ladders shown in the Introduction, the lower/upper inside corners of each, respectively, are $(2,2)/(3,2)$; $(4,2)/\{(2,4), (3,3)\}$; and $(3,2)/(3,2)$. The ladder L_3 has a coincidental inside corner at $(3, 2)$.

³ We use A. Conca's [8] notation/description for lower and upper. Thus, $(i, j) \leq (h, k)$ if and only if $i \leq h$ and $j \leq k$. In particular, $(1, 1)$ is lowest since $(1, 1) \leq (h, k)$ for all h, k .

The inside corners determine the rank of the free abelian group $\text{Cl}(R_t(Y))$. To describe how, we use the height-1 prime ideals of $R_2(Y)$ shown below [8, §2]:

$$\begin{aligned} \mathfrak{p}_i &= (x_{pq} \in R_2(Y) \mid p \leq c_j \text{ and } q \leq d_j) & j = 1, \dots, k \\ \mathfrak{q}_i &= (x_{a_{i-1}q} \in R_2(Y)) & i = 1, \dots, h + 1 \\ \mathfrak{q}'_i &= (x_{pb_i} \in R_2(Y)) & i = 1, \dots, h + 1. \end{aligned}$$

Fact 2.6. The facts below were established in [8, §2].

- (1) The set $\{[\mathfrak{q}_1], \dots, [\mathfrak{q}_{h+1}], [\mathfrak{p}_1], \dots, [\mathfrak{p}_k]\}$ is a basis of $\text{Cl}(R_2(Y))$.
- (2) The canonical class is described as $[\omega_R] = \sum_{i=1}^{h+1} \lambda_i [\mathfrak{q}_i] + \sum_{j=1}^k \delta_j [\mathfrak{p}_j]$, where $\lambda_i = a_i + b_i - a_{i-1} - b_{i-1}$ for all $i = 1, \dots, h + 1$ and $\delta_j = a_{i_j} + b_{i_j} - c_j - d_j$ for all $j = 1, \dots, k$, where $i_j = \min\{i : a_i > c_j\}$.
- (3) The ideals \mathfrak{q}'_i are useful for computations. In particular, $[\mathfrak{q}_i] + [\mathfrak{q}'_i] + \sum_{j \in I_i} [\mathfrak{p}_j] = 0$ for all $i = 1, \dots, h + 1$, where $I_i = \{j : 1 \leq j \leq k, (a_{i-1}, b_i) \leq (c_j, d_j)\}$, [8, (i) in Proposition 2.3]. If $I_i = \emptyset$, then $[\mathfrak{q}'_i] = -[\mathfrak{q}_i]$.

Finally, since we are interested in (non)-Gorenstein rings, we note:

Fact 2.7. The ring $R_2(Y)$ is Gorenstein if and only if $m = n$ and all inside corners (i, j) of Y satisfy $i + j = m + 1$ [8, Proposition 2.5]. In particular, if Y is an $m \times n$ matrix and $m, n > 1$, then $R_2(Y)$ is Gorenstein if and only if $m = n$ [5, Corollary 8.9].

2.2. Background-Bass classes

The definition of Bass class originates with H.-B. Foxby [4]. (See also [6].) We present here only those few results we need.

Definition 2.8. Let A be a commutative Noetherian ring with identity, and let M, N be A -modules such that $M \in \mathfrak{S}_0(A)$. Then N is in the **Bass class** with respect to M , written $N \in \mathcal{B}_M(A)$, if

- (i) $\text{Ext}_A^{\geq 1}(M, N) = 0 = \text{Tor}_{\geq 1}^A(M, (\text{Hom}_A(M, N)))$; and
- (ii) $M \otimes_A \text{Hom}_A(M, N) \xrightarrow{\zeta_N^M} N$ is an isomorphism, where $\zeta_N^M(m \otimes \varphi) = \varphi(m)$.

Example 2.9. Let A be a commutative Noetherian ring with identity, and let $M, M' \in \mathfrak{S}_0(R)$. Then $M \in \mathcal{B}_M(A)$ by [15, Corollary 3.2.2(a)]. Also, if $M' \in \mathcal{B}_M(A)$, then [15, Proposition 4.1.1(b)] implies that $\text{Hom}_R(M, M') \in \mathfrak{S}_0(A)$.

Lemma 2.10 ([2, Theorem 4.3]). *Let R and S be algebras finitely generated over a field \mathbf{k} . Set $T = R \otimes_{\mathbf{k}} S$, and let $M, M' \in \mathfrak{S}_0(R)$ and $N, N' \in \mathfrak{S}_0(S)$. Then $M \otimes_{\mathbf{k}} N \in \mathcal{B}_{M' \otimes_{\mathbf{k}} N'}(T)$*

if and only if $M \in \mathcal{B}_{M'}(R)$ and $N \in \mathcal{B}_{N'}(S)$. Likewise, $M' \otimes_k N' \in \mathcal{B}_{M \otimes_k N}(T)$ if and only if $M' \in \mathcal{B}_M(R)$ and $N' \in \mathcal{B}_N(S)$.

Proposition 2.11. *Let R be a standard graded ring, $R_0 = k$ a field, $R = R_0[R_1]$, and $\mathfrak{m} = \langle R_1 \rangle$. Let M, M' be finitely-generated graded R -modules. Then:*

- (1) $M \in \mathfrak{S}_0(R)$ if and only if $M_{\mathfrak{m}} \in \mathfrak{S}_0(R_{\mathfrak{m}})$;
- (2) $M \cong R$ if and only if $M_{\mathfrak{m}} \cong R_{\mathfrak{m}}$;
- (3) M is dualizing for R if and only if $M_{\mathfrak{m}}$ is dualizing for $R_{\mathfrak{m}}$;
- (4) For $M \in \mathfrak{S}_0(R)$, we have $M' \in \mathcal{B}_M(A)$ if and only if $M'_{\mathfrak{m}} \in \mathcal{B}_{M_{\mathfrak{m}}}(A_{\mathfrak{m}})$;
- (5) For $M, M' \in \mathfrak{S}_0(R)$, we have $M \cong M'$ if and only if $M_{\mathfrak{m}} \cong M'_{\mathfrak{m}}$; and
- (6) If R is a normal domain and $\underline{f} \in R$ is a homogeneous R -regular sequence, then the map $\beta: \mathfrak{S}_0(R) \rightarrow \mathfrak{S}_0(R/\underline{f}R)$ given by $C \mapsto C/\underline{f}C$ is well-defined and injective.

Proof. Much of this is a variation on standard localization results. For instance, [15, Proposition 2.2.3] says that a finitely generated R -module C is semidualizing for R if and only if for all maximal ideals \mathfrak{m} the localization $C_{\mathfrak{m}}$ is semidualizing for $R_{\mathfrak{m}}$. One modifies the proof of this result, using the fact that $-\otimes_R R_{\mathfrak{m}}$ is faithfully exact on the category of graded R -modules, to establish part (1). Part (4) is verified similarly, from the proof of [15, Proposition 3.5.4].

The non-trivial implication in part (2) follows from the fact that M and $M_{\mathfrak{m}}$ have the same minimal numbers of generators (over R and $R_{\mathfrak{m}}$, respectively) followed by an application of [15, Corollary 2.1.14]. For the non-standard implication in part (3), use the isomorphism $\text{Ext}_{R_{\mathfrak{m}}}^i(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}, M_{\mathfrak{m}}) \cong \text{Ext}_R^i(R/\mathfrak{m}, M)$ to compare injective dimensions over R and $R_{\mathfrak{m}}$, with part (1).

The non-trivial implication in part (5) merits a little more explanation. Assume that $M, M' \in \mathfrak{S}_0(R)$ satisfy $M_{\mathfrak{m}} \cong M'_{\mathfrak{m}}$. Example 2.9 implies that $M'_{\mathfrak{m}} \cong M_{\mathfrak{m}} \in \mathcal{B}_{M_{\mathfrak{m}}}(R_{\mathfrak{m}})$ and furthermore that $\text{Hom}_R(M, M')_{\mathfrak{m}} \cong \text{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, M'_{\mathfrak{m}}) \in \mathfrak{S}_0(R_{\mathfrak{m}})$. Since $\text{Hom}_R(M, M')$ is finitely generated and graded, part (1) implies that $\text{Hom}_R(M, M') \in \mathfrak{S}_0(R)$. Returning to the isomorphism $M_{\mathfrak{m}} \cong M'_{\mathfrak{m}}$, we conclude that

$$\text{Hom}_R(M, M')_{\mathfrak{m}} \cong \text{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, M'_{\mathfrak{m}}) \cong \text{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, M_{\mathfrak{m}}) \cong R_{\mathfrak{m}}$$

so $\text{Hom}_R(M, M') \cong R$ by part (2). Since part (4) implies that $M' \in \mathcal{B}_M(R)$, it follows by definition of $\mathcal{B}_M(R)$ that

$$M' \cong M \otimes_R \text{Hom}_R(M, M') \cong M \otimes_R R \cong M$$

as desired.

For part (6), assume that R is a normal domain and $\underline{f} \in R$ is a homogeneous R -regular sequence. Fact 2.3(2) implies that $\mathfrak{S}_0(R) \subseteq \text{Cl}(R)$. Thus, since R is standard graded over k , every class of $\text{Cl}(R)$ is represented by a graded module, so every semidualizing

R -module has the structure of a graded R -module. Hence, the map $\mathfrak{S}_0(R) \rightarrow \mathfrak{S}_0(R/\underline{f}R)$ given by $C \mapsto C/\underline{f}C$ is well-defined by [15, Corollary 3.4.3]. To see that this map is injective,⁴ suppose that $M, M' \in \mathfrak{S}_0(R)$ are such that $M/\underline{f}M \cong M'/\underline{f}M'$. Then $M_{\mathfrak{m}}/\underline{f}M_{\mathfrak{m}} \cong (M/\underline{f}M)_{\overline{\mathfrak{m}}} \cong (M'/\underline{f}M')_{\overline{\mathfrak{m}}} \cong M'_{\mathfrak{m}}/\underline{f}M'_{\mathfrak{m}}$. By [15, Proposition 4.2.18], we have $M_{\mathfrak{m}} \cong M'_{\mathfrak{m}}$, hence by part (5), $M \cong M'$. \square

Proposition 2.12. *Let R and S be standard graded rings with $R_0 = \mathfrak{k} = S_0$ a field, $R = R_0[R_1]$, and $S = S_0[S_1]$. Set $T = R \otimes_{\mathfrak{k}} S$, which is standard graded with T_+ maximal. Then there is an injective map $\alpha : \mathfrak{S}_0(R) \times \mathfrak{S}_0(S) \rightarrow \mathfrak{S}_0(T)$ defined by $\alpha([M], [N]) = [M \otimes_{\mathfrak{k}} N]$.*

Proof. The map α is well-defined by [15, Proposition 2.3.6]. For the injectivity of α , let $M, M' \in \mathfrak{S}_0(R)$ and $N, N' \in \mathfrak{S}_0(S)$ such that $M \otimes_{\mathfrak{k}} N \cong M' \otimes_{\mathfrak{k}} N'$. We need to show that $M \cong M'$ and $N \cong N'$. By assumption, $M \otimes_{\mathfrak{k}} N \in \mathcal{B}_{M' \otimes_{\mathfrak{k}} N'}(T)$ and vice versa. Thus, by Lemma 2.10, $M \in \mathcal{B}_{M'}(R), N \in \mathcal{B}_{N'}(S)$ and likewise, $M' \in \mathcal{B}_M(R), N' \in \mathcal{B}_N(S)$. Consequently, $M_{\mathfrak{m}} \in \mathcal{B}_{M'}(R_{\mathfrak{m}})$ and $M'_{\mathfrak{m}} \in \mathcal{B}_M(R_{\mathfrak{m}})$, hence $M_{\mathfrak{m}} \cong M'_{\mathfrak{m}}$, and thus, $M \cong M'$, as per Proposition 2.11(5). Likewise, $N \cong N'$. (See also [2, Theorems 4.3 and 4.6].) \square

Corollary 2.13. *For $t \times t$ ladder determinantal rings R_1, R_2 with ladders Y_1, Y_2 , respectively, let Z be the ladder constructed by identifying the lower left variable y_1 of Y_1 with the upper right variable y_2 of Y_2 . Then we have*

$$|\mathfrak{S}_0(R_1)| \cdot |\mathfrak{S}_0(R_2)| \leq |\mathfrak{S}_0(R_t(Z))| = \left| \mathfrak{S}_0 \left(\frac{R_1 \otimes_{\mathfrak{k}} R_2}{(y_1 - y_2)} \right) \right|.$$

Proof. The rings R_1, R_2 satisfy the assumptions of Proposition 2.12. Also, $R_1 \otimes_{\mathfrak{k}} R_2$ is a normal domain since it is a ladder determinantal ring over the disconnected ladder $Y = Y_1 \cup Y_2$. Thus, the nonzero homogeneous element $f = y_1 - y_2$ is regular and satisfies $R_t(Z) \cong (R_1 \otimes_{\mathfrak{k}} R_2)/(y_1 - y_2)$. The result now immediately follows from the composition of the injective maps $\alpha : \mathfrak{S}_0(R_1) \times \mathfrak{S}_0(R_2) \rightarrow \mathfrak{S}_0(R_1 \otimes_{\mathfrak{k}} R_2)$ and $\beta : \mathfrak{S}_0(R_1 \otimes_{\mathfrak{k}} R_2) \rightarrow \mathfrak{S}_0((R_1 \otimes_{\mathfrak{k}} R_2)/f)$ from Propositions 2.12 and 2.11(6). \square

3. Proof of Main Theorem

We will prove our main result, Theorem 3.12, in this section through a series of inductions. Because many of the arguments proceed in a similar manner, in certain cases only highlights are provided. We begin with some additional notation and an example that will be carried throughout the section.

⁴ The map is not a homomorphism, as $\mathfrak{S}_0(-)$ has no useful group structure, so we can not just check a kernel condition here; see [15, Remark 2.3.5].

3.1. Preliminaries-notation

Notation 3.1. Let Z_0, Z_1 be ladders in matrices of minimal size $m_0 \times n_0, m_1 \times n_1$, respectively. We define $Z_0 \# Z_1$ to be the ladder with a coincidental inside corner formed by identifying the variable $X_{m_0 1}$ of Z_0 with the variable $X_{1 n_1}$ of Z_1 . We repeat this process to get $Z_0 \# Z_1 \# \dots \# Z_w$.

Example 3.2. The ladder L_3 in the Introduction is $Z_0 \# Z_1$ of two 3×2 matrices Z_0, Z_1 ; i.e., ladders with no inside corners. If the elements of Z_0 and Z_1 are indexed as below, then we identify X_{32} with X'_{32} .

$$\begin{array}{cc} X_{12} & X_{13} & X'_{31} & X'_{32} \\ X_{22} & X_{23} & X'_{41} & X'_{42} \\ X_{32} & X_{33} & X'_{51} & X'_{52} \\ & Z_0 & & Z_1 \end{array}$$

Then $R_2(L_3) = R_2(Z_0 \# Z_1) = \frac{R_2(Z_0) \otimes R_2(Z_1)}{(x_{32} - x'_{32})}$. In terms of Corollary 2.13, we have $|\mathfrak{S}_0(R_2(Z_0))| \cdot |\mathfrak{S}_0(R_2(Z_1))| \leq |\mathfrak{S}_0(R_2(Z_0 \# Z_1))|$.

Notation 3.3. Let Z_0, Z_1, \dots, Z_w be ladders with no coincidental inside corners, and let $Z = Z_0 \# Z_1 \# \dots \# Z_w$. We will use double indices to label the corners of Z and the generators of $\text{Cl}(R_2(Z))$. The ladder Z_u will have h_u lower inside corners and k_u upper inside corners. The inside corners of Z which are inside corners of Z_u will be written as $(a_{ui}, b_{ui}), (c_{uj}, d_{uj})$, for $1 \leq i \leq h_u$ and $1 \leq j \leq k_u$. Additionally, there are inside corners of Z , which are not inside corners of any Z_u , but which are variables coincidental to some Z_u and Z_{u+1} . In particular, for all $0 \leq u < w$ we have $(a_{u, h_u+1}, b_{u, h_u+1}) = (c_{u, k_u+1}, d_{u, k_u+1}) = (a_{u+1, 0}, b_{u+1, 0}) = (c_{u+1, 0}, d_{u+1, 0})$. Similarly, we label the ideals of Z as $\mathfrak{q}_{ui}, \mathfrak{p}_{uj}$. Moreover, we write the ideals that contain the variable at the u -th coincidental inside corner, where $1 \leq u \leq w$, as \mathfrak{q}_{u1} and \mathfrak{p}_{u0} . That is,

$$\begin{aligned} \mathfrak{p}_{uj} &= (x_{pq} \in R_2(Z) \mid p \leq c_{uj} \text{ and } q \leq d_{uj}) && \text{for all } 1 \leq j \leq k_u \text{ and } 0 \leq u \leq w, \\ \mathfrak{p}_{u0} &= (x_{pq} \in R_2(Z) \mid p \leq c_{u0} \text{ and } q \leq d_{u0}) && \text{for all } 1 \leq u \leq w, \text{ and} \\ \mathfrak{q}_{ui} &= (x_{a_{\{u, i-1\}, q}} \in R_2(Y) \mid q \in \mathbb{N}) && \text{for all } 1 \leq i \leq h_u + 1 \text{ and } 0 \leq u \leq w. \end{aligned}$$

We will also use $\mathfrak{q}_{ui}, \mathfrak{p}_{uj}$ to denote the *restrictions* of these ideals in $R_2(Z_u)$ for $j \neq 0$. On the other hand, we will identify $[\omega_{R_2(Z_u)}] \in \text{Cl}(R_2(Z_u))$ with its *image* in $\text{Cl}(R_2(Z))$. That is, we write

$$[\omega_{R_2(Z_u)}] = \sum_{i=1}^{h_u+1} \lambda_{ui} [\mathfrak{q}_{ui}] + \sum_{j=1}^{k_u} \delta_{uj} [\mathfrak{p}_{uj}] \text{ in } \text{Cl}(R_2(Z_u)), \text{ but}$$

$$[\omega_{R_2(Z_u)}] = \lambda_{u1}([\mathbf{p}_{u0}] + [\mathbf{q}_{u1}]) + \sum_{i=2}^{h_u+1} \lambda_{ui}[\mathbf{q}_{ui}] + \sum_{j=1}^{k_u} \delta_{uj}[\mathbf{p}_{uj}] \text{ in } \text{Cl}(R_2(Z)).$$

Note that in $\text{Cl}(R_2(Z))$, the class $[\mathbf{p}_{u0}] + [\mathbf{q}_{u1}] = [\mathbf{p}_{u0} \cap \mathbf{q}_{u1}]$ is the image of $[\mathbf{q}_{u1}] \in \text{Cl}(R_2(Z_u))$ by [13, Lemma 2.2]. We then have $[\omega_{R_2(Z)}] = [\omega_{R_2(Z_0)}] + \cdots + [\omega_{R_2(Z_u)}]$ in $\text{Cl}(R_2(Z))$.

Example 3.2 (continued). Recall $L_3 = Z_0 \# Z_1$ with corners $(a_{00}, b_{00}) = (c_{00}, d_{00}) = (1, 3)$, $(a_{01}, b_{01}) = (a_{10}, b_{10}) = (c_{01}, d_{01}) = (c_{10}, d_{10}) = (3, 2)$ and $(a_{11}, b_{11}) = (c_{11}, d_{11}) = (5, 1)$. The ring $R_2(Z)$ has ideals $\mathbf{q}_{01} = (x_{12}, x_{13})$, $\mathbf{q}_{11} = (x_{31}, x_{32}, x_{33})$ and $\mathbf{p}_{10} = (x_{12}, x_{22}, x_{31}, x_{32})$. The ring $R_2(Z_0)$ has ideal $\mathbf{q}_{01} = (x_{12}, x_{13})$ and the ring $R_2(Z_1)$ has ideal $\mathbf{q}_{11} = (x'_{31}, x'_{32})$.

We identify $[\omega_{R_2(Z_0)}] = [(x_{12}, x_{13})] = [\mathbf{q}_{01}] \in \text{Cl}(R_2(Z_0))$ with $[\mathbf{q}_{01}] \in \text{Cl}(R_2(L_3))$. We identify $[\omega_{R_2(Z_1)}] = [(x'_{31}, x'_{32})] = [\mathbf{q}_{11}] \in \text{Cl}(R_2(L_3))$ with $[\mathbf{q}_{11}] + [\mathbf{p}_{10}] = [\mathbf{q}_{11} \cap \mathbf{p}_{10}] = [(x_{31}, x_{32}, x_{33}) \cap (x_{12}, x_{22}, x_{31}, x_{32})] = [(x_{31}, x_{32})] \in \text{Cl}(R_2(L_3))$. With such identification, we have $[\omega_{R_2(L_3)}] = [\mathbf{q}_{01}] + [\mathbf{q}_{11}] + [\mathbf{p}_{10}] = [\omega_{R_2(Z_1)}] + [\omega_{R_2(Z_0)}]$.

Notation 3.4. When we are considering a ladder Y and would like to discuss a new related ladder, we will use the notation $Y^\bullet, Y^\dagger, \tilde{Y}$, etc., to denote the new ladders. The notation R^\bullet, R^\dagger , etc., will always denote the associated ladder determinantal ring $R_2(Y^\bullet), R_2(Y^\dagger)$, respectively.

Definition 3.5. ([13, Definition 3.4]) The **antitranspose** of a ladder Y is the ladder \tilde{Y} obtained by **antitransposing** Y , so that $\tilde{Y}_{ij} = X_{a_{h+1-j}+a_0, b_0-i+b_{h+1}}$. The ladder \tilde{Y} has corners $(\tilde{a}_0, \tilde{b}_0) = (b_{h+1}, a_{h+1})$, $(\tilde{a}_1, \tilde{b}_1) = (b_0 - d_1 + b_{h+1}, a_{h+1} - c_1 + a_0), \dots, (\tilde{a}_k, \tilde{b}_k) = (b_0 - d_k + b_{h+1}, a_{h+1} - c_k + a_0)$, $(\tilde{a}_{k+1}, \tilde{b}_{k+1}) = (b_0, a_0)$, $(\tilde{c}_1, \tilde{d}_1) = (b_0 - b_1 + b_{h+1}, a_{h+1} - a_1 + a_0), \dots, (\tilde{c}_h, \tilde{d}_h) = (b_0 - b_h + b_{h+1}, a_{h+1} - a_h + a_0)$.

3.2. Base cases

We begin establishing the main result by proving some base cases. In each of these statements (3.6-3.11), the number of semidualizing modules of $R = R_2(Y)$ is either 1, 2, or 4; we are setting $Y = Z_0 \# Z_1$, hence $\mathfrak{S}_0(R) = 2^{\varepsilon_0 + \varepsilon_1}$, where $\varepsilon_i = 0$ if $R_2(Z_i)$ is Gorenstein and $\varepsilon_i = 1$ otherwise.

Proposition 3.6. Let $Y = Z_0 \# Z_1$ be a 2-connected ladder with exactly one coincidental inside corner, where Z_0, Z_1 are matrices of indeterminates, as shown below. Let $R = R_2(Y)$. Then $\mathfrak{S}_0(R) = \{[R], [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1)}], [\omega_R]\}$, where $[R]$ is the 0 class and $[\omega_R] = [\omega_{R_2(Z_0)}] + [\omega_{R_2(Z_1)}]$. In particular, $|\mathfrak{S}_0(R)| = |\mathfrak{S}_0(R_2(Z_0))| \cdot |\mathfrak{S}_0(R_2(Z_1))|$.

$$\begin{array}{ccccccc}
 & & & X_{1,b_{01}} & X_{1,b_{01}+1} & \cdots & X_{1,n} \\
 & & & \vdots & \vdots & Z_0 & \vdots \\
 & & & X_{a_{01}-1,b_{01}} & X_{a_{01}-1,b_{01}+1} & \cdots & X_{a_{01}-1,n} \\
 X_{a_{10},1} & \cdots & X_{a_{10},b_{10}-1} & X_{a_{10},b_{10}} & X_{a_{10},b_{10}+1} & \cdots & X_{a_{10},n} \\
 X_{a_{10}+1,1} & \cdots & X_{a_{10}+1,b_{10}-1} & X_{a_{10}+1,b_{10}} & & & \\
 \vdots & Z_1 & \vdots & \vdots & & & \\
 X_{m,1} & \cdots & X_{m,b_{10}-1} & X_{m,b_{10}} & & & \\
 & & & & & (a_{10}, b_{10}) = (a_{01}, b_{01}) \\
 & & & & & = (c_{10}, d_{10}) = (c_{01}, d_{01})
 \end{array}$$

Proof. Let Y be the ladder shown above, where $2 \leq a_{10} \leq m - 1, 2 \leq b_{10} \leq n - 1$. This ladder is 2-connected and we have $(a_{00}, b_{00}) = (c_{00}, d_{00}) = (1, n), (a_{01}, b_{01}) = (a_{10}, b_{10}) = (c_{01}, d_{01}) = (c_{10}, d_{10}),$ and $(a_{11}, b_{11}) = (c_{11}, d_{11}) = (m, 1)$. With $\mathfrak{q}_{01}, \mathfrak{q}_{11},$ and \mathfrak{p}_{10} the ideals shown below, the class group of R is $\text{Cl}(R) \cong \mathbb{Z}[\mathfrak{q}_{01}] \oplus \mathbb{Z}[\mathfrak{q}_{11}] \oplus \mathbb{Z}[\mathfrak{p}_{10}],$ where

$$\begin{aligned}
 \mathfrak{q}_{01} &= (x_{1,b_{01}}, x_{1,b_{01}+1}, \dots, x_{1,n}), \\
 \mathfrak{q}_{11} &= (x_{a_{10},1}, \dots, x_{a_{10},b_{10}}, \dots, x_{a_{10},n}), \text{ and} \\
 \mathfrak{p}_{10} &= (x_{1,b_{01}}, x_{2,b_{01}}, \dots, x_{a_{01},b_{01}}, x_{a_{10},1}, x_{a_{10},2}, \dots, x_{a_{10},b_{10}-1}).
 \end{aligned}$$

The canonical class of R is $[\omega_R] = \lambda_{01}[\mathfrak{q}_{01}] + \lambda_{11}[\mathfrak{q}_{11}] + \lambda_{11}[\mathfrak{p}_{10}],$ where $\lambda_{01} = a_{01} + b_{01} - 1 - n, \lambda_{11} = m + 1 - a_{10} - b_{10}$ and $\delta_{10} = \lambda_{11}$.

The proof will proceed by inverting variables in Y . We will let C_1, C_2, \dots denote possible semidualizing modules of R .

Step 1. First, let Y^\bullet be the ladder obtained by deleting rows $a_{00}, a_{00} + 1, \dots, a_{01} - 1$ and columns $b_{01} + 1, b_{01} + 2, \dots, b_{00}$ of Y ; that is, $Y^\bullet = Z_1$. Invert $x_{1,b_{01}}$ in R and let ρ^\bullet be the composition of the following natural surjections:

$$\text{Cl}(R) \rightarrow \text{Cl}(R_{x_{1,b_{01}}}) \xrightarrow{\cong} \text{Cl}(R^\bullet).$$

In particular, $\text{Cl}(R^\bullet) \cong \mathbb{Z}[\mathfrak{q}_{11}],$ where (the new) \mathfrak{q}_{11} is the ideal generated by the (images in R^\bullet of the) variables in the first row of $Y^\bullet,$ by [5, Corollary 8.4], and $[\omega_{R^\bullet}] = \lambda_{11}[\mathfrak{q}_{11}]$ by [5, (7.10),(8.8)].

Under the natural map $\rho^\bullet: \text{Cl}(R) \rightarrow \text{Cl}(R^\bullet),$ we have $\rho^\bullet([\mathfrak{q}_{01}]) = 0, \rho^\bullet([\mathfrak{p}_{10}]) = 0$ and $\rho^\bullet([\mathfrak{q}_{11}]) = [\mathfrak{q}_{11}],$ so $\text{Ker}(\rho^\bullet) = \mathbb{Z}[\mathfrak{q}_{01}] \oplus \mathbb{Z}[\mathfrak{p}_{10}].$ The determinantal ring R^\bullet has semidualizing modules R^\bullet and ω_{R^\bullet} only. Since the localization of a semidualizing module is also a semidualizing module, the only possible semidualizing modules of R are in $\varphi^{-1}([R^\bullet]) = \mathbb{Z}[\mathfrak{q}_{01}] \oplus \mathbb{Z}[\mathfrak{p}_{10}]$ or $\varphi^{-1}([\omega_{R^\bullet}]) = \mathbb{Z}[\mathfrak{q}_{01}] \oplus \mathbb{Z}[\mathfrak{p}_{10}] + \lambda_{11}[\mathfrak{q}_{11}].$ Thus, the possible semidualizing modules of R are $[C_1] = r[\mathfrak{q}_{01}] + s[\mathfrak{p}_{10}]$ and $[C_2] = u[\mathfrak{q}_{01}] + v[\mathfrak{p}_{10}] + \lambda_{11}[\mathfrak{q}_{11}] (= u[\mathfrak{q}_{01}] + v[\mathfrak{p}_{10}] + [\omega_{R_2(L)}] - \lambda_{11}[\mathfrak{p}_{10}]),$ where $r, s, u, v \in \mathbb{Z}.$

Step 2. Next, obtain a ladder Y^\dagger by deleting rows $1, \dots, a_{01} - 1$ and columns $b_{01} + 1, \dots, n$ of Y ; in fact, $Y^\dagger = Z_1$. Invert $x_{a_{01},n}$ in R and let ρ^\dagger be the composition of the following natural surjections:

$$\text{Cl}(R) \rightarrow \text{Cl}(R_{x_{a_{01},n}}) \xrightarrow{\cong} \text{Cl}(R^\dagger).$$

Under the natural map $\rho^\dagger: \text{Cl}(R) \rightarrow \text{Cl}(R^\dagger)$, we have $[\mathfrak{q}_{01}], [\mathfrak{q}_{11}] \mapsto 0$ and $[\mathfrak{p}_{10}] \mapsto [\mathfrak{q}_{11}]$ (the new \mathfrak{q}_{11}). Again $\mathfrak{S}_0(R^\dagger) = \{[R^\dagger], [\omega_{R^\dagger}]\}$, where $[\omega_{R^\dagger}] = \lambda_{11}[\mathfrak{q}_{11}]$.

To determine the semidualizing modules of R , consider the possible images of $[C_1], [C_2]$ under ρ^\dagger :

$$\rho^\dagger(r[\mathfrak{q}_{01}] + s[\mathfrak{p}_{10}]) = 0 \Rightarrow s = 0 \quad \text{and} \quad \rho^\dagger(r[\mathfrak{q}_{01}] + s[\mathfrak{p}_{10}]) = \lambda_{11}[\mathfrak{q}_{11}] \Rightarrow s = \lambda_{11},$$

and similarly $v = 0$ or λ_{11} . Hence, the possible semidualizing modules of R are $[C_3] = r[\mathfrak{q}_{01}]$, $[C_4] = r[\mathfrak{q}_{01}] + \lambda_{11}[\mathfrak{p}_{10}]$, $[C_5] = u[\mathfrak{q}_{01}] + \lambda_{11}[\mathfrak{q}_{11}]$ and $[C_6] = u[\mathfrak{q}_{01}] + \lambda_{11}[\mathfrak{p}_{10}] + \lambda_{11}[\mathfrak{q}_{11}]$.

Step 3. Thirdly, obtain $Y^{\bullet\bullet}$ by deleting rows $a_{10} + 1, \dots, m$ and columns $1, \dots, b_{10} - 1$ of Y ; that is, $Y^{\bullet\bullet} = Z_0$. Invert $x_{a_{10},1}$ in R . Under the natural map $\rho^{\bullet\bullet}: \text{Cl}(R) \rightarrow \text{Cl}(R^{\bullet\bullet})$, we have $[\mathfrak{q}_{11}], [\mathfrak{p}_{10}] \mapsto 0$ and $[\mathfrak{q}_{01}] \mapsto [\mathfrak{q}_{01}]$. Since $R^{\bullet\bullet}$ is a determinantal ring, we know that $\mathfrak{S}_0(R^{\bullet\bullet}) = \{[R^{\bullet\bullet}], [\omega_{R^{\bullet\bullet}}]\}$, where $[\omega_{R^{\bullet\bullet}}] = \lambda_{01}[\mathfrak{q}_{01}]$.

If $\rho^{\bullet\bullet}([C_3]) = r[\mathfrak{q}_{01}] = 0$, then $r = 0$, and $[C_3] = 0$ is a trivial semidualizing module of R . If $\rho^{\bullet\bullet}([C_3]) = \lambda_{01}[\mathfrak{q}_{01}]$, then $r = \lambda_{01}$. Doing the same for C_4, C_5, C_6 , we get the following possible nontrivial semidualizing modules of R .

$$\begin{aligned} [C_7] &= \lambda_{01}[\mathfrak{q}_{01}] = [\omega_{R_2(Z_0)}] \\ [C_8] &= \lambda_{11}[\mathfrak{p}_{10}] \\ [C_9] &= \lambda_{01}[\mathfrak{q}_{01}] + \lambda_{11}[\mathfrak{p}_{10}] = [\omega_R] - [C_{10}] \\ [C_{10}] &= \lambda_{11}[\mathfrak{q}_{11}] \\ [C_{11}] &= \lambda_{01}[\mathfrak{q}_{01}] + \lambda_{11}[\mathfrak{q}_{11}] = [\omega_R] - [C_8] \\ [C_{12}] &= \lambda_{11}[\mathfrak{p}_{10}] + \lambda_{11}[\mathfrak{q}_{11}] = [\omega_{R_2(Z_1)}] \end{aligned}$$

Hence, it remains to show that C_8, C_9, C_{10}, C_{11} can not be nontrivial semidualizing modules of R .

Step 4. Fourthly, obtain the ladder $Y^{\dagger\dagger}$ by deleting rows $a_{10} + 1, \dots, m$ and columns $1, \dots, b_{10} - 1$ of Y ; in fact, $Y^{\dagger\dagger} = Z_0$. Invert $x_{m,b_{10}}$ in R . Under the natural map $\rho^{\dagger\dagger}: \text{Cl}(R) \rightarrow \text{Cl}(R^{\dagger\dagger})$, we have $[\mathfrak{q}_{01}] \mapsto [\mathfrak{q}_{01}]$, $[\mathfrak{q}_{11}] \mapsto [(x_{a_{01},b_{01}}, \dots, x_{a_{01},n})] = [\mathfrak{q}_{01}]$, and $[\mathfrak{p}_{10}] \mapsto [(x_{1,b_{01}}, x_{2,b_{01}}, \dots, x_{a_{01},b_{01}})] = -[\mathfrak{q}_{01}]$. Again, $\mathfrak{S}_0(R^{\dagger\dagger}) = \{[R^{\dagger\dagger}], [\omega_{R^{\dagger\dagger}}]\}$, where $[\omega_{R^{\dagger\dagger}}] = \lambda_{01}[\mathfrak{q}_{01}]$.

We can now show that the modules C_8, C_9, C_{10}, C_{11} can not be nontrivial semidualizing modules of R . If $\rho^{\dagger\dagger}([C_8]) = -\lambda_{11}[\mathfrak{q}_{01}] = 0$ (equivalently, $\rho^{\dagger\dagger}([C_{11}]) = [\omega_{R^{\dagger\dagger}}]$), then $\lambda_{11} = 0$, so $[C_8] = 0$ is a trivial semidualizing module. If $\rho^{\dagger\dagger}([C_8]) = [\omega_{R^{\dagger\dagger}}] = \lambda_{01}[\mathfrak{q}_{01}]$ (equivalently, $\rho^{\dagger\dagger}([C_{11}]) = 0$), then $\lambda_{11} = -\lambda_{01}$. Similarly, if $\rho^{\dagger\dagger}([C_{10}]) = \lambda_{11}[\mathfrak{q}_{01}] = 0$ or $[\omega_{R^{\dagger\dagger}}]$ (equivalently, $\rho^{\dagger\dagger}([C_9]) = [\omega_{R^{\dagger\dagger}}]$ or 0 respectively), then $\lambda_{11} = 0$ or λ_{01} respectively. So we only need to show that $\lambda_{11} = \pm\lambda_{01} \neq 0$ leads to a contradiction.

Case 1. $\lambda_{11} = -\lambda_{01} > 0$. We have $[C_8] = \lambda_{11}[\mathfrak{p}_{10}]$ and $[C_{11}] = -\lambda_{11}[\mathfrak{q}_{01}] + \lambda_{11}[\mathfrak{q}_{11}]$. By Fact 2.6(3), we have $[C_{11}] = \lambda_{11}([\mathfrak{q}'_{01}] + [\mathfrak{p}_{10}] + [\mathfrak{q}_{11}])$. By [13, Lemma 2.1],

$$[C_8] = [\mathfrak{p}_{10}^{\lambda_{11}}] \text{ and}$$

$$[C_{11}] = [(\mathfrak{q}'_{01})^{\lambda_{11}} \cap \mathfrak{p}_{10}^{\lambda_{11}} \cap \mathfrak{q}_{11}^{\lambda_{11}}].$$

Let us identify C_8 with the ideal $\mathfrak{p}_{10}^{\lambda_{11}}$, and likewise for C_{11} . Then under the multiplication map $\mu: C_8 \otimes C_{11} \rightarrow C_8 C_{11}$, we have

$$\mu(x_{a_{10}, b_{10}}^{\lambda_{11}} \otimes x_{1, b_{01}} x_{a_{10}, 1} x_{a_{10}, b_{10}}^{\lambda_{11}-1}) = \mu(x_{a_{10}, 1} x_{1, b_{01}} x_{a_{10}, b_{10}}^{\lambda_{11}-1} \otimes x_{a_{10}, b_{10}}^{\lambda_{11}}).$$

Hence μ is not injective, contradicting Fact 2.3 (1), so C_8, C_{11} are not semidualizing modules. Since $\lambda_{11} \neq \lambda_{01}$ in this case, the modules C_9, C_{10} are not semidualizing either, so the only remaining possible classes of nontrivial semidualizing modules are $[C_7] = [\omega_{R_2(Z_0)}]$ and $[C_{12}] = [\omega_{R_2(Z_1)}]$.

Case 2. $\lambda_{11} = \lambda_{01} > 0$. By [13, Lemma 2.1], we have

$$[C_9] = \lambda_{11}[\mathfrak{q}_{01}] + \lambda_{11}[\mathfrak{p}_{10}] = [\mathfrak{q}_{01}^{\lambda_{11}} \cap \mathfrak{p}_{10}^{\lambda_{11}}] \text{ and}$$

$$[C_{10}] = [\mathfrak{q}_{11}^{\lambda_{11}}].$$

As in the previous case, we identify C_9, C_{10} with the corresponding ideals on the right. Then under the multiplication map $\mu: C_9 \otimes C_{10} \rightarrow C_9 C_{10}$, we have

$$\begin{aligned} \mu(x_{1, b_{01}}^{\lambda_{11}-1} x_{1, n} x_{a_{10}, 1} \otimes x_{a_{01}, b_{01}} x_{a_{01}, n}^{\lambda_{11}-1}) &= x_{a_{10}, 1} x_{1, b_{01}}^{\lambda_{11}-1} x_{1, n} x_{a_{01}, b_{01}} x_{a_{01}, n}^{\lambda_{11}-1} \\ &= x_{a_{10}, 1} x_{1, b_{01}}^{\lambda_{11}} x_{a_{01}, n}^{\lambda_{11}} \\ &= \mu(x_{a_{10}, 1} x_{1, b_{01}}^{\lambda_{11}} \otimes x_{a_{01}, n}^{\lambda_{11}}). \end{aligned}$$

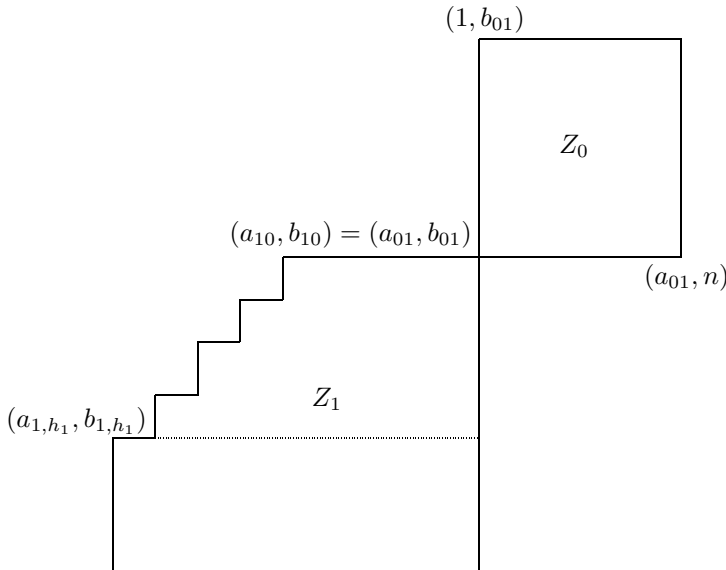
Hence μ is not injective, and we reach the same conclusion as in Case 1.

Case 3. $\lambda_{01} = -\lambda_{11} > 0$. In this case, we take the antitranspose \tilde{Y} of Y . The coincidental inside corner of \tilde{Y} is at $(n + 1 - b_{01}, m + 1 - a_{01}) = (n + 1 - b_{10}, m + 1 - a_{10})$. For \tilde{Y} , we have $\tilde{\lambda}_{01} = (n + 1 - b_{01}) + (m + 1 - a_{01}) - (m + 1) = n + 1 - a_{01} - b_{01} = -\lambda_{01}$ and $\tilde{\lambda}_{11} = (n + 1) - (n + 1 - b_{10}) - (m + 1 - a_{10}) = a_{10} + b_{10} - m - 1 = -\lambda_{11}$. Then $\tilde{\lambda}_{11} = -\tilde{\lambda}_{01} > 0$, and we can use Case 1.

Case 4. $\lambda_{01} = \lambda_{11} < 0$. Then we antitranspose Y and use Case 2.

In summary, we have established that $\mathfrak{S}_0(R) \subseteq \{[R], [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1)}], [\omega_R]\}$, where $[\omega_R] = [\omega_{R_2(Z_0)}] + [\omega_{R_2(Z_1)}]$, and hence $|\mathfrak{S}_0(R)| \leq |\mathfrak{S}_0(R_2(Z_0))| \cdot |\mathfrak{S}_0(R_2(Z_1))|$. The reverse inequality is given by Corollary 2.13, which completes the proof. \square

Lemma 3.7. Let $Y = Z_0 \# Z_1$ be a 2-connected ladder, where Z_0 is a matrix of indeterminates and Z_1 is a one-sided ladder. Let $R = R_2(Y)$. Then $\mathfrak{S}_0(R) = \{[R], [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1)}], [\omega_R]\}$. In particular, $|\mathfrak{S}_0(R)| = |\mathfrak{S}_0(R_2(Z_0))| \cdot |\mathfrak{S}_0(R_2(Z_1))|$.



Proof. Since Z_0 is a matrix, $h_0 = k_0 = 0$. Now if $h_1 = 0$ for the ladder Z_1 , then the ladder Y can be antitransposed to obtain a one-sided ladder \tilde{Z}_1 with $k_1 = 0$. Hence we may assume that $k_1 = 0$ for the ladder Z_1 , and that $Z_0 \# Z_1$ takes the shape above. Set $Y = Z_0 \# Z_1$ and $R = R_2(Y)$. The class group of R is $\text{Cl}(R) \cong \mathbb{Z}[\mathfrak{q}_{01}] \oplus \mathbb{Z}[\mathfrak{q}_{11}] \oplus \mathbb{Z}[\mathfrak{q}_{12}] \oplus \dots \oplus \mathbb{Z}[\mathfrak{q}_{1, h_1+1}] \oplus \mathbb{Z}[\mathfrak{p}_{10}]$; i.e., it is free of rank $h_1 + 3$. The canonical class of R is

$$[\omega_R] = \lambda_{01}[\mathfrak{q}_{01}] + \sum_{i=1}^{h_1+1} \lambda_{1i}[\mathfrak{q}_{1i}] + \delta_{10}[\mathfrak{p}_{10}],$$

where $\lambda_{01} = a_{01} + b_{01} - 1 - n$, $\lambda_{1i} = a_{1,i} + b_{1,i} - a_{1,i-1} - b_{1,i-1}$ and $\delta_{10} = \lambda_{11}$.

The Lemma is proved by induction on h_1 . The base case of $h_1 = 0$ is given by Proposition 3.6. Thus, let $h_1 > 0$ and assume that the Lemma holds for any 2-connected ladder $\tilde{Z}_0 \# \tilde{Z}_1$, where \tilde{Z}_0 is a matrix and \tilde{Z}_1 is a one-sided ladder with $h_1 - 1$ lower inside corners. As before, C_1, C_2 will denote possible semidualizing modules of R .

Step 5. Obtain the ladder Y^\bullet by deleting rows $1, \dots, a_{01} - 1$ and columns $b_{01} + 1, \dots, b_{00}$ of Y ; that is, $Y^\bullet = Z_1$. Invert $x_{1, b_{01}}$ in R . Under the natural map $\rho^\bullet: \text{Cl}(R) \rightarrow \text{Cl}(R^\bullet)$, where $R^\bullet = R_2(Z_1)$, we have $[q_{01}], [p_{10}] \mapsto 0$ and $[q_{1i}] \mapsto [q_{1i}]$ for all $1 \leq i \leq h_1 + 1$. We know $\text{Cl}(R^\bullet) \cong \mathbb{Z}^{h_1+1}$, generated by the ideals q_{1i} for $1 \leq i \leq h_1 + 1$, so $\text{Ker}(\rho^\bullet) = \mathbb{Z}[q_{01}] \oplus \mathbb{Z}[p_{10}]$. By the One-Sided Ladder Theorem [13], $\mathfrak{S}_0(R^\bullet) = \{0, [\omega_{R^\bullet}]\}$, where $[\omega_{R^\bullet}] = \sum_{i=1}^{h_1+1} \lambda_{1i}[q_{1i}]$. Hence the possible classes of semidualizing modules of R are $[C_1] = r[q_{01}] + s[p_{10}]$ and $[C_2] = u[q_{01}] + v[p_{10}] + \sum_{i=1}^{h_1+1} \lambda_{1i}[q_{1i}] (= u[q_{01}] + v[p_{10}] + [\omega_{R^\bullet}] - \lambda_{11}[p_{10}])$, where $r, s, u, v \in \mathbb{Z}$.

Step 6. Next, obtain $Y^{\bullet\bullet}$ by deleting rows $a_{1, h_1} + 1, \dots, a_{1, h_1+1}$ and columns $1, \dots, b_{1, h_1} - 1$ of Y . Then $Y^{\bullet\bullet} = Z_0 \# Z_1^{\bullet\bullet}$, where $Z_1^{\bullet\bullet}$ is a one-sided ladder with one fewer inside corner than Z_1 . Invert $x_{a_{1, h_1}, 1}$ in R . Under the natural map $\rho^{\bullet\bullet}: \text{Cl}(R) \rightarrow \text{Cl}(R^{\bullet\bullet})$, we have $[q_{1, h_1+1}] \mapsto 0$, $[q_{01}] \mapsto [q_{01}]$, $[p_{10}] \mapsto [p_{10}]$ and $[q_{1i}] \mapsto [q_{1i}]$ for all $1 \leq i \leq h_1$. Since $Z_1^{\bullet\bullet}$ has $h_1 - 1$ lower inside corners, the induction hypothesis gives $\mathfrak{S}_0(R^{\bullet\bullet}) = \{0, [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1^{\bullet\bullet})}], [\omega_{R^{\bullet\bullet}}]\}$. We consider several cases.

Case 5. If $\rho^{\bullet\bullet}([C_1]) = 0$, then $r = s = 0$, and $[C_1] = 0$.

If $\rho^{\bullet\bullet}([C_2]) = 0$, then $u = v = 0$ and $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1$. Then $[C_2] = \lambda_{1, h_1+1}[q_{1, h_1+1}] = \lambda_{11}[p_{10}] + \sum_{i=1}^{h_1+1} \lambda_{1i}[q_{1i}] = [\omega_{R_2(Z_1)}]$.

Case 6. If $\rho^{\bullet\bullet}([C_1]) = [\omega_{R_2(Z_0)}] = \lambda_{01}[q_{01}]$, then $r = \lambda_{01}$ and $s = 0$, so $[C_1] = \lambda_{01}[q_{01}] = [\omega_{R_2(Z_0)}]$.

If $\rho^{\bullet\bullet}([C_2]) = [\omega_{R_2(Z_0)}]$, then $u = \lambda_{01}$, $v = 0$, and $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1$, so that $[C_2] = \lambda_{01}[q_{01}] + \lambda_{11}[p_{10}] + \sum_{i=1}^{h_1+1} \lambda_{1i}[q_{1i}] = [\omega_R]$.

Case 7. If $\rho^{\bullet\bullet}([C_1]) = [\omega_{R_2(Z_1^{\bullet\bullet})}] = \lambda_{11}[p_{10}] + \sum_{i=1}^{h_1} \lambda_{1i}[q_{1i}]$, then $r = 0$, $s = \lambda_{11}$, and $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1$. So $[C_1] = 0$.

If $\rho^{\bullet\bullet}([C_2]) = [\omega_{R_2(Z_1^{\bullet\bullet})}]$, then $u = 0$ and $v = \lambda_{11}$, so that $[C_2] = \lambda_{11}[p_{10}] + \sum_{i=1}^{h_1+1} \lambda_{1i}[q_{1i}] = [\omega_{R_2(Z_1)}]$.

Case 8. If $\rho^{\bullet\bullet}([C_1]) = [\omega_{R^{\bullet\bullet}}] = \lambda_{01}[q_{01}] + \lambda_{11}[p_{10}] + \sum_{i=1}^{h_1} \lambda_{1i}[q_{1i}]$, then $r = \lambda_{01}$, $s = \lambda_{11}$, and $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1$. So $[C_1] = [\omega_{R_2(Z_0)}]$.

If $\rho^{\bullet\bullet}([C_2]) = [\omega_{R^{\bullet\bullet}}]$, then $r = \lambda_{01}$ and $s = \lambda_{11}$, so that $[C_2] = [\omega_R]$.

We have established that $\mathfrak{S}_0(R) \subseteq \{[R], [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1)}], [\omega_R]\}$, and hence $|\mathfrak{S}_0(R)| \leq |\mathfrak{S}_0(R_2(Z_0))| \cdot |\mathfrak{S}_0(R_2(Z_1))|$. The reverse inequality is given by Corollary 2.13. \square

Lemma 3.8. Let $Y = Z_0 \# Z_1$ be a 2-connected ladder, where Z_0 is a matrix of indeterminates and Z_1 is any 2-connected ladder with no coincidental inside corners. Let $R = R_2(Y)$. Then $\mathfrak{S}_0(R) = \{[R], [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1)}], [\omega_R]\}$. In particular, $|\mathfrak{S}_0(R)| = |\mathfrak{S}_0(R_2(Z_0))| \cdot |\mathfrak{S}_0(R_2(Z_1))|$.

Proof-outline. The proof mimics that above, inducting on h_1 . The base case of $h_1 = 0$ is given by Lemma 3.7, and the possible semidualizing modules are determined by the Two-Sided Ladder Theorem [13]. Otherwise, the argument proceeds in a similar manner. \square

Lemma 3.9. *Let $Y = Z_0 \# Z_1$ be a 2-connected ladder, where Z_0, Z_1 are one-sided ladders with $h_0 = k_1 = 0$ or $k_0 = h_1 = 0$. Let $R = R_2(Y)$. Then $\mathfrak{S}_0(R) = \{[R], [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1)}], [\omega_R]\}$. In particular, $|\mathfrak{S}_0(R)| = |\mathfrak{S}_0(R_2(Z_0))| \cdot |\mathfrak{S}_0(R_2(Z_1))|$.*

Proof-outline. The proof is again similar. It may be assumed that $h_0 = k_1 = 0$, for if $Y = Z_0 \# Z_1$ with $k_0 = h_1 = 0$, then Y can be antitransposed. Inducting on h_1 , with the base case of $h_1 = 0$ given by Lemma 3.7, proceed in the manner above. \square

Lemma 3.10. *Let $Y = Z_0 \# Z_1$ be a 2-connected ladder, where Z_0 is a one-sided ladder and Z_1 is any 2-connected ladder with no coincidental inside corners. Let $R = R_2(Y)$. Then $\mathfrak{S}_0(R) = \{[R], [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1)}], [\omega_R]\}$. In particular, $|\mathfrak{S}_0(R)| = |\mathfrak{S}_0(R_2(Z_0))| \cdot |\mathfrak{S}_0(R_2(Z_1))|$.*

Proof. By antitransposing if necessary, we may assume that $k_0 = 0$ for the ladder Z_0 . The class group of R has $\{[q_{01}], \dots, [q_{0,h_0+1}], [q_{11}], \dots, [q_{1,h_1+1}], [p_{10}], \dots, [p_{1k_1}]\}$ as a basis; i.e., $\text{Cl}(R) \cong \mathbb{Z}^{h_0+h_1+k_1+3}$. The canonical class of R is

$$[\omega_R] = \sum_{i=1}^{h_0+1} \lambda_{0i}[q_{0i}] + \sum_{i=1}^{h_1+1} \lambda_{1i}[q_{1i}] + \lambda_{11}[p_{10}] + \sum_{j=1}^{k_1} \delta_{1j}[p_{1j}].$$

We will prove the Lemma by double induction on $(h_0, h_1) \in \mathbb{N}^2$. First, we induct on h_0 . The case $h_0 = 0$ is given by Lemma 3.8. So we may assume that $h_0 > 0$ and that this Lemma holds by induction for $(h_0 - 1, g)$ for any $g \in \mathbb{N}$. Next, we induct on h_1 . The case $h_1 = 0$ is given by Lemma 3.9. Thus, let $h_1 > 0$ and assume by induction that this Lemma holds for (h_0, g) for all $g < h_1$. As before, C_1, C_2, \dots will denote possible semidualizing modules of R .

Step 7. Obtain the ladder Y^\bullet by deleting rows $1, \dots, a_{01} - 1$ and columns $b_{01} + 1, \dots, b_{00}$ of Y . Then $Y^\bullet = Z_0^\bullet \# Z_1$, where Z_0^\bullet is a one-sided ladder with one fewer inside corner than Z_0 . Invert $x_{1,b_{01}}$ in R . Under the natural map $\rho^\bullet: \text{Cl}(R) \rightarrow \text{Cl}(R^\bullet)$, where $R^\bullet = R_2(Y^\bullet)$, we have $[q_{01}] \mapsto 0$, and all other basis elements of $\text{Cl}(R)$ are mapped to their same representations in $\text{Cl}(R^\bullet)$, so $\text{Ker}(\rho^\bullet) = \mathbb{Z}[q_{01}]$. Since Z_0^\bullet has $h_0 - 1$ lower inside corners, the induction hypothesis gives $\mathfrak{S}_0(R^\bullet) = \{0, [\omega_{R_2(Z_0^\bullet)}], [\omega_{R_2(Z_1)}], [\omega_{R^\bullet}]\}$. Thus, R has four possible classes of semidualizing modules $[C_1], [C_2], [C_3], [C_4]$, where

$$[C_1] - r[q_{01}] = 0,$$

$$\begin{aligned}
 [C_2] - s[\mathfrak{q}_{01}] &= \sum_{i=2}^{h_0+1} \lambda_{0i}[\mathfrak{q}_{0i}] = [\omega_{R_2(Z_0)}] - \lambda_{01}[\mathfrak{q}_{01}], \\
 [C_3] - u[\mathfrak{q}_{01}] &= \lambda_{11}[\mathfrak{p}_{10}] + \sum_{i=1}^{h_1+1} \lambda_{1i}[\mathfrak{q}_{1i}] + \sum_{j=1}^{k_1} \delta_{1j}[\mathfrak{p}_{1j}] = [\omega_{R_2(Z_1)}] \text{ and} \\
 [C_4] - v[\mathfrak{q}_{01}] &= \sum_{i=2}^{h_0+1} \lambda_{0i}[\mathfrak{q}_{0i}] + \lambda_{11}[\mathfrak{p}_{10}] + \sum_{i=1}^{h_1+1} \lambda_{1i}[\mathfrak{q}_{1i}] + \sum_{j=1}^{k_1} \delta_{1j}[\mathfrak{p}_{1j}] \\
 &= [\omega_R] - \lambda_{01}[\mathfrak{q}_{01}],
 \end{aligned}$$

such that $r, s, u, v \in \mathbb{Z}$, and the classes $0, [\omega_{R_2(Z_0^*)}], [\omega_{R_2(Z_1)}], [\omega_{R^*}]$ have the same representations in $\text{Cl}(R^*)$ as on the right hand side.

Step 8. Let $\kappa_1 = \min\{i \mid c_{1i} \geq a_{1,h_1}\}$, as in [13, Notation 3.1]. We obtain $Y^{\bullet\bullet}$ by deleting rows $a_{1,h_1} + 1, \dots, a_{1,h_1+1}$ and columns $1, \dots, b_{1,h_1} - 1$ of Y . Then $Y^{\bullet\bullet} = Z_0 \# Z_1^{\bullet\bullet}$, where $Z_1^{\bullet\bullet}$ is a ladder with $h_1 - 1$ lower inside corners and $\kappa_1 - 1$ upper inside corners. Invert $x_{a_{1,h_1},1}$ in R . Under the natural map $\rho^{\bullet\bullet}: \text{Cl}(R) \rightarrow \text{Cl}(R^{\bullet\bullet})$, we have $[\mathfrak{q}_{1,h_1+1}], [\mathfrak{p}_{1,\kappa_1}], \dots, [\mathfrak{p}_{1,k}] \mapsto 0$, and all other basis elements of $\text{Cl}(R)$ are mapped to their same representations in $\text{Cl}(R^{\bullet\bullet})$. Since $Z_1^{\bullet\bullet}$ has $h_1 - 1$ lower inside corners, the induction hypothesis gives $\mathfrak{S}_0(R^{\bullet\bullet}) = \{0, [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1^{\bullet\bullet})}], [\omega_{R^{\bullet\bullet}}]\}$, where

$$\begin{aligned}
 [\omega_{R_2(Z_0)}] &= \sum_{i=1}^{h_0+1} \lambda_{0i}[\mathfrak{q}_{0i}], \\
 [\omega_{R_2(Z_1^{\bullet\bullet})}] &= \lambda_{11}[\mathfrak{p}_{10}] + \sum_{i=1}^{h_1} \lambda_{1i}[\mathfrak{q}_{1i}] + \sum_{j=1}^{\kappa_1-1} \delta_{1j}[\mathfrak{p}_{1j}],
 \end{aligned}$$

and $[\omega_{R^{\bullet\bullet}}] = [\omega_{R_2(Z_0)}] + [\omega_{R_2(Z_1^{\bullet\bullet})}]$. We consider several cases.

Case 9. If $\rho^{\bullet\bullet}([C_1]) = 0$, then $r = 0$, so $[C_1] = 0$.

If $\rho^{\bullet\bullet}([C_2]) = 0$, then $s = 0$ and $\lambda_{0i} = 0$ for all $2 \leq i \leq h_0 + 1$, so $[C_2] = 0$.

If $\rho^{\bullet\bullet}([C_3]) = 0$, then $u = 0$, $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1$ and $\delta_{1j} = 0$ for all $1 \leq j \leq \kappa_1 - 1$. Then $[C_3] = \lambda_{11}[\mathfrak{p}_{10}] + \sum_{i=1}^{h_1+1} \lambda_{1i}[\mathfrak{q}_{1i}] + \sum_{j=1}^{k_1} \delta_{1j}[\mathfrak{p}_{1j}] = [\omega_{R_2(Z_1)}]$.

If $\rho^{\bullet\bullet}([C_4]) = 0$, then $v = 0$, $\lambda_{0i} = 0$ for all $2 \leq i \leq h_0 + 1$, $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1$ and $\delta_{1j} = 0$ for all $1 \leq j \leq \kappa_1 - 1$, so $[C_4] = [\omega_{R_2(Z_1)}]$.

Case 10. If $\rho^{\bullet\bullet}([C_1]) = [\omega_{R_2(Z_0)}]$, then $r = \lambda_{01}$ and $\lambda_{0i} = 0$ for all $2 \leq i \leq h_0 + 1$. Then $[C_1] = \sum_{i=1}^{h_0+1} \lambda_{0i}[\mathfrak{q}_{0i}] = [\omega_{R_2(Z_0)}]$.

If $\rho^{\bullet\bullet}([C_2]) = [\omega_{R_2(Z_0)}]$, then $s = \lambda_{01}$, so $[C_2] = [\omega_{R_2(Z_0)}]$.

If $\rho^{\bullet\bullet}([C_3]) = [\omega_{R_2(Z_0)}]$, then $u = \lambda_{01}$, $\lambda_{0i} = 0$ for all $2 \leq i \leq h_0 + 1$, $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1$ and $\delta_{1j} = 0$ for all $1 \leq j \leq \kappa_1 - 1$, so $[C_3] = [\omega_R]$.

If $\rho^{\bullet\bullet}([C_4]) = [\omega_{R_2(Z_0)}]$, then $v = \lambda_{01}$, $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1$ and $\delta_{1j} = 0$ for all $1 \leq j \leq \kappa_1 - 1$, so $[C_4] = [\omega_R]$.

Case 11. If $\rho^{\bullet\bullet}([C_1]) = [\omega_{R_2(Z_1^{\bullet\bullet})}]$, then $r = 0$, so $[C_1] = 0$.

If $\rho^{\bullet\bullet}([C_2]) = [\omega_{R_2(Z_1^{\bullet\bullet})}]$, then $s = 0$ and $\lambda_{0i} = 0$ for all $2 \leq i \leq h_0 + 1$, so $[C_2] = 0$.

If $\rho^{\bullet\bullet}([C_3]) = [\omega_{R_2(Z_1^{\bullet\bullet})}]$, then $u = 0$, so $[C_3] = [\omega_{R_2(Z_1)}]$.

If $\rho^{\bullet\bullet}([C_4]) = [\omega_{R_2(Z_1^{\bullet\bullet})}]$, then $v = 0$ and $\lambda_{0i} = 0$ for all $2 \leq i \leq h_0 + 1$, so $[C_4] = [\omega_{R_2(Z_1)}]$.

Case 12. If $\rho^{\bullet\bullet}([C_1]) = [\omega_{R^{\bullet\bullet}}]$, then $r = \lambda_{01}$ and $\lambda_{0i} = 0$ for all $2 \leq i \leq h_0 + 1$, so $[C_1] = [\omega_{R_2(Z_0)}]$.

If $\rho^{\bullet\bullet}([C_2]) = [\omega_{R^{\bullet\bullet}}]$, then $s = \lambda_{01}$, so $[C_2] = [\omega_{R_2(Z_0)}]$.

If $\rho^{\bullet\bullet}([C_3]) = [\omega_{R^{\bullet\bullet}}]$, then $u = \lambda_{01}$ and $\lambda_{0i} = 0$ for all $2 \leq i \leq h_0 + 1$, so $[C_3] = [\omega_R]$.

If $\rho^{\bullet\bullet}([C_4]) = [\omega_{R^{\bullet\bullet}}]$, then $v = \lambda_{01}$, so $[C_4] = [\omega_R]$.

We have now shown that $\mathfrak{S}_0(R) \subseteq \{[R], [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1)}], [\omega_R]\}$, and the reverse inclusion is given by Corollary 2.13. \square

Proposition 3.11. *Let $Y = Z_0 \# Z_1$ be a 2-connected ladder, where Z_0, Z_1 are 2-connected ladders with no coincidental inside corners. Let $R = R_2(Y)$. Then $\mathfrak{S}_0(R) = \{[R], [\omega_{R_2(Z_0)}], [\omega_{R_2(Z_1)}], [\omega_R]\}$. In particular, $|\mathfrak{S}_0(R)| = |\mathfrak{S}_0(R_2(Z_0))| \cdot |\mathfrak{S}_0(R_2(Z_1))|$.*

Proof. The proof mimics that of Lemma 3.10, inducting on $(h_0, h_1) \in \mathbb{N}^2$. First, induct on h_0 , where the case $h_0 = 0$ is given by Lemma 3.10; then assume that $h_0 > 0$ and that the Proposition holds by induction for $(h_0 - 1, g)$ for any $g \in \mathbb{N}$. Next, induct on h_1 , where the base case $h_1 = 0$ is given by Lemma 3.10. With $h_1 > 0$, assume by induction that the Proposition holds for (h_0, g) for all $g < h_1$. One difference from the previous proof is in step 1. Here we let $\kappa_2 = \max\{j \mid d_{0j} \geq b_{01}\}$ as in [13, Notation 3.1], and obtain the ladder Y^\bullet by deleting rows $1, \dots, a_{01} - 1$ and columns $b_{01} + 1, \dots, b_{00}$ of Y . The remainder of the proof is now similar to that of Lemma 3.10. \square

3.3. The general argument

Theorem 3.12. *Let $Y = Z_0 \# \dots \# Z_w$ be a 2-connected ladder, where each Z_u is a 2-connected ladder with no coincidental inside corners. Let $R = R_2(Y)$. Then $\mathfrak{S}_0(R) = \{\sum_{u=0}^w \theta_u [\omega_{R_2(Z_u)}] \mid \theta_u = 0 \text{ or } 1\}$. In particular, $|\mathfrak{S}_0(R)| = |\mathfrak{S}_0(R_2(Z_0))| \cdot \dots \cdot |\mathfrak{S}_0(R_2(Z_w))| = 2^{\varepsilon_0 + \dots + \varepsilon_w}$, where $\varepsilon_u = 0$ if $R_2(Z_u)$ is Gorenstein and $\varepsilon_u = 1$ otherwise.*

Proof. An outline of the proof of the Theorem is as follows. First, we induct on the number of coincidental inside corners $w \in \mathbb{N}$. The case $w = 0$ is given by the Two-Sided Ladder Theorem [13]. The case $w = 1$ is given by Proposition 3.11. Thus, we may assume that $w > 1$.

Next, for each $w > 1$, we need to consider in turn the six different combinations of Z_0 and Z_w as in Proposition 3.6 to Proposition 3.11, viz. Z_0 and Z_w are matrices

of indeterminates, Z_0 is a matrix of indeterminates and Z_w is a 2-connected one-sided ladder, and so on, until Z_0 and Z_w are arbitrary 2-connected ladders. The proofs for the six different combinations are similar, as outlined above, hence we will show here only the proofs of two combinations.

Let us consider the case when Z_0, Z_w are matrices of indeterminates, for example.

Step 9. Obtain the ladder Y^\bullet by deleting rows $1, \dots, a_{01} - 1$ and columns $b_{01} + 1, \dots, b_{00}$ of Y ; that is, $Y^\bullet = Z_1 \# \dots \# Z_w$. Invert $x_{1, b_{01}}$ in R . Under the natural map $\rho^\bullet: \text{Cl}(R) \rightarrow \text{Cl}(R^\bullet)$, where $R^\bullet = R_2(Y^\bullet)$, we have $\text{Ker}(\rho^\bullet) = \mathbb{Z}[\mathfrak{q}_{01}] \oplus \mathbb{Z}[\mathfrak{p}_{10}]$. Let $[K] = r[\mathfrak{q}_{01}] + s[\mathfrak{p}_{10}] \in \text{Cl}(R)$, where $r, s \in \mathbb{Z}$. Induction on w gives $\mathfrak{S}_0(R^\bullet) = \{ \sum_{u=1}^w \theta_u [\omega_{R_2(Z_u)}] \mid \theta_u = 0 \text{ or } 1 \}$. Hence the possible classes of semidualizing modules of R have the form $[C] = [K] + \varphi_1([\omega_{R_2(Z_1)}] - \lambda_{11}[\mathfrak{p}_{10}]) + \sum_{u=2}^w \varphi_u [\omega_{R_2(Z_u)}]$, where $\varphi_u = 0$ or 1 for all $1 \leq u \leq w$ (see, for example, Lemma 3.7).

Step 10. Next, obtain $Y^{\bullet\bullet}$ by deleting rows $a_{w0} + 1, \dots, m$ and columns $1, \dots, b_{w0} - 1$ of Y . Then $Y^{\bullet\bullet} = Z_0 \# \dots \# Z_{w-1}$. Invert $x_{a_{w0}, 1}$ in R . Under the natural map $\rho^{\bullet\bullet}: \text{Cl}(R) \rightarrow \text{Cl}(R^{\bullet\bullet})$, we have $[\omega_{R_2(Z_w)}] \mapsto 0$. Induction on w gives $\mathfrak{S}_0(R^{\bullet\bullet}) = \{ \sum_{u=0}^{w-1} \theta_u [\omega_{R_2(Z_u)}] \mid \theta_u = 0 \text{ or } 1 \}$.

Step 11. Let $[C]$ be as in Step 9. Assume that $\rho^{\bullet\bullet}([C]) \in \mathfrak{S}_0(R^{\bullet\bullet})$. We show that we get candidates

$$\begin{aligned} & \{ [C] \in \text{Cl}(R) \mid [C] \text{ is a possible semidualizing module of } R \} \\ & = \{ \sum_{u=0}^w \theta_u [\omega_{R_2(Z_u)}] \mid \theta_u = 0 \text{ or } 1 \}. \end{aligned}$$

(\subseteq): We solve the equation

$$\begin{aligned} \rho^{\bullet\bullet}([K] + \varphi_1([\omega_{R_2(Z_1)}] - \lambda_{11}[\mathfrak{p}_{10}]) + \sum_{u=2}^w \varphi_u [\omega_{R_2(Z_u)}]) &= \sum_{u=0}^{w-1} \psi_u [\omega_{R_2(Z_u)}], \text{ or} \\ r[\mathfrak{q}_{01}] + s[\mathfrak{p}_{10}] + \varphi_1([\omega_{R_2(Z_1)}] - \lambda_{11}[\mathfrak{p}_{10}]) + \sum_{u=2}^{w-1} \varphi_u [\omega_{R_2(L_u)}] &= \sum_{u=0}^{w-1} \psi_u [\omega_{R_2(Z_u)}], \end{aligned}$$

where $\varphi_u, \psi_u = 0$ or 1 . We need to find the coefficient of $[\omega_{R_2(Z_u)}]$ in $[C]$ as in Lemma 3.7, for example. We show how to find the coefficient of $[\omega_{R_2(Z_1)}]$ in $[C]$ in the case $\varphi_1 \neq \psi_1$.

If $\varphi_1 = 0$ and $\psi_1 = 1$, then $s = \lambda_{11}$, $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1 + 1$ and $\delta_{1j} = 0$ for all $1 \leq j \leq k_1$. So $s = 0$, and the coefficient of $[\omega_{R_2(Z_1)}]$ in $[C]$ is 0 .

If $\varphi_1 = 1$ and $\psi_1 = 0$, then $s = 0$, $\lambda_{1i} = 0$ for all $1 \leq i \leq h_1 + 1$ and $\delta_{1j} = 0$ for all $1 \leq j \leq k_1$. So the coefficient of $[\omega_{R_2(Z_1)}]$ in $[C]$ is 0 .

The coefficient of $[\omega_{R_2(Z_u)}]$ in $[C]$ is easier to find in all other cases. We get

$$[C] = \psi_0[\omega_{R_2(Z_0)}] + \sum_{u=1}^{w-1} \min(\varphi_u, \psi_u)[\omega_{R_2(Z_u)}] + \varphi_w[\omega_{R_2(Z_w)}],$$

and certainly $\min(\varphi_u, \psi_u) = 0$ or 1 .

(\supseteq): If $[D] = \sum_{u=0}^w \theta_u[\omega_{R_2(Z_u)}]$ is in $\text{Cl}(R)$, where the coefficients $\theta_u = 0$ or 1 , then $\rho^{\bullet\bullet}([D]) = \sum_{u=0}^{w-1} \theta_u[\omega_{R_2(Z_u)}] \in \mathfrak{S}_0(R^{\bullet\bullet})$. Hence $[D]$ is a possible semidualizing module of R .

Corollary 2.13 now shows that all modules in $\{\sum_{u=0}^w \theta_u[\omega_{R_2(Z_u)}] \mid \theta_u = 0 \text{ or } 1\}$ are in fact semidualizing modules of R .

Next, we consider the case when Z_0, Z_w are arbitrary 2-connected ladders, assuming that the Theorem holds for all five other possible combinations of Z_0 and Z_w . We induct on $(h_0, h_w) \in \mathbb{N}^2$, as in Proposition 3.11, with $h_0, h_w > 0$.

Step 12. Let $\kappa_2 = \max\{j \mid d_{0j} \geq b_{01}\}$. Obtain the ladder Y^\bullet by deleting rows $1, \dots, a_{01} - 1$ and columns $b_{01} + 1, \dots, b_{00}$ of Y . Then $Y^\bullet = Z_0^\bullet \# Z_1 \# \dots \# Z_w$, where Z_0^\bullet has one fewer lower inside corner than Z_0 . Invert $x_{1, b_{01}}$ in R . Under the natural map $\rho^\bullet: \text{Cl}(R) \rightarrow \text{Cl}(R^\bullet)$, where $R^\bullet = R_2(Y^\bullet)$, we have $\text{Ker}(\rho^\bullet) = \mathbb{Z}[\mathfrak{q}_{01}] \oplus \mathbb{Z}[\mathfrak{p}_{01}] \oplus \dots \oplus \mathbb{Z}[\mathfrak{p}_{0\kappa_2}]$. Let $K = r[\mathfrak{q}_{01}] + \sum_{j=1}^{\kappa_2} s_j[\mathfrak{p}_{0j}]$, where $r, s_j \in \mathbb{Z}$. Induction on h_0 gives $\mathfrak{S}_0(R^\bullet) = \{\theta_0[\omega_{R_2(Z_0^\bullet)}] + \sum_{u=1}^w \theta_u[\omega_{R_2(Z_u)}] \mid \theta_u = 0 \text{ or } 1\}$. Thus, the possible classes of semidualizing modules of R have the form $[C] = [K] + \varphi_0([\omega_{R_2(Z_0)}] - \lambda_{01}[\mathfrak{q}_{01}] - \sum_{j=1}^{\kappa_2} \delta_{0j}[\mathfrak{p}_{0j}]) + \sum_{u=1}^w \varphi_u[\omega_{R_2(Z_u)}]$, where $\varphi_u = 0$ or 1 for all $0 \leq u \leq w$, as in Proposition 3.11.

Step 13. Let $\kappa_1 = \min\{i \mid c_{wi} \geq a_{w, h_w}\}$. We obtain $Y^{\bullet\bullet}$ by deleting rows $a_{w, h_w} + 1, \dots, a_{w, h_w+1}$ and columns $1, \dots, b_{w, h_w} - 1$ of Y . Then $Y^{\bullet\bullet} = Z_0 \# \dots \# Z_{w-1} \# Z_w^{\bullet\bullet}$, where $Z_w^{\bullet\bullet}$ is a ladder with $h_w - 1$ lower inside corners. Invert $x_{a_{w, h_w}, 1}$ in R . Under the natural map $\rho^{\bullet\bullet}: \text{Cl}(R) \rightarrow \text{Cl}(R^{\bullet\bullet})$, we have $\text{Ker}(\rho^{\bullet\bullet}) = \mathbb{Z}[\mathfrak{q}_{w, h_w+1}] \oplus \mathbb{Z}[\mathfrak{p}_{w, \kappa_1}] \oplus \dots \oplus \mathbb{Z}[\mathfrak{p}_{w, k_w}]$. Induction on h_w gives $\mathfrak{S}_0(R^{\bullet\bullet}) = \{\theta_w[\omega_{R_2(Z_w^{\bullet\bullet})}] + \sum_{u=0}^{w-1} \theta_u[\omega_{R_2(Z_u)}] \mid \theta_u = 0 \text{ or } 1\}$.

Step 14. Assume that $\rho^{\bullet\bullet}([C]) \in \mathfrak{S}_0(R^{\bullet\bullet})$. We show that we get candidates

$$\begin{aligned} & \{[C] \in \text{Cl}(R) \mid [C] \text{ is a possible semidualizing module of } R\} \\ &= \left\{ \sum_{u=0}^w \theta_u[\omega_{R_2(Z_u)}] \mid \theta_u = 0 \text{ or } 1 \right\}. \end{aligned}$$

(\subseteq): We solve the equation $\rho^{\bullet\bullet}([C]) = \psi_w[\omega_{R_2(Z_w^{\bullet\bullet})}] + \sum_{u=0}^{w-1} \psi_u[\omega_{R_2(Z_u)}]$, where $\rho^{\bullet\bullet}([C])$ equals

$$[K] + \varphi_0 \left([\omega_{R_2(Z_0)}] - \lambda_{01}[\mathfrak{q}_{01}] - \sum_{j=1}^{\kappa_2} \delta_{0j}[\mathfrak{p}_{0j}] \right) + \sum_{u=1}^{w-1} \varphi_u[\omega_{R_2(Z_u)}] + \varphi_w[\omega_{R_2(Z_w^{\bullet\bullet})}],$$

and $\varphi_u, \psi_u = 0$ or 1 . Then certainly $\min(\varphi_u, \psi_u) = 0$ or 1 and (see Lemma 3.10)

$$[C] = \psi_0[\omega_{R_2(Z_0)}] + \sum_{u=1}^{w-1} \min(\varphi_u, \psi_u)[\omega_{R_2(Z_u)}] + \varphi_w[\omega_{R_2(Z_w)}].$$

(\supseteq): If $[D] = \sum_{u=0}^w \theta_u[\omega_{R_2(Z_u)}] \in \text{Cl}(R)$, where $\theta_u = 0$ or 1 , then

$$\rho^{\bullet\bullet}([D]) = \theta_w[\omega_{R_2(Z_w^{\bullet\bullet})}] + \sum_{u=0}^{w-1} \theta_u[\omega_{R_2(Z_u)}] \in \mathfrak{S}_0(R^{\bullet\bullet}).$$

Hence $[D]$ is a possible semidualizing module of R .

Again Corollary 2.13 completes the induction on w . \square

Corollary 3.13. *For any $N \in \mathbb{N}$, there exist ladders Y such that $|\mathfrak{S}_0(R_2(Y))| = 2^N$. In fact, infinitely many such ladders exist.*

Proof. Let $N \in \mathbb{N}$, and for $i = 0, \dots, N - 1$, let $m_i, n_i > 1$ be pairwise distinct integers. Let Z_i be the matrix of variables of size $m_i \times n_i$. Then each (ladder) determinantal ring $R_2(Z_i)$ is not Gorenstein (see Fact 2.7). Setting $Y = Z_0 \# \dots \# Z_{N-1}$, it follows from Theorem 3.12 that $|\mathfrak{S}_0(R_2(Y))| = 2^N$. The same result holds for more general ladders: let Z_i be a ladder of size $m_i \times n_i$ where m_i, n_i are (not necessarily distinct) integers greater than 1. If $m_i \neq n_i$, or all inside corners (r, s) of Z_i satisfy $r + s = m_i + 1$, then $R_2(Z_i)$ is not Gorenstein, per Fact 2.7. \square

Example 3.2 (concluded). For the ladder $L_3 = Z_0 \# Z_1$, of two 3×2 matrices, in the Introduction, $|\mathfrak{S}_0(R_2(L_3))| = 4$. Set $R = R_2(L_3)$. Then $\mathfrak{S}_0(R) = \{[R], [(x_{12}, x_{13})], [(x_{31}, x_{32})], [\omega_R]\}$.

Example 3.14. Finally, for the ladder $Y = L_1 \# L_2 \# L_3$, where the ladders L_i are those from the Introduction, $|\mathfrak{S}_0(R_2(Y))| = 8$ by Theorem 3.12. More specifically, only $R_2(L_2)$ is Gorenstein (see Fact 2.7) and $|\mathfrak{S}_0(R_2(L_1))| = 2$ by the Two-Sided Ladder Theorem [13]. The previous example finishes off the calculation.

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