On semidualizing modules of ladder determinantal rings

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Abstract We identify all semidualizing modules over certain classes of ladder determinantal rings over a field k. Specifically, given a ladder of variables Y, we show that the ring $\mathsf{k}[Y]/I_t(Y)$ has only trivial semidualizing modules up to isomorphism in the following cases: (1) Y is a one-sided ladder, and (2) Y is a two-sided ladder with $t = 2$ and no coincidental inside corners.

1. Introduction

Let R be a commutative noetherian ring and let k be a field. A finitely generated R-module C is *semidualizing* if $\mathrm{Hom}_R(C, C) \cong R$ and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$. The set of isomorphism classes of semidualizing R-modules is denoted $\mathfrak{S}_0(R)$. See Section 2 for background information on these modules.

Semidualizing modules arise in several different contexts. Hans-Bjorn Foxby [7] introduced them to provide a useful generalization of the dualities with respect to a free module of rank 1 and with respect to a dualizing/canonical module. Other applications include progress by Luchezar Avramov and Foxby [2] and Sean Sather-Wagstaff [16] on composition questions for local ring homomorphisms, and advances on a question of Craig Huneke on growth of Bass numbers of local rings by Sather-Wagstaff [17].

Despite the utility of semidualizing modules, very little is known about the set $\mathfrak{S}_0(R)$. Only recently have Saeed Nasseh and Sather-Wagstaff [10] shown that this set is finite. Anders Frankild and Sather-Wagstaff show that the set has even cardinality when R is local, complete, Cohen-Macaulay and not Gorenstein in [8]. At this time, we only have more information than this in very special cases: Olgur Celikbas and Hailong Dao [3] deal with certain Veronese subrings; William Sanders [14] handles some rings of invariants; Sather-Wagstaff treats determinantal rings in [15]; and Nasseh, Sather-Wagstaff and Ryo Takahashi [11, 12] handle the rings that specialize to nontrivial fiber products (this includes the well-known but seemingly undocumented result for rings of minimal multiplicity).

In particular, the following question [15, Question 4.13] of Sather-Wagstaff is still open: If R is a local ring, must the cardinality $|\mathfrak{S}_0(R)|$ be a power of 2? Each of the special cases in the previous paragraph answers this question in the affirmative for its

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certain class of rings. In fact, in most cases the rings admit only *trivial semidualizing modules*, namely, the free module of rank 1 and a dualizing module; exceptions occur for determinantal rings with coefficients in non-Gorenstein rings.

We provide more special-case evidence of an affirmative answer to Sather-Wagstaff's question by studying the semidualizing modules of ladder determinantal rings. Roughly speaking, a ladder is a subset Y of an $m \times n$ matrix X of indeterminates that (possibly) excludes matrix entries as in the examples depicted below:

$$\begin{matrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} \\ X_{41} & X_{42} & X_{43} & X_{44} & \\ X_{51} & X_{52} & X_{53} & & \end{matrix} \qquad \begin{matrix} & X_{12} & X_{13} \\ & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & \\ X_{51} & X_{52} & \end{matrix}$$

O: one-sided [5, Example 4.10] $\qquad$ T: ladder with coincidental corner

The associated ladder determinantal ring of t-minors is $R_t(Y) = \mathsf{k}[Y]/I_t(Y)$, where $I_t(Y)$ is the ideal generated by the $t \times t$ minors of X lying entirely in Y. These are essentially coordinate rings for Schubert varieties [1]. See the paper of Aldo Conca [5] and our Section 2 for background on these rings, including information on their divisor class groups that is crucial for our work.

The main results of Sections 3 and 4 of the current paper are as follows. They show that many ladder determinantal rings have only trivial semidualizing modules. See, however, the next paper in our work [18] for the study of ladder determinantal rings with nontrivial semidualizing modules; Example 4.11(3) contains a sample computation.

ONE-SIDED LADDER THEOREM (Theorem 3.6)
Let Y be a one-sided ladder.[1] *The ring $R_t(Y)$ has only trivial semidualizing modules; i.e., $|\mathfrak{S}_0(R_t(Y))| \leqslant 2$.*

For two-sided ladders, we focus specifically on the 2×2 case.

TWO-SIDED LADDER THEOREM [$t = 2$, NO COINCIDENTAL CORNERS] (Theorem 4.10)
Let Y be a 2-connected ladder such that no lower inside corner and upper inside corner coincide. Then the ring $R_2(Y)$ has only trivial semidualizing modules; i.e., $|\mathfrak{S}_0(R_2(Y))| \leqslant 2$.

2. Background

Divisor class groups
For a normal domain R, the isomorphism class of an R-module M is denoted $[M]$, and the set of isomorphism classes of rank-1 reflexive modules is the *divisor class*

1. In Definition 2.8, we require all of our one- and two-sided ladders to be path-connected.

group of R, denoted $\mathrm{Cl}(R)$. This is an abelian group under the operations $[M]+[N] = [(M \otimes_R N)^{**}]$, where $(-)^* = \mathrm{Hom}_R(-, R)$, and $[M]-[N] = [\mathrm{Hom}_R(N, M)]$, with additive identity $[R]$. Equivalently, $\mathrm{Cl}(R)$ is the set of isomorphism classes of height-1 reflexive ideals.

Semidualizing modules/ideals

Recall the definition of the semidualizing property and the notation $\mathfrak{S}_0(R)$ from the introduction of this paper. By [15, Proposition 3.4], if R is a normal domain, then each semidualizing R-module is reflexive of rank 1, so there is an inclusion $\mathfrak{S}_0(R) \subseteq \mathrm{Cl}(R)$. A *semidualizing ideal* is an ideal of the ring R that is semidualizing as an R-module.

REMARK 2.1

For our purposes, it is important to note that the property of being semidualizing is preserved under localization, since the defining conditions are preserved by flat base change.

FACT 2.2 ([15, Proposition 3.3])

Let $\mathfrak{a}$ and $\mathfrak{b}$ be semidualizing ideals such that $\mathfrak{a} \otimes \mathfrak{b}$ is semidualizing. The natural multiplication map $\mathfrak{a} \otimes_R \mathfrak{b} \to \mathfrak{a}\mathfrak{b}$ is an isomorphism.

A *dualizing R-module* is a semidualizing R-module of finite injective dimension. The ring R has a dualizing module if and only if it is Cohen–Macaulay and a homomorphic image of a Gorenstein ring with finite Krull dimension; see [7], [13], and [19]. Thus, a dualizing module is a canonical module over a Cohen–Macaulay ring. To say that a ring R admits only *trivial semidualizing modules* means $\mathfrak{S}_0(R) = \{[R], [\omega_R]\}$ if R has a dualizing module ω_R, and it means that $\mathfrak{S}_0(R) = \{[R]\}$ if R does not have a dualizing module.

FACT 2.3

If R is Cohen–Macaulay with a dualizing module ω_R, and if C is a semidualizing R-module, then $\mathrm{Hom}_R(C, \omega_R)$ is semidualizing. Moreover, the natural evaluation map $\gamma\colon \mathrm{Hom}_R(C, \omega_R) \otimes_R C \to \omega_R$ given by $\gamma(\varphi \otimes c) = \varphi(c)$ is an isomorphism, and $\mathrm{Tor}_i^R(\mathrm{Hom}_R(C, \omega_R), C) = 0$ for all $i \geqslant 1$; see [4, Theorem 2.11, Proposition 4.4, and Observation 4.10]. If, in addition, R is a normal domain and $C \neq \omega_R$ are height-1 reflexive ideals, then $\mathrm{Hom}_R(C, \omega_R)$ is naturally isomorphic to a height-1 reflexive ideal C', and we have $\omega_R \xrightarrow[\gamma^{-1}]{\cong} C \otimes_R C' \xrightarrow{\cong} CC'$, where the second isomorphism is the multiplication map from Fact 2.2.

Ladder determinantal rings

We will recall the terminology and results in [5] and [6] and also introduce some new terminology.

Let $X = (X_{ij})$ be an $m \times n$ matrix of indeterminates. A *ladder* in X is a subset Y satisfying the following property: if $X_{ij}, X_{pq} \in Y$ satisfy $i \leqslant p$ and $j \leqslant q$, then

$X_{iq}, X_{pj} \in Y$. Recall that $R_t(Y) = \mathsf{k}[Y]/I_t(Y)$ is the associated *ladder determinantal ring*, where $I_t(Y)$ is the ideal generated by the $t \times t$ minors of X lying entirely in Y. As in [5, p. 121(b)], to avoid trivialities, we assume without loss of generality that $X_{m1}, X_{1n} \in Y$ and furthermore that each row of X contains an element of Y, as does each column of X. One such ladder is as follows:

$$\text{(L)} \qquad \begin{array}{ccccc} & X_{12} & X_{13} & X_{14} & X_{15} \\ & X_{22} & X_{23} & X_{24} & X_{25} \\ X_{31} & X_{32} & X_{33} & & \\ X_{41} & X_{42} & X_{43} & & \\ X_{51} & X_{52} & & & \end{array}$$

Bruns and Trung [9, Corollary 4.10] show that the ring $R_t(Y)$ is Cohen–Macaulay, and it is a normal domain by [5, Proposition 3.3]. For each $X_{ij} \in Y$, we let x_{ij} denote its residue in $R_t(Y)$.

The *lower inside corners* of Y are the points (a,b) with $X_{ab}, X_{a-1b}, X_{ab-1} \in Y$, but $X_{a-1b-1} \in X \smallsetminus Y$; these are denoted $X_{a_i b_i}$, or simply (a_i, b_i), with $1 < a_1 < \cdots < a_h < m$. For notational convenience, we also set $(a_0, b_0) = (1, n)$ and $(a_{h+1}, b_{h+1}) = (m, 1)$. Likewise, the *upper inside corners* of a ladder Y are the points (c,d) such that $X_{cd}, X_{c+1d}, X_{cd+1} \in Y$, but $X_{c+1d+1} \in X \smallsetminus Y$; these are denoted $X_{c_j d_j}$, or simply (c_j, d_j), with $1 < c_1 < \cdots < c_k < m$. The ladder Y has *coincidental corners* if $(a_i, b_i) = (c_j, d_j)$ for some i, j. For notational convenience, we also set $(c_0, d_0) = (1, n)$ and $(c_{k+1}, d_{k+1}) = (m, 1)$.

For instance, the ladder (L) above has $h = 1$ and $k = 2$, with $(a_0, b_0) = (1,5) = (c_0, d_0)$ and $(a_2, b_2) = (5,1) = (c_3, d_3)$, and the variables at inside corners are boxed in the next display:

$$\begin{array}{cccc} & X_{12} & X_{13} & X_{14} & X_{15} \\ & X_{22} & X_{23} & X_{24} & X_{25} \\ X_{31} & \boxed{X_{32}} & X_{33} & & \\ X_{41} & X_{42} & X_{43} & & \\ X_{51} & X_{52} & & & \end{array} \qquad \begin{array}{ccccc} & X_{12} & X_{13} & X_{14} & X_{15} \\ & X_{22} & \boxed{X_{23}} & X_{24} & X_{25} \\ X_{31} & X_{32} & X_{33} & & \\ X_{41} & \boxed{X_{42}} & X_{43} & & \\ X_{51} & X_{52} & & & \end{array}$$

lower inside corner ... upper inside corners

One point of identifying the inside corners is to describe $\mathrm{Cl}(R_t(Y))$. In particular, for $t = 2$, the following ideals of $R_2(Y)$ are height-1 primes by [5, Proposition 2.1 and Corollary 2.3]:

$$\mathfrak{p}_j = \left(x_{pq} \in R_2(Y) \;\middle|\; p \leqslant c_j \text{ and } q \leqslant d_j\right) \quad j = 1, \ldots, k$$

$$\mathfrak{q}_i = \left(x_{a_{i-1}q} \in R_2(Y)\right) \quad i = 1, \ldots, h+1$$

$$\mathfrak{q}'_i = \left(x_{pb_i} \in R_2(Y)\right) \quad i = 1, \ldots, h+1.$$

FACT 2.4

The following facts were established in [5]:

(1) $\mathrm{Cl}(R_2(Y))$ *is a free abelian group of rank* $h+k+1$ *with basis* $[\mathfrak{q}_1], \ldots, [\mathfrak{q}_{h+1}], [\mathfrak{p}_1], \ldots, [\mathfrak{p}_k]$*, by [5, Corollary 2.3].*

(2) *With* $t=2$*, set* $\lambda_i = a_i + b_i - a_{i-1} - b_{i-1}$ *for all* $i = 1, \ldots, h+1$ *and* $\delta_j = a_{i_j} + b_{i_j} - c_j - d_j$ *for all* $j = 1, \ldots, k$*, where* $i_j = \min\{i : a_i > c_j\}$*. Then the canonical class is* $[\omega_R] = \sum_{i=1}^{h+1} \lambda_i [\mathfrak{q}_i] + \sum_{j=1}^{k} \delta_j [\mathfrak{p}_j]$ *by [5, Proposition 2.4].*

(3) *The relations between the classes of the ideals* $\mathfrak{q}_i$*,* $\mathfrak{q}'_i$*,* $\mathfrak{p}_j$*, described in the proof of [5, Corollary 2.3(i)], are as follows. For all* $i = 1, \ldots, h+1$ *if* $I_i = \{j : 1 \leqslant j \leqslant k,\ a_{i-1} \leqslant c_j, \text{ and } b_i \leqslant d_j\}$*, where* I_i *may be empty, then* $[\mathfrak{q}_i] + [\mathfrak{q}'_i] + \sum_{j \in I_i} [\mathfrak{p}_j] = 0$*.*

For the specific ladder (L) above, we have

$$\mathfrak{p}_1 = (x_{12}, x_{13}, x_{22}, x_{23}) \qquad \delta_1 = 3+2-2-3$$

$$\mathfrak{p}_2 = (x_{12}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}) \qquad \delta_2 = 5+1-4-2$$

$$\mathfrak{q}_1 = (x_{12}, x_{13}, x_{14}, x_{15}) \qquad \lambda_1 = 3+2-1-5$$

$$\mathfrak{q}_2 = (x_{31}, x_{32}, x_{33}) \qquad \lambda_2 = 5+1-3-2$$

$$\mathfrak{q}'_1 = (x_{12}, x_{22}, x_{32}, x_{42}, x_{52})$$

$$\mathfrak{q}'_2 = (x_{31}, x_{41}, x_{51})$$

$$\mathrm{Cl}\big(R_2(L)\big) \cong \mathbb{Z}^4 \cong \mathbb{Z}[\mathfrak{p}_1] \oplus \mathbb{Z}[\mathfrak{p}_2] \oplus \mathbb{Z}[\mathfrak{q}_1] \oplus \mathbb{Z}[\mathfrak{q}_2]$$

$$[\omega_{R_2(L)}] = [\mathfrak{q}_2] - [\mathfrak{q}_1].$$

A ladder Y is *t-disconnected* [6, p. 457] if there exist two subladders $\varnothing \neq Z_1, Z_2 \subseteq Y$ such that $Z_1 \cap Z_2 = \varnothing$, $Z_1 \cup Z_2 = Y$, and every t-minor of Y is contained in Z_1 or Z_2. In this case, we say that Z_1, Z_2 form a *t-disconnection* of Y. A ladder Y is *t-connected* if it is not t-disconnected. (For instance, the ladder (L) above is 2-connected and is vacuously t-disconnected for each $t \geqslant 3$ since it has no 3-minors. The one-sided ladder O from the introduction is 2- and 3-connected, but not 4-connected.) A *block submatrix* of Y is a rectangular subladder, that is, a subset of Y consisting of all the X_{pq} with $u \leqslant p \leqslant v$ and $r \leqslant q \leqslant s$ for some u, v, r, s.

We use the following natural definitions below to compute semidualizing modules in some one-sided cases where Y is not t-connected.

DEFINITION 2.5

A *path* in a ladder Y is a nonempty (but possibly one-element) list of variables $X_{i_0 j_0}, X_{i_1 j_1}, \ldots, X_{i_\ell j_\ell}$ in Y, such that for all $0 \leqslant u < \ell$, either

(1) $i_u = i_{u+1}$ and $|j_u - j_{u+1}| = 1$, or

(2) $j_u = j_{u+1}$ and $|i_u - i_{u+1}| = 1$.

In such a case, we say that there is a path *from* $X_{i_0 j_0}$ *to* $X_{i_\ell j_\ell}$, or *between* $X_{i_0 j_0}$ *and* $X_{i_\ell j_\ell}$. We write $X_{i_0 j_0} \sim X_{i_\ell j_\ell}$ to denote that there is a path from $X_{i_0 j_0}$ to $X_{i_\ell j_\ell}$.

The *path-components* of a ladder Y are the equivalence classes of $\sim$. A ladder Y is *path-connected* if there is a path between any two variables in Y, or equivalently, Y has only one path-component. A ladder is *path-disconnected* if it is not path-connected.

LEMMA 2.6
Every path-component of a ladder is also a ladder.

Proof
Let Y be a ladder and Y_1 a path-component of Y. Let $X_{ij}, X_{pq} \in Y_1$ with $i \leqslant p$ and $j \leqslant q$. Let Z be the block submatrix of X with corners X_{ij} and X_{pq}. By the defining condition for Y being a ladder, we have $Z \subseteq Y$, so there are paths in Y between all of the variables X_{ij}, X_{pq}, X_{iq}, X_{pj}. Hence $X_{iq}, X_{pj} \in Y_1$, and Y_1 is a subladder of Y. □

The ladder Y in Example 3.7 shows that the converse of the next result fails with $t = 3$; examples for other t-values are similarly easy to construct.

LEMMA 2.7
A t-connected ladder is path-connected for any $t > 0$.

Proof
The only 1-connected ladder is the one consisting of one variable, so the result is easy in this case. Thus, we let $t > 1$ and assume that Y is t-connected. By way of contradiction, suppose that Y were path-disconnected. Let Y_1 be a path-component of Y and $Y_2 = Y \smallsetminus Y_1$. Then Y_1, Y_2 are ladders. Since Y is t-connected, there is a t-minor given by a set of variables M of Y such that $M \cap Y_1 \neq \varnothing$ and $M \cap Y_2 \neq \varnothing$. Let Z be the smallest block submatrix of X that contains M. Then $Z \subseteq Y$, so $Y_1 \cup Z$ is path-connected and properly contains Y_1, a contradiction. Therefore, Y is path-connected. □

DEFINITION 2.8
A ladder Y is *one-sided* if it is path-connected and $h = 0$ or $k = 0$; i.e., it has no lower inside corners or no upper inside corners. When this is the case, we usually assume that $h = 0$, by symmetry. A ladder Y is *two-sided* if it is path-connected and $h, k > 0$.

Since all ladders in [6] are assumed to be t-connected, our definitions of one-sided and two-sided ladders are compatible with those in [6, pp. 457–458] by Lemma 2.7.

3. One-sided ladders

In this section, we prove the One-Sided Ladder Theorem from the introduction. Note that the results of this section will be applied in the next section. In particular, Lemmas 3.1–3.2 apply to arbitrary 2-connected ladders (one- or two-sided).

We recall that the ideals $\mathfrak{q}_i$, $\mathfrak{q}'_i$, $\mathfrak{p}_j$ were defined on page 168.

LEMMA 3.1

Let Y be a 2-connected ladder and set $R = R_2(Y) = \mathsf{k}[Y]/I_2(Y)$. Then for all $1 \leqslant i \leqslant h+1$, $1 \leqslant j \leqslant k$ and $e \in \mathbb{N}$, we have $\mathfrak{q}_i^{(e)} = \mathfrak{q}_i^e$, $(\mathfrak{q}_i')^{(e)} = (\mathfrak{q}_i')^e$ and $\mathfrak{p}_j^{(e)} = \mathfrak{p}_j^e$ in R.

Proof

Let J denote any fixed ideal $\mathfrak{q}_i$, $\mathfrak{q}_i'$ or $\mathfrak{p}_j$. For any polynomial $f \in \mathsf{k}[Y]$, we let $\bar{f}$ denote its residue class in R. We define a grading on $\mathsf{k}[Y]$ by letting $\deg(X_{ij}) = 1$ if $x_{ij} \in J$ and $\deg(X_{ij}) = 0$ otherwise. We note that the generators $X_{i_1 j_1} X_{i_2 j_2} - X_{i_1 j_2} X_{i_2 j_1} \in I_2(Y)$, where $i_1 \leqslant i_2$ and $j_1 \leqslant j_2$, are homogeneous binomials of degree 0, 1, or 2. Hence R inherits the same grading from $\mathsf{k}[Y]$.

Given a polynomial $f \in \mathsf{k}[Y]$, let us write $f = f_r + f_{r+1} + \cdots + f_t$, where f_s is homogeneous of degree s for $r \leqslant s \leqslant t$. We note that whenever $e \in \mathbb{N}$ and $\bar{f_r} \neq 0$, we have $\bar{f} \in J^e$ if and only if $r \geqslant e$.

Now fix $e \in \mathbb{N}$ and let $f \in \mathsf{k}[Y]$. Let $f = f_r + f_{r+1} + \cdots + f_t$ with $\bar{f_r} \neq 0$. Suppose that $\bar{f} \in J^{(e)}$. Then there is $g \in \mathsf{k}[Y]$ with $g = g_0 + g_1 + \cdots + g_s$ and $\bar{g_0} \neq 0$, such that $\bar{f}\bar{g} \in J^e$. Since R is a domain, we have $\bar{f_r}\bar{g_0} \neq 0$. Since $\deg(f_r g_0) = r$ and $\bar{f}\bar{g} \in J^e$, we have $r \geqslant e$. Hence, $\bar{f} \in J^e$. Therefore, $J^{(e)} = J^e$. □

The $\mathfrak{q}_i$, $\mathfrak{q}_i'$, $\mathfrak{p}_j$ are residues in $R_2(Y)$ of ideals in $\mathsf{k}[Y]$ [5, §2]. In particular, $\mathfrak{q}_i$ is the residue class of $Q_i = (X_{a_{i-1},j} :$ for all j such that $X_{a_{i-1},j} \in Y) + I_2(Y)$.

LEMMA 3.2

Let Y be a 2-connected ladder. Let J be any one of the ideals Q_i^e, $(Q_i')^e$ or P_j^e, where $e \geqslant 1$. Let M be any monomial ideal in $\mathsf{k}[Y]$. Then $\bar{J} \cap \bar{M} \subseteq R = R_2(Y)$ is generated by the least common multiples of the monomial generators of J and M. In other words, $\bar{J} \cap \bar{M} = \overline{J \cap M}$.

Proof

Let $f \in \mathsf{k}[Y]$ be such that $\bar{f} \in \bar{J} \cap \bar{M}$. By collecting terms from $I_2(Y)$, we may assume that $f = f_J + f_I = f_M$, where $f_J \in J$, $f_I \in I_2(Y)$ and $f_M \in M$. Consider a term t_J that appears in f_J. Suppose that there is cancellation between t_J and a term $rm(X_{i_1,j_1} X_{i_2,j_2} - X_{i_1,j_2} X_{i_2,j_1})$ in f_I, where $r \in \mathsf{k}$, m is a monomial, $i_1 < i_2$ and $j_1 < j_2$. Then considering the grading in Lemma 3.1, since $X_{i_1,j_1} X_{i_2,j_2} - X_{i_1,j_2} X_{i_2,j_1}$ is homogeneous of degree 0, 1, or 2, we have $rm(X_{i_1,j_1} X_{i_2,j_2} - X_{i_1,j_2} X_{i_2,j_1}) \in J$. So by rearranging terms, we may assume that no monomial in f_I belongs to J. Now if a term t_J appears in f_J, then it does not cancel with any term in f_I, so t_J appears in f_M since $f_J + f_I = f_M$. Therefore, t_J is a common multiple of one monomial generator in J and one in M. Finally, we recall that the intersection of two monomial ideals is generated by the least common multiples of their respective monomial generators. □

The proofs below will involve new ladders obtained from a given ladder Y.

NOTATION 3.3

When we are considering a ladder Y and would like to discuss a new related ladder $\tilde{Y}$, we denote the corners of $\tilde{Y}$ as $(\tilde{a}_i, \tilde{b}_i)$ and $(\tilde{c}_j, \tilde{d}_j)$. The notation $\tilde{R}$ will denote the associated ladder determinantal ring $R_2(\tilde{Y})$, with prime ideals such as $\tilde{\mathfrak{p}}_1$, $\tilde{\mathfrak{p}}_2$, $\tilde{\mathfrak{q}}_1$, $\tilde{\mathfrak{q}}_2$, etc. Similar protocols apply for ladders $\check{Y}$.

THEOREM 3.4

Let $R = R_2(Y)$ be a one-sided 2-connected ladder determinantal ring. Then $|\mathfrak{S}_0(R)| \leqslant 2$.

Proof

Since Y is one-sided, we assume without loss of generality that $h = 0$. The proof is by induction on the number k of (upper) inside corners of Y. The case $k = 0$ is given by [15, Theorem 4.2]; therefore, let $k > 0$. Per Fact 2.4, we have $[\omega_R] = \sum_{j=0}^{k} \delta_j [\mathfrak{p}_j]$, where $\mathfrak{p}_0 = \mathfrak{q}_1$ and $\delta_j = a_1 + b_1 - c_j - d_j$ for all $0 \leqslant j \leqslant k$.

Consider the ladder $\tilde{Y}$ obtained by deleting rows $c_0, c_0 + 1, \ldots, c_1 - 1$ and columns $d_1 + 1, d_1 + 2, \ldots, d_0$ of Y. Then $[\omega_{\tilde{R}}] = \sum_{j=0}^{k-1} \delta_{j+1} [\tilde{\mathfrak{p}}_j]$. Let us invert $x_{c_1 d_0}$ in R, and let $\tilde{\varphi}$ be the composition of the following natural surjections:

$$\mathrm{Cl}(R) \to \mathrm{Cl}(R_{x_{c_1 d_0}}) \xrightarrow{\cong} \mathrm{Cl}(\tilde{R}).$$

The maps here come from the flat maps

$$R \to R_{x_{c_1 d_0}} \xrightarrow{\cong} \tilde{R}[X_{c_0 d_0}, \ldots, X_{c_1 d_0}, \ldots, X_{c_1, d_1+1}]_{X_{c_1 d_0}} \leftarrow \tilde{R}.$$

In particular, the maps on divisor class groups respect semidualizing modules by [15, Lemma 3.10(a)].

We have $\tilde{\varphi}([\mathfrak{p}_0]) = 0$ and $\tilde{\varphi}([\mathfrak{p}_i]) = [\tilde{\mathfrak{p}}_{i-1}]$ for $1 \leqslant j \leqslant k$; hence, $\operatorname{Ker} \tilde{\varphi} = \mathbb{Z}[\mathfrak{p}_0]$. By our induction hypothesis, the only semidualizing modules of $\tilde{R}$ are $[\tilde{R}]$ and $[\omega_{\tilde{R}}]$. Since the localization of a semidualizing module is also a semidualizing module, the only possible semidualizing modules of R are in $\tilde{\varphi}^{-1}([\tilde{R}]) = \mathbb{Z}[\mathfrak{p}_0]$ or $\tilde{\varphi}^{-1}([\omega_{\tilde{R}}]) = \mathbb{Z}[\mathfrak{p}_0] + \sum_{j=1}^{k} \delta_j [\mathfrak{p}_j]$. Let us write the possible semidualizing modules of R as $[N_1] = r[\mathfrak{p}_0]$ and $[N_2] = s[\mathfrak{p}_0] + \sum_{j=1}^{k} \delta_j [\mathfrak{p}_j]$, where $r, s \in \mathbb{Z}$.

Next, we invert $x_{c_{k+1} d_k}$ and obtain $\check{Y}$ by deleting rows $c_k + 1, c_k + 2, \ldots, c_{k+1}$ and columns $d_{k+1}, d_{k+1} + 1, \ldots, d_k - 1$ of Y. Then $\mathrm{Cl}(\check{R})$ is generated by the basis elements $[\check{\mathfrak{p}}_0]$, $[\check{\mathfrak{p}}_1]$, $\ldots$, $[\check{\mathfrak{p}}_{k-1}]$, and $[\omega_{\check{R}}] = \sum_{j=0}^{k-1} (c_k + d_k - c_j - d_j)[\check{\mathfrak{p}}_j]$.

Under the natural map $\check{\varphi}\colon \mathrm{Cl}(R) \to \mathrm{Cl}(\check{R})$, we have $\check{\varphi}([\mathfrak{p}_j]) = [\check{\mathfrak{p}}_j]$ for all $0 \leqslant j \leqslant k - 1$ and $\check{\varphi}([\mathfrak{p}_k]) = [\check{\mathfrak{q}}_1'] = -\sum_{j=0}^{k-1} [\check{\mathfrak{p}}_j]$ by [5, Proposition 2.1]. By our induction hypothesis, the only classes of semidualizing modules of $\check{R}$ are $[\check{R}]$ and $[\omega_{\check{R}}]$. By assumption, the classes of semidualizing modules of R are of the form $[N_1]$ and $[N_2]$, so we must have $\check{\varphi}([N_i]) = 0$ or $[\omega_{\check{R}}]$ for $i = 1, 2$.

If $0 = \check{\varphi}([N_1]) = r[\check{\mathfrak{p}}_0]$, then $[N_1] = 0$. Similarly, if

$$[\omega_{\check{R}}] = \check{\varphi}\big([N_2]\big) = (s - \delta_k)[\check{\mathfrak{p}}_0] + \sum_{j=1}^{k-1} (\delta_j - \delta_k)[\check{\mathfrak{p}}_j],$$

then $[N_2] = [\omega_R]$.

If $[\omega_{\check{R}}] = \check{\varphi}([N_1]) = r[\check{\mathfrak{p}}_0]$, then $r = c_k + d_k - c_0 - d_0 = \delta_0 - \delta_k$, and $c_1 + d_1 = c_2 + d_2 = \cdots = c_k + d_k$; i.e., all inside corners lie on the same "antidiagonal", by which we simply mean the same line (and do not require that the matrix be square). In this case, $[N_1]$ gives us the possible nontrivial semidualizing module $[M_1] = (\delta_0 - \delta_k)[\mathfrak{p}_0] = (\delta_0 - \delta_1)[\mathfrak{p}_0]$. Hence, $M_2 = \operatorname{Hom}_R(M_1, \omega_R)$ is also semidualizing for R with $[M_2] = \delta_1 \sum_{j=0}^{k} [\mathfrak{p}_j]$.

On the other hand, if $\check{\varphi}([N_2]) = (s - \delta_k)[\check{\mathfrak{p}}_0] + \sum_{j=1}^{k-1} (\delta_j - \delta_k)[\check{\mathfrak{p}}_j]$, then $s = \delta_k$ and $\delta_j = \delta_k$ for all $1 \leqslant j \leqslant k-1$. In this case, $[N_2]$ gives us the possible nontrivial semidualizing module $[M_2] = \delta_1 \sum_{j=0}^{k} [\mathfrak{p}_j]$, where all inside corners lie on the same antidiagonal, so $[M_2] = [\omega_R] - [M_1]$. Hence, $M_1 = \operatorname{Hom}_R(M_2, \omega_R)$ is also semidualizing for R with $[M_1] = (\delta_0 - \delta_1)[\mathfrak{p}_0]$.

Let us write $\zeta = \delta_0 - \delta_1$ and $\delta = \delta_1$, so that $[M_1] = \zeta[\mathfrak{p}_0]$ and $[M_2] = \delta \sum_{j=0}^{k} [\mathfrak{p}_j]$. Since $[M_1] + [M_2] = [\omega_R]$, it suffices to show that we get a contradiction if $\zeta\delta \neq 0$.

CASE 1

$\zeta, \delta > 0$. Using Lemmas 3.1 and 3.2, one can check that

$$[M_1] = [\mathfrak{p}_0^{\zeta}] = \big[(x_{a_0 d_{k+1}}, x_{a_0 d_{k+1}+1}, \ldots, x_{a_0 d_0})^{\zeta}\big] \qquad \text{and}$$

$$[M_2] = \Big[\bigcap_{j=0}^{k} \mathfrak{p}_j^{\delta}\Big] = \big[(x_{a_0 d_{k+1}}, x_{a_0 d_{k+1}+1}, \ldots, x_{a_0 d_k})^{\delta}\big].$$

Let us identify M_1 with the ideal $(x_{a_0 d_{k+1}}, x_{a_0 d_{k+1}+1}, \ldots, x_{a_0 d_0})^{\zeta}$ above and M_2 with $(x_{a_0 d_{k+1}}, x_{a_0 d_{k+1}+1}, \ldots, x_{a_0 d_k})^{\delta}$. Via the multiplication map $M_1 \otimes_R M_2 \xrightarrow{\mu} M_1 M_2$, we have

$$\mu(x_{a_0 d_{k+1}}^{\zeta} \otimes x_{a_0 d_{k+1}}^{\delta-1} x_{a_0 d_{k+1}+1}) = \mu(x_{a_0 d_{k+1}}^{\zeta-1} x_{a_0 d_{k+1}+1} \otimes x_{a_0 d_{k+1}}^{\delta}).$$

So μ is not injective. If M_1, M_2 are semidualizing modules of R, then we get a contradiction by Fact 2.2.

CASE 2

$\zeta, \delta < 0$. By [5, Corollary 2.3(i)], we have $\sum_{i=0}^{k+1} [\mathfrak{p}_i] = 0$, where $[\mathfrak{p}_{k+1}] = [\mathfrak{q}_1']$, giving

$$[M_1] = \zeta[\mathfrak{p}_0] = -\zeta \sum_{i=1}^{k+1} [\mathfrak{p}_i] \qquad \text{and}$$

$$[M_2] = \delta \sum_{i=0}^{k} [\mathfrak{p}_i] = -\delta[\mathfrak{p}_{k+1}].$$

We may then use Case 1 by symmetry.

CASE 3

$\zeta > 0$, $\delta < 0$. Again [5, Corollary 2.3(i)] and Lemma 3.1 give

$$[M_1] = [\mathfrak{p}_0^{\zeta}] = \big[(x_{a_0b_1}, x_{a_0b_1+1}, \ldots, x_{a_0b_0})^{\zeta}\big],$$

$$[M_2] = |\delta|[\mathfrak{p}_{k+1}] = [\mathfrak{p}_{k+1}^{|\delta|}] = \big[(x_{a_0b_1}, x_{a_0+1b_1}, \ldots, x_{a_1b_1})^{|\delta|}\big], \qquad \text{and}$$

$$[\omega_R] = [M_1] + [M_2] = [\mathfrak{p}_0^{\zeta} \cap \mathfrak{p}_{k+1}^{|\delta|}].$$

Let us identify M_1, M_2, ω_R with the ideals on the right, as in Case 1.

Now we use the fact that if M_2 is semidualizing, then so is $\operatorname{Hom}_R(M_2, \omega_R)$, and we have isomorphisms

$$M_1 \otimes_R M_2 \cong \operatorname{Hom}_R(M_2, \omega_R) \otimes_R M_2 \xrightarrow{\cong} \omega_R,$$

where the second map is given by evaluation. In particular, it follows that the modules $M_1 \otimes_R M_2$ and ω_R have minimal generating sets of the same size.

Let $\operatorname{mingen}(\mathfrak{p}_0^{\zeta})$ denote the set of all monomials $m_1 = x_{a_0b_1}^{p_0} x_{a_0b_1+1}^{p_1} \cdots x_{a_0b_0}^{p_{b_0-b_1}}$ in R of degree ζ. This is a minimal generating set for $\mathfrak{p}_0^{\zeta}$. Similarly, the set $\operatorname{mingen}(\mathfrak{p}_{k+1}^{|\delta|})$ of all monomials $m_2 = x_{a_0b_1}^{q_0} x_{a_0+1b_1}^{q_1} \cdots x_{a_1b_1}^{q_{a_1-a_0}}$ in R of degree $|\delta|$ is a minimal generating set for $\mathfrak{p}_{k+1}^{|\delta|}$. By abuse of notation, we write $\operatorname{lcm}(m_1, m_2)$ for the monomial

$$\operatorname{lcm}(m_1, m_2) = x_{a_0b_1}^{\max(p_0,q_0)} x_{a_0b_1+1}^{p_1} \cdots x_{a_0b_0}^{p_{b_0-b_1}} x_{a_0+1b_1}^{q_1} \cdots x_{a_1b_1}^{q_{a_1-a_0}}.$$

Then Lemma 3.2 shows that the function

$$\operatorname{lcm}\colon \operatorname{mingen}(\mathfrak{p}_0^{\zeta}) \times \operatorname{mingen}(\mathfrak{p}_{k+1}^{|\delta|}) \to \mathfrak{p}_0^{\zeta} \cap \mathfrak{p}_{k+1}^{|\delta|}$$

has its image equal to a generating set for $\mathfrak{p}_0^{\zeta} \cap \mathfrak{p}_{k+1}^{|\delta|}$. The previous paragraph shows that each minimal generating set for $\mathfrak{p}_0^{\zeta} \cap \mathfrak{p}_{k+1}^{|\delta|}$ must have the same size as $\operatorname{mingen}(\mathfrak{p}_0^{\zeta}) \times \operatorname{mingen}(\mathfrak{p}_{k+1}^{|\delta|})$, so the lcm map here must be injective with image equal to a minimal generating set for $\mathfrak{p}_0^{\zeta} \cap \mathfrak{p}_{k+1}^{|\delta|}$. However, this is not the case because the following computation exhibits an lcm in $\mathfrak{m}(\mathfrak{p}_0^{\zeta} \cap \mathfrak{p}_{k+1}^{|\delta|})$ where $\mathfrak{m}$ is the homogeneous maximal ideal of R:

$$\begin{aligned}\operatorname{lcm}(x_{a_0b_1}^{\zeta-1} x_{a_0b_1+1}, x_{a_0b_1}^{|\delta|-1} x_{a_0+1b_1}) &= x_{a_0b_1}^{\max(\zeta,|\delta|)-1} x_{a_0b_1+1} x_{a_0+1b_1} \\ &= x_{a_0b_1}^{\max(\zeta,|\delta|)} x_{a_0+1,b_1+1} \\ &= \operatorname{lcm}(x_{a_0b_1}^{\zeta}, x_{a_0b_1}^{|\delta|}) x_{a_0+1,b_1+1}.\end{aligned}$$

Hence, this lcm cannot be part of a minimal generating set for $\mathfrak{p}_0^{\zeta} \cap \mathfrak{p}_{k+1}^{|\delta|}$.

CASE 4

$\zeta < 0$, $\delta > 0$. Then $[M_1] = \zeta[\mathfrak{p}_0] = |\zeta| \sum_{j=1}^{k+1} [\mathfrak{p}_j]$. Since $[M_2] = \delta \sum_{j=0}^{k} [\mathfrak{p}_j]$, we may assume that $\delta \geqslant |\zeta|$ by symmetry, so that $\delta_1 > \delta_0 \geqslant 0$. Then using Lemmas 3.1 and 3.2

we have

$$[M_1] = \Big[\bigcap_{j=1}^{k+1} \mathfrak{p}_j^{|\zeta|}\Big] = \big[(x_{c_0 d_{k+1}}, x_{c_0+1 d_{k+1}}, \dots, x_{c_1 d_{k+1}})^{|\zeta|}\big],$$

$$[M_2] = \Big[\bigcap_{j=0}^{k} \mathfrak{p}_j^{\delta}\Big] = \big[(x_{c_0 d_{k+1}}, x_{c_0 d_{k+1}+1}, \dots, x_{c_0 d_k})^{\delta}\big], \text{ and}$$

$$[\omega_R] = \delta_0[\mathfrak{p}_0] + \delta_1 \sum_{j=1}^{k} [\mathfrak{p}_j] = \Big[\mathfrak{p}_0^{\delta_0} \cap \bigcap_{j=1}^{k} \mathfrak{p}_j^{\delta_1}\Big] = \Big[\Big(\prod_{u=1}^{\delta_0} x_{c_0 j_u} \prod_{v=1}^{\delta_1-\delta_0} x_{i_v, j_{\delta_0+v}}\Big)\Big],$$

where $c_0 \leqslant i_1, i_2, \dots, i_{\delta_1-\delta_0} \leqslant c_1$ and $d_{k+1} \leqslant j_1, j_2, \dots, j_{\delta_1} \leqslant d_k$. We again identify M_1, M_2, ω_R with the ideals shown above. Here, the multiplication map

$$(x_{c_0 d_{k+1}}, x_{c_0+1 d_{k+1}}, \dots, x_{c_1 d_{k+1}})^{\delta_1-\delta_0} \otimes_R (x_{c_0 d_{k+1}}, x_{c_0 d_{k+1}+1}, \dots, x_{c_0 d_k})^{\delta_1}$$
$$\to x_{c_0 d_{k+1}}^{\delta_1-\delta_0} \omega_R \cong \omega_R$$

may actually give an isomorphism. So to get a contradiction, we will use the fact that if M_2 is semidualizing, then

$$\operatorname{Tor}_1^R(M_1, M_2) \cong \operatorname{Tor}_1^R\big(\operatorname{Hom}_R(M_2, \omega_R), M_2\big) = 0.$$

Consider a minimal free resolution of M_2 as follows:

$$0 \leftarrow M_2 \xleftarrow{\partial_0} R^{\beta_0} \xleftarrow{\partial_1} R^{\beta_1} \xleftarrow{\partial_2} R^{\beta_2} \leftarrow \cdots, \tag{3.4.1}$$

where

$$\partial_0 = \begin{pmatrix} x_{c_0 d_{k+1}}^{\delta} & x_{c_0 d_{k+1}}^{\delta-1} x_{c_0 d_{k+1}+1} & \cdots & x_{c_0 d_{k+1}} x_{c_0 d_{k+1}+1}^{\delta-1} & x_{c_0 d_{k+1}+1}^{\delta} & \cdots \end{pmatrix},$$

$$\partial_1 = \begin{pmatrix} x_{c_0 d_{k+1}+1} & x_{c_0+1 d_{k+1}+1} & \cdots & x_{c_{k+1} d_{k+1}+1} & \cdots \\ -x_{c_0 d_{k+1}} & -x_{c_0+1 d_{k+1}} & \cdots & -x_{c_{k+1} d_{k+1}} & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots \end{pmatrix}, \quad \text{etc.}$$

Now we truncate (3.4.1) and tensor with M_1 to get

$$0 \leftarrow M_1^{\beta_0} \xleftarrow{\partial_1 \otimes M_1} M_1^{\beta_1} \xleftarrow{\partial_2 \otimes M_1} M_1^{\beta_2} \leftarrow \cdots$$

We see that $\mathbf{x} = (x_{c_0 d_{k+1}}^{|\zeta|-1} x_{c_0+1 d_{k+1}}, -x_{c_0 d_{k+1}}^{|\zeta|}, 0, \dots, 0)^T \in \operatorname{Ker}(\partial_1 \otimes M_1)$. This is a minimal generator of $M_1^{\beta_1}$. However, since (3.4.1) is a minimal resolution, the entries of ∂_2 are in the homogeneous maximal ideal of R, so $\mathbf{x} \notin \operatorname{Im}(\partial_2 \otimes M_1)$, giving us our final contradiction. □

COROLLARY 3.5

Let Y be a one-sided t-connected ladder. Then $|\mathfrak{S}_0(R_t(Y))| \leqslant 2$ for any $t \geqslant 1$.

Proof
We induct on t. If $t = 1$ or if Y contains no $t \times t$ minors, then $R_t(Y)$ is Gorenstein, so the result is trivial. Since the case of $t = 2$ is handled above, suppose that Y contains $t \times t$ minors for $t \geqslant 3$, and assume that for all one-sided $(t-1)$-connected ladders, the associated ladder determinantal rings of $(t-1) \times (t-1)$ minors have only trivial semidualizing modules. Let Z be the ladder obtained from Y by deleting the first row and first column, which is necessarily $(t-1)$-connected. By [5, Proposition 4.1(2) and the proof of Theorem 4.9(b)], there is an isomorphism $\mathrm{Cl}(R_t(Y)) \to \mathrm{Cl}(R_{t-1}(Z))$. As in the proof of Theorem 3.4, the class of any semidualizing module for $R_t(Y)$ must map to the class of a semidualizing module for $R_{t-1}(Z)$, and the result follows. □

Next, we address the case of one-sided ladders that are not necessarily t-connected. Recall that one-sided ladders are, by definition, path-connected.

THEOREM 3.6 (One-Sided Ladder Theorem)
Let Y be a one-sided ladder. The ring $R_t(Y)$ has only trivial semidualizing modules; i.e., $|\mathfrak{S}_0(R_t(Y))| \leqslant 2$.

Proof
The field $R_1(Y) = \mathsf{k}$ has $\mathfrak{S}_0(R_1(Y)) = \mathfrak{S}_0(\mathsf{k}) = \{[\mathsf{k}]\}$. Thus, we may assume that $t > 1$, and furthermore that $h = 0$ and $k > 0$. If Y contains no t-minors, then $R_t(Y)$ is a polynomial ring over k, which is Gorenstein, so $\mathfrak{S}_0(R_t(Y)) = \{[R]\}$ in this case. Thus, we assume that Y contains a t-minor. Since Y is path-connected, it is straightforward to show that $X_{11} \in Y$ and, moreover, that all the variables X_{1j} and X_{i1} are in Y.

Let $j_1 = \max\{j \mid c_j < t\}$ and $j_2 = \min\{j \mid d_j < t\}$. If $j_1 \geqslant j_2$, then Y contains no t-minors, so we must have $j_1 < j_2$. Let $Y' = \{X_{ij} \in Y \mid i \leqslant c_{j_2} \text{ and } j \leqslant d_{j_1}\}$ and $Z = Y \smallsetminus Y'$. Then Y' is a one-sided t-connected ladder and Z contains no t-minors. It follows that $R_t(Y) = R_t(Y')[Z]$. Then $|\mathfrak{S}_0(R_t(Y))| = |\mathfrak{S}_0(R_t(Y'))|$ by [15, Corollary 3.11(a)]. Now apply Corollary 3.5. □

We end this section with an example that illustrates two aspects of Theorem 3.6 and its proof.

EXAMPLE 3.7
The following ladder Y is 2-connected and path-connected:

$$Y: \begin{array}{ccccc} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} \\ X_{31} & X_{32} & X_{33} & X_{34} & \end{array}$$

However, it is 3-disconnected because the variables X_{14}, X_{24} are not used in any 3-minor. In the notation of the proof of Theorem 3.6 with $t = 3$, this yields $Z = \{X_{14}, X_{24}\}$, and Y' is the next ladder which is 3-connected:

$$Y': \begin{array}{cccc} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \end{array}$$

and $R_3(Y) = R_3(Y')[Z]$, so $|\mathfrak{S}_0(R_3(Y))| = |\mathfrak{S}_0(R_3(Y'))| = 2$ by [15, Theorem 4.2]. Similarly, we have $|\mathfrak{S}_0(R_4(O))| = 1$ for the ladder O from the introduction. Also, the path-connected condition in our definition of "one-sided" is necessary for Theorem 3.6 as the next ladder has no corners, but $|R_2(Y'')| = 4$ by [15, Theorem 4.5]:

$$Y'': \quad \begin{array}{cccccc} & & & X_{14} & X_{15} & X_{16} \\ & & & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & & & \\ X_{41} & X_{42} & X_{43} & & & \end{array}$$

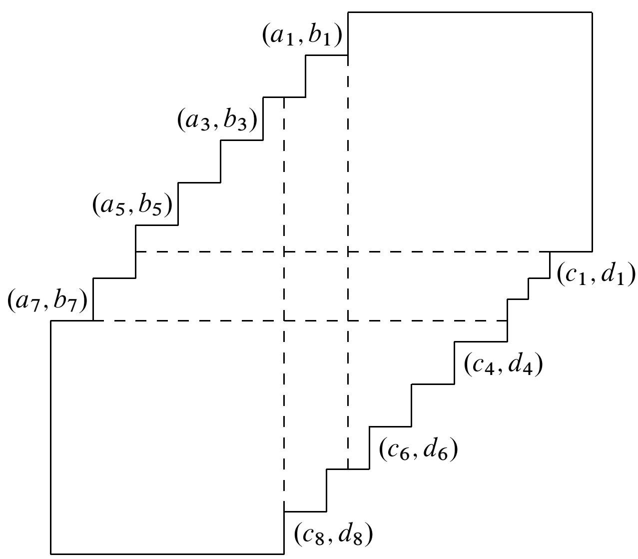
4. Size-2 minors of two-sided ladders with no coincidental corners

In this section, we study ladders which are 2-connected. (Minors of size 2×2 are special in the sense that $R_2(Y)$ is an algebra with straightening laws, or ASL, on the poset Y, as per [5, p. 121], but $R_{t>2}(Y)$ is not. We will consider more general ladders in another project [18], including ladders with coincidental corners.) In particular, throughout this section, Y will be a 2-connected ladder without coincidental corners and $R_2(Y)$ the associated ladder determinantal ring.

As in the previous section, we will use the notation $\tilde{Y}$ for ladders obtained from the given ladder Y. The notation $\tilde{R}$ will always denote the associated ladder determinantal ring $R_2(\tilde{Y})$. See Notation 3.3. In order to provide an upper bound on $|\mathfrak{S}_0(R_2(Y))|$, we will need the additional notation defined below.

NOTATION 4.1

Consider an arbitrary ladder with the following sample in mind:



Let $\eta_1 = \min\{j \mid b_j \leqslant d_k\}$, $\eta_2 = \max\{i \mid a_i \leqslant c_1\}$, $\kappa_1 = \min\{i \mid c_i \geqslant a_h\}$, and $\kappa_2 = \max\{j \mid d_j \geqslant b_1\}$. For example, in the above ladder, we have $h = 7$, $k = 8$, $\eta_1 = 3$, $\eta_2 = 5$, $\kappa_1 = 4$, and $\kappa_2 = 6$.

As a second example, for the ladder (L) on page 168, $\eta_1 = 1$, $\eta_2 = 0$, $\kappa_1 = 2$, and $\kappa_2 = 2$.

REMARK 4.2
Note that a ladder is one-sided if and only if $\eta_1 = 0$ or $\kappa_1 = 0$; i.e., if and only if $k = 0$ or $h = 0$, respectively.

PROPOSITION 4.3
Let $R = R_2(Y)$ for a two-sided 2-connected ladder Y with $h \geqslant 1$ lower inside corners and $k \geqslant 1$ upper inside corners, such that no two inside corners coincide. Assume that for all 2-connected ladders Z with fewer than $h + k$ inside corners, where no two coincide, the associated ladder determinantal ring $R_2(Z)$ has only trivial semidualizing modules. Then $|\mathfrak{S}_0(R)| \leqslant 4$.

Proof
As per Fact 2.4, $[\omega_R] = \sum_{i=1}^{h+1} \lambda_i [\mathfrak{q}_i] + \sum_{j=1}^{k} \delta_j [\mathfrak{p}_j]$. The letters M_i, N_i will be used to denote (possible) semidualizing modules of R. First, we invert $x_{a_0 b_1}$ and obtain the ladder $\tilde{Y}$ by deleting rows $a_0, a_0 + 1, \dots, a_1 - 1$ and columns $b_1 + 1, b_1 + 2, \dots, b_0$ of Y. The kernel of the natural map $\varphi\colon \mathrm{Cl}(R) \to \mathrm{Cl}(\tilde{R})$ is generated by $[\mathfrak{q}_1]$, $[\mathfrak{p}_1]$, $[\mathfrak{p}_2]$, …, $[\mathfrak{p}_{\kappa_2}]$. (Note that it is possible for $\kappa_2 = 0$, in which case, the kernel is generated only by $[\mathfrak{q}_1]$. The argument below allows for this possibility.) By assumption, the semidualizing modules of $\tilde{R}$ are $[\tilde{R}]$ and its canonical class; hence, the possible semidualizing modules of R are

$$[N_1] = r_1[\mathfrak{q}_1] + \sum_{j=1}^{\kappa_2} s_j[\mathfrak{p}_j], \qquad \text{and}$$

$$[N_2] = r_1[\mathfrak{q}_1] + \sum_{j=1}^{\kappa_2} s_j[\mathfrak{p}_j] + \sum_{i=2}^{h+1} \lambda_i[\mathfrak{q}_i] + \sum_{j=\kappa_2+1}^{k} \delta_j[\mathfrak{p}_j],$$

where $r_1, s_j \in \mathbb{Z}$ and $[\omega_{\tilde{R}}] = \sum_{i=1}^{h} \lambda_{i+1}[\tilde{\mathfrak{q}}_i] + \sum_{j=1}^{k-\kappa_2} \delta_{j+\kappa_2}[\tilde{\mathfrak{p}}_j]$.

Next, we invert $x_{a_h b_{h+1}}$ and obtain (a new) $\tilde{Y}$ by deleting rows $a_h + 1, a_h + 2, \dots, a_{h+1}$ and columns $b_{h+1}, b_{h+1} + 1, \dots, b_h - 1$ of Y. The kernel of the natural map $\varphi\colon \mathrm{Cl}(R) \to \mathrm{Cl}(\tilde{R})$ is generated by $[\mathfrak{q}_{h+1}]$, $[\mathfrak{p}_{\kappa_1}]$, $[\mathfrak{p}_{\kappa_1+1}]$, …, $[\mathfrak{p}_k]$, and $[\omega_{\tilde{R}}] = \sum_{i=1}^{h} \lambda_i[\tilde{\mathfrak{q}}_i] + \sum_{j=1}^{\kappa_1-1} \delta_j[\tilde{\mathfrak{p}}_j]$.

Suppose that $\varphi([N_1]) = 0$. If $\kappa_1 > \kappa_2$, then $0 = \varphi([N_1]) = r_1[\tilde{\mathfrak{q}}_1] + \sum_{j=1}^{\kappa_2} s_j[\tilde{\mathfrak{p}}_j]$, so $[N_1] = 0$. Since we are seeking nontrivial semidualizing modules, we may assume here that $\kappa_1 \leqslant \kappa_2$, in which case $0 = \varphi([N_1]) = r_1[\tilde{\mathfrak{q}}_1] + \sum_{j=1}^{\kappa_1-1} s_j[\tilde{\mathfrak{p}}_j]$, and hence, $[N_1]$ equals

$$[N_3] = \sum_{j=\kappa_1}^{\kappa_2} s_j[\mathfrak{p}_j], \quad \text{where } \kappa_1 \leqslant \kappa_2.$$

Suppose that $\varphi([N_1]) = [\omega_{\tilde{R}}]$. Because neither relation among the κ_i may be discarded, we allow for both cases (where the notation $\sum_r^s$ for $s < r$ is simply a vacuous sum). Then $[N_1]$ equals

$$[N_4] = \lambda_1[\mathfrak{q}_1] + \sum_{j=1}^{\min(\kappa_1-1,\kappa_2)} \delta_j[\mathfrak{p}_j] + \sum_{j=\kappa_1}^{\kappa_2} s_j[\mathfrak{p}_j],$$

where $\lambda_i = 0$ for all $1 < i < h+1$, and $\delta_j = 0$ for $\kappa_2 < j$ and $j < \kappa_1$, a condition which may or may not be satisfied. (In particular, it's not satisfied if $\kappa_1 \leqslant \kappa_2 + 1$.)

Suppose that $\varphi([N_2]) = 0$. Because neither relation among the κ_i may be discarded, we allow for both cases. We have

$$\varphi\big([N_2]\big) = r_1[\tilde{\mathfrak{q}}_1] + \sum_{j=1}^{\min(\kappa_1-1,\kappa_2)} s_j[\tilde{\mathfrak{p}}_j] + \sum_{i=2}^{h} \lambda_i[\tilde{\mathfrak{q}}_i] + \sum_{j=\kappa_2+1}^{\max(\kappa_1-1,\kappa_2)} \delta_j[\tilde{\mathfrak{p}}_j];$$

hence, $[N_2]$ equals

$$[N_5] = \lambda_{h+1}[\mathfrak{q}_{h+1}] + \sum_{j=\kappa_1}^{\kappa_2} s_j[\mathfrak{p}_j] + \sum_{j=\max(\kappa_1-1,\kappa_2)+1}^{k} \delta_j[\mathfrak{p}_j],$$

where $\lambda_i = 0$ for all $1 < i < h+1$, and $\delta_j = 0$ for $\kappa_2 < j$ and $j < \kappa_1$, a condition which may or may not be satisfied. (In particular, it's not satisfied if $\kappa_1 \leqslant \kappa_2 + 1$.)

Suppose that $\varphi([N_2]) = [\omega_{\tilde{R}}]$. If $\kappa_1 > \kappa_2$, then $[N_2] = [\omega_R]$. So we may assume that $\kappa_1 \leqslant \kappa_2$, in which case $[N_2]$ equals

$$[N_6] = \sum_{j=1}^{h+1} \lambda_i[\mathfrak{q}_i] + \sum_{j=1}^{\kappa_1-1} \delta_j[\mathfrak{p}_j] + \sum_{j=\kappa_1}^{\kappa_2} s_j[\mathfrak{p}_j] + \sum_{j=\kappa_2+1}^{k} \delta_j[\mathfrak{p}_j], \quad \text{where } \kappa_1 \leqslant \kappa_2.$$

Now we invert $x_{c_1 d_0}$ and obtain $\tilde{Y}$ by deleting rows $c_0, c_0+1, \dots, c_1-1$ and columns $d_1+1, d_1+2, \dots, d_0$ of Y. The kernel of the natural map $\varphi\colon \mathrm{Cl}(R) \to \mathrm{Cl}(\tilde{R})$ is generated by $[\mathfrak{q}_1], [\mathfrak{q}_2], \dots, [\mathfrak{q}_{\eta_2+1}]$, and $\varphi([\mathfrak{p}_1]) = [\tilde{\mathfrak{q}}_1]$. Let us write $[\tilde{\mathfrak{p}}_0] = [\tilde{\mathfrak{q}}_1]$. We have

$$[\omega_{\tilde{R}}] = \sum_{i=2}^{h-\eta_2+1} \lambda_{i+\eta_2}[\tilde{\mathfrak{q}}_i] + \delta_1[\tilde{\mathfrak{q}}_1] + \sum_{j=1}^{k-1} \delta_{j+1}[\tilde{\mathfrak{p}}_j].$$

If $\varphi([N_3]) = 0$, then $0 = \varphi([N_3]) = \sum_{j=\kappa_1}^{\kappa_2} s_j[\tilde{\mathfrak{p}}_{j-1}]$ implies that $[N_3] = 0$. If $\varphi([N_3]) = [\omega_{\tilde{R}}]$, then $[N_3]$ gives us the possibly nontrivial semidualizing module

$$[N_7] = \sum_{j=\kappa_1}^{\kappa_2} \delta_j[\mathfrak{p}_j], \quad \text{where } \kappa_1 \leqslant \kappa_2,$$

and $\lambda_i = 0$ for all $i > \eta_2 + 1$, and $\delta_j = 0$ for all $j < \kappa_1$ or $j > \kappa_2$.

If $\varphi([N_4]) = 0$, then since $0 = \varphi([N_4]) = \sum_{j=1}^{\min(\kappa_1-1,\kappa_2)} \delta_j[\tilde{\mathfrak{p}}_{j-1}] + \sum_{j=\kappa_1}^{\kappa_2} s_j[\tilde{\mathfrak{p}}_{j-1}]$, $[N_4]$ gives us the candidate

$$[N_8] = \lambda_1[\mathfrak{q}_1],$$

where $\lambda_i = 0$ for all $1 < i < h+1$, and $\delta_j = 0$ for all j such that $\kappa_2 < j < \kappa_1$ or $1 \leqslant j \leqslant \min(\kappa_1 - 1, \kappa_2)$. Whether $\kappa_1 > \kappa_2$ or $\kappa_1 \leqslant \kappa_2$, we conclude that $\delta_j = 0$ for all $j < \kappa_1$.

Suppose that $\varphi([N_4]) = [\omega_{\tilde{R}}]$. If $\eta_2 < h$, then $\lambda_i = 0$ for all $i > 1$, and $[N_4] = [\omega_R]$. Therefore, we may assume that $\eta_2 = h$. Thus, the case $[N_4]$ gives us the candidate

$$[N_9] = \lambda_1[\mathfrak{q}_1] + \sum_{j=1}^{\kappa_2} \delta_j[\mathfrak{p}_j],$$

where $\eta_2 = h$, which implies $\kappa_1 = 1$, and $\lambda_i = 0$ for all $1 < i < h+1$ and $\delta_j = 0$ for $j > \kappa_2$.

Suppose that $\varphi([N_5]) = 0$. If $\eta_2 < h$, then none of the terms in the expression for $[N_5]$ is in $\operatorname{Ker}\varphi$, in which case $[N_5] = 0$. Therefore, we may assume that $\eta_2 = h$, in which case the only term in the expression for $[N_5]$ that is in $\operatorname{Ker}\varphi$ is $\lambda_{h+1}[\mathfrak{q}_{h+1}]$. Thus, $[N_5]$ gives us the candidate

$$[N_{10}] = \lambda_{h+1}[\mathfrak{q}_{h+1}] = [\omega_R] - [N_9].$$

Suppose that $\varphi([N_5]) = [\omega_{\tilde{R}}]$. If $\eta_2 < h$ and $\kappa_1 \leqslant \kappa_2$, then we have

$$\varphi\big([N_5]\big) = \lambda_{h+1}[\tilde{\mathfrak{q}}_{h-\eta_2+1}] + \sum_{j=\kappa_1-1}^{\kappa_2-1} s_{j+1}[\tilde{\mathfrak{p}}_j] + \sum_{j=\kappa_2}^{k-1} \delta_{j+1}[\tilde{\mathfrak{p}}_j].$$

If $\eta_2 < h$ and $\kappa_1 > \kappa_2$, then we have

$$\varphi\big([N_5]\big) = \lambda_{h+1}[\tilde{\mathfrak{q}}_{h-\eta_2+1}] + \sum_{j=\kappa_1-1}^{k-1} \delta_{j+1}[\tilde{\mathfrak{p}}_j].$$

If $\eta_2 = h$ and $\kappa_1 \leqslant \kappa_2$, then we have

$$[\omega_{\tilde{R}}] = \delta_1[\tilde{\mathfrak{q}}_1] + \sum_{j=1}^{k-1} \delta_{j+1}[\tilde{\mathfrak{p}}_j] \qquad \text{and}$$

$$\varphi\big([N_5]\big) = \sum_{j=\kappa_1-1}^{\kappa_2-1} s_{j+1}[\tilde{\mathfrak{p}}_j] + \sum_{j=\kappa_2}^{k-1} \delta_{j+1}[\tilde{\mathfrak{p}}_j].$$

If $\eta_2 = h$ and $\kappa_1 > \kappa_2$, then we have

$$[\omega_{\tilde{R}}] = \delta_1[\tilde{\mathfrak{q}}_1] + \sum_{j=1}^{k-1} \delta_{j+1}[\tilde{\mathfrak{p}}_j] \qquad \text{and} \qquad \varphi\big([N_5]\big) = \sum_{j=\kappa_1-1}^{k-1} \delta_{j+1}[\tilde{\mathfrak{p}}_j].$$

In all cases, $[N_5]$ gives us the candidate

$$[N_{11}] = \lambda_{h+1}[\mathfrak{q}_{h+1}] + \sum_{j=\kappa_1}^{k} \delta_j[\mathfrak{p}_j] = [\omega_R] - [N_8],$$

where $\lambda_i = 0$ for all $1 < i < h+1$, and $\delta_j = 0$ for $j < \kappa_1$.

Suppose that $\varphi([N_6]) = 0$. Then

$$0 = \varphi\big([N_6]\big) = \sum_{i=2}^{h-\eta_2+1} \lambda_{i+\eta_2}[\tilde{\mathfrak{q}}_i] + \sum_{j=1}^{\kappa_1-1} \delta_j[\tilde{\mathfrak{p}}_{j-1}] + \sum_{j=\kappa_1}^{\kappa_2} s_j[\tilde{\mathfrak{p}}_{j-1}] + \sum_{j=\kappa_2+1}^{k} \delta_j[\tilde{\mathfrak{p}}_{j-1}],$$

where $\kappa_1 \leqslant \kappa_2$. So $[N_6]$ gives us the candidate

$$[N_{12}] = \sum_{i=1}^{\eta_2+1} \lambda_i [\mathfrak{q}_i] = [\omega_R] - [N_7], \quad \text{where } \kappa_1 \leqslant \kappa_2,$$

$\lambda_i = 0$ for all $i > \eta_2 + 1$ and $\delta_j = 0$ for all $j < \kappa_1$ or $j > \kappa_2$.

If $\varphi([N_6]) = [\omega_{\tilde{R}}]$, then $[N_6] = [\omega_R]$.

Finally, we invert $x_{c_{k+1}d_k}$ and obtain $\tilde{Y}$ by deleting rows $c_k + 1, c_k + 2, \ldots, c_{k+1}$ and columns $d_{k+1}, d_{k+1} + 1, \ldots, d_k - 1$ of Y. Then $\mathrm{Cl}(\tilde{R})$ is generated by the basis elements $[\tilde{\mathfrak{q}}_1], [\tilde{\mathfrak{q}}_2], \ldots, [\tilde{\mathfrak{q}}_{\eta_1}], [\tilde{\mathfrak{p}}_1], [\tilde{\mathfrak{p}}_2], \ldots, [\tilde{\mathfrak{p}}_{k-1}]$, and

$$2[\omega_{\tilde{R}}] = \sum_{i=1}^{\eta_1} \tilde{\lambda}_i [\tilde{\mathfrak{q}}_i] + \sum_{j=1}^{k-1} \tilde{\delta}_j [\tilde{\mathfrak{p}}_j], \quad \text{where}$$

$$\tilde{\lambda}_i = \lambda_i \quad \text{for all } i < \eta_1,$$

$$\tilde{\lambda}_{\eta_1} = c_k + d_k - a_{\eta_1 - 1} - b_{\eta_1 - 1},$$

$$\tilde{\delta}_j = \delta_j \quad \text{if } c_j < a_{\eta_1 - 1}, \qquad \text{and}$$

$$\tilde{\delta}_j = c_k + d_k - c_j - d_j \quad \text{otherwise.}$$

Suppose that $\varphi([N_7]) = 0$ (equivalently, $\varphi([N_{12}]) = [\omega_{\tilde{R}}]$) under the natural map $\varphi\colon \mathrm{Cl}(R) \to \mathrm{Cl}(\tilde{R})$. If $\kappa_2 \neq k$, then $0 = \varphi([N_7]) = \sum_{j=\kappa_1}^{\kappa_2} \delta_j [\tilde{\mathfrak{p}}_j]$, so $[N_7] = 0$. If $\kappa_2 = k$ (equivalently, $\eta_1 = 1$), then

$$0 = \varphi\big([N_7]\big) = \delta_k [\tilde{\mathfrak{q}}'_1] + \sum_{j=\kappa_1}^{k-1} \delta_j [\tilde{\mathfrak{p}}_j] = -\delta_k [\tilde{\mathfrak{q}}_1] + \sum_{j=\kappa_1}^{k-1} (\delta_j - \delta_k)[\tilde{\mathfrak{p}}_j],$$

where the last equality follows from [5, Corollary 2.3(i), with $I_1 = \{1, \ldots, k-1\}$]. Since the $[\tilde{\mathfrak{q}}_1]$, $[\tilde{\mathfrak{p}}_j]$ are basis elements, we again get $[N_7] = 0$. Thus, $\varphi([N_7]) = 0$ produces no candidate for a semidualizing module.

Suppose that $\varphi([N_{12}]) = 0$ (equivalently, $\varphi([N_7]) = [\omega_{\tilde{R}}]$). If $\eta_2 < \eta_1$, then $\varphi([N_{12}]) = \sum_{i=1}^{\eta_2+1} \lambda_i [\tilde{\mathfrak{q}}_i]$ implies that $[N_{12}] = 0$. If $\eta_2 \geqslant \eta_1$, then

$$0 = \varphi\big([N_{12}]\big) = \sum_{i=1}^{\eta_1 - 1} \lambda_i [\tilde{\mathfrak{q}}_i] + \Big(\sum_{i=\eta_1}^{\eta_2+1} \lambda_i\Big)[\tilde{\mathfrak{q}}_{\eta_1}]$$

(where if $\eta_1 = 1$, then the first sum is vacuous). Since the $[\tilde{\mathfrak{q}}_1], \ldots, [\tilde{\mathfrak{q}}_{\eta_1}]$ are basis elements, we must have $\lambda_i = 0$ for $1 \leqslant i \leqslant \eta_1 - 1$ and $\sum_{i=\eta_1}^{\eta_2+1} \lambda_i = 0$.

Thus, $[N_{12}]$, and hence its pair $[N_7]$, give us, respectively, the candidates

$$[M_1] = \sum_{i=\eta_1}^{\eta_2+1} \lambda_i [\mathfrak{q}_i] \quad \text{and} \quad [M_2] = \sum_{j=\kappa_1}^{\kappa_2} \delta_j [\mathfrak{p}_j] = [\omega_R] - [M_1],$$

where $\eta_1 \leqslant \eta_2$, $\kappa_1 \leqslant \kappa_2$, $\lambda_i = 0$ for all $i < \eta_1$ or $i > \eta_2 + 1$, $\delta_j = 0$ for all $j < \kappa_1$ or $j > \kappa_2$, and $\lambda_{\eta_1} + \lambda_{\eta_1+1} + \cdots + \lambda_{\eta_2+1} = 0$. In other words, the corners (a_0, b_0), (a_{h+1}, b_{h+1}) together with all inside corners, except (a_i, b_i), (c_j, d_j) for $\eta_1 \leqslant i \leqslant \eta_2$

and $\kappa_1 \leqslant j \leqslant \kappa_2$, all lie on the same antidiagonal. Furthermore, we have $a_{\eta_1} \leqslant a_{\eta_2} \leqslant c_1 \leqslant c_{\kappa_1} \leqslant c_{\kappa_2}$ and $b_{\eta_2} \leqslant b_{\eta_1} \leqslant d_k \leqslant d_{\kappa_2} \leqslant d_{\kappa_1}$, and by assumption, no two inside corners coincide. Hence, $(a_i, b_i) \lneq (c_j, d_j)$ for all $\eta_1 \leqslant i \leqslant \eta_2$ and $\kappa_1 \leqslant j \leqslant \kappa_2$.

Suppose that $\varphi([N_{10}]) = 0$ (equivalently, $\varphi([N_9]) = [\omega_{\tilde{R}}]$). Since $\eta_2 = h$ in this case, we have $0 = \varphi([N_{10}]) = \lambda_{h+1}[\tilde{\mathfrak{q}}_{\eta_1}]$, so $[N_{10}] = 0$. Thus, $\varphi([N_{10}]) = 0$ produces no candidate for a semidualizing module.

Suppose that $\varphi([N_9]) = 0$ (equivalently, $\varphi([N_{10}]) = [\omega_{\tilde{R}}]$). If $\kappa_2 < k$, then $\varphi([N_9]) = \lambda_1[\tilde{\mathfrak{q}}_1] + \sum_{j=1}^{\kappa_2} \delta_j[\tilde{\mathfrak{p}}_j]$ implies that $[N_9] = 0$ since the $[\tilde{\mathfrak{q}}_i]$, $[\tilde{\mathfrak{p}}_j]$ are basis elements. If $\kappa_2 = k$, then

$$0 = \varphi\big([N_9]\big) = \lambda_1[\tilde{\mathfrak{q}}_1] + \delta_k[\tilde{\mathfrak{q}}_1'] + \sum_{j=1}^{k-1} \delta_j[\tilde{\mathfrak{p}}_j] = (\lambda_1 - \delta_k)[\tilde{\mathfrak{q}}_1] + \sum_{j=1}^{k-1}(\delta_j - \delta_k)[\tilde{\mathfrak{p}}_j],$$

where the last equality follows from [5, Corollary 2.3(i), with $I_1 = \{1, \ldots, k-1\}$]. Since the $[\tilde{\mathfrak{q}}_1]$, $[\tilde{\mathfrak{p}}_j]$ are basis elements, it follows that $\lambda_1 = \delta_1 = \delta_2 = \cdots = \delta_k$. Therefore, $[N_9]$, $[N_{10}]$ give us the candidates

$$[M_3] = \lambda_1[\mathfrak{q}_1] + \sum_{j=1}^{k} \lambda_1[\mathfrak{p}_j] = -\lambda_1[\mathfrak{q}_1'] \qquad \text{and}$$
$$[M_4] = \lambda_{h+1}[\mathfrak{q}_{h+1}] = [\omega_R] - [M_3],$$

where $\eta_2 = h$, $\kappa_1 = 1$, $\kappa_2 = k$; hence, $\eta_1 = 1$, and $\lambda_1 = \delta_1 = \delta_2 = \cdots = \delta_k$, and $\lambda_2 = \lambda_3 = \cdots = \lambda_h = 0$. Since $\eta_1 = \kappa_1 = 1$, we have $(a_i, b_i) \lneq (c_j, d_j)$ for all $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant k$.

If $\varphi([N_8]) = 0$, then $\varphi([N_8]) = \lambda_1[\tilde{\mathfrak{q}}_1]$ implies $[N_8] = 0$. Suppose that $\varphi([N_8]) = [\omega_{\tilde{R}}]$. In this case, we have $\lambda_i = \delta_j = 0$ for all $1 < i < h+1$ and $j < \kappa_1$, $\lambda_1 = \tilde{\lambda}_1$, and $\tilde{\lambda}_i = \tilde{\delta}_j = 0$ for all $1 < i < \eta_1 + 1$ and $j < k$. Since $\lambda_i = 0$ for all $1 < i < h+1$, the corners (a_i, b_i) for $1 \leqslant i \leqslant h$ all lie on the same antidiagonal. Since $\lambda_1 = \tilde{\lambda}_1$ and $\tilde{\lambda}_i = 0$ for all $1 < i < \eta_1 + 1$, where $\tilde{\lambda}_i = \lambda_i = a_i + b_i - a_{i-1} - b_{i-1}$ for $2 \leqslant i \leqslant \eta_1 - 1$ and $\tilde{\lambda}_{\eta_1} = c_k + d_k - a_{\eta_1 - 1} - b_{\eta_1 - 1}$, the corners $(a_1, b_1), \ldots, (a_h, b_h)$ and (c_k, d_k) lie on the same antidiagonal (whether or not $\eta_1 = 1$). By definition (recall Fact 2.4), $\delta_j := a_{i_j} + b_{i_j} - c_j - d_j = 0$ for all $j < \kappa_1 = \min\{i \mid c_i \geqslant a_h\}$; i.e., $i_j \leqslant h$, where $i_j = \min\{i : a_i > c_j\}$, the corners $(a_1, b_1), \ldots, (a_h, b_h)$, (c_k, d_k), and (c_j, d_j) for $1 \leqslant j < \kappa_1$ all lie on the same antidiagonal. Since $\tilde{\delta}_j = c_k + d_k - c_j - d_j = 0$ for all j such that $c_j \geqslant a_{\eta_1 - 1}$, and $c_{\kappa_1} \geqslant a_h \geqslant a_{\eta_1 - 1}$, the corners (c_j, d_j) for $\kappa_1 \leqslant j \leqslant k$ all lie on the same antidiagonal. Thus, $\lambda_{h+1} := a_{h+1} + b_{h+1} - a_h - b_h = a_{h+1} + b_{h+1} - c_j - d_j = \delta_j$ for $\kappa_1 \leqslant j \leqslant k$. Hence, $[N_8]$ and $[N_{11}]$ give us the candidates

$$[M_5] = \lambda_1[\mathfrak{q}_1] \qquad \text{and}$$
$$[M_6] = \lambda_{h+1}[\mathfrak{q}_{h+1}] + \sum_{j=\kappa_1}^{k} \lambda_{h+1}[\mathfrak{p}_j] = -\lambda_{h+1}[\mathfrak{q}_{h+1}'] = [\omega_R] - [M_5],$$

(the last equality again follows from [5, Corollary 2.3(i), with $I_1 = \{1, \ldots, k-1\}$], and) where all inside corners lie on the same antidiagonal.

To summarize, the possible semidualizing modules of R are listed below, along with the conditions in which they have the potential to exist based upon the analysis above:

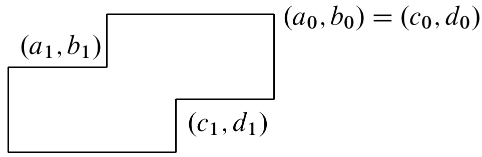
$$[M_1],[M_2],[M_3],[M_4] \quad \text{if } (a_i,b_i) \lneqq (c_j,d_j) \text{ for all } 1 \leqslant i \leqslant h \text{ and } 1 \leqslant j \leqslant k,$$

$$[M_5],[M_6] \quad \text{if } (a_i,b_i) \nleq (c_j,d_j) \text{ for all } 1 \leqslant i \leqslant h \text{ and } 1 \leqslant j \leqslant k, \text{ and}$$

$$[M_1],[M_2] \quad \text{otherwise.}$$

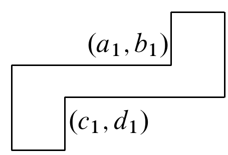
□

With this summary in hand, it is convenient to address two cases based upon the shape of the ladder. In particular, we use the descriptives "thick" and "thin." The most basic case of each such ladder is outlined below, where there is exactly one lower, and one upper, inside corner. Casually speaking, a "thick" ladder is one in which every lower inside corner (a_i,b_i) is strictly less than every upper inside corner (c_j,d_j) (i.e., the case of $[M_1]-[M_4]$ above), while a "thin" ladder is the diametric opposite of this; i.e., one such that $(a_i,b_i) \nleq (c_j,d_j)$, for all $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant k$. Note that for the latter, it is possible for upper and lower inside corners to lie on the same antidiagonal. For the ladder on the right, it is necessary that $a_1 < c_1$ if $b_1 > d_1$. The result for the case $a_1 > c_1$ follows by symmetry.



$a_1 \leqslant c_1,\ b_1 \leqslant d_1, (a_1,b_1) \neq (c_1,d_1)$

basic thick ladder



$b_1 > d_1$

basic thin ladder

DEFINITION 4.4

Let Y be a ladder. We say that we *antitranspose* Y to mean that we form the *antitranspose* $\tilde{Y}$, where $\tilde{Y}_{ij} = X_{a_{h+1}-j+a_0,b_0-i+b_{h+1}}$. The ladder $\tilde{Y}$ has corners

$$(\tilde{a}_0,\tilde{b}_0) = (b_{h+1},a_{h+1}),$$

$$(\tilde{a}_1,\tilde{b}_1) = (b_0-d_1+b_{h+1},a_{h+1}-c_1+a_0),\ldots,$$

$$(\tilde{a}_k,\tilde{b}_k) = (b_0-d_k+b_{h+1},a_{h+1}-c_k+a_0),$$

$$(\tilde{a}_{k+1},\tilde{b}_{k+1}) = (b_0,a_0),$$

$$(\tilde{c}_1,\tilde{d}_1) = (b_0-b_1+b_{h+1},a_{h+1}-a_1+a_0),\ldots,$$

$$(\tilde{c}_h,\tilde{d}_h) = (b_0-b_h+b_{h+1},a_{h+1}-a_h+a_0).$$

THEOREM 4.5 (Thick Ladder Theorem)

Let Y be a two-sided 2-connected ladder, with $h \geqslant 1$ lower inside corners and $k \geqslant 1$ upper inside corners, such that $(a_i,b_i) \lneqq (c_j,d_j)$ for all $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant k$. Let $R = R_2(Y)$. Then $|\mathfrak{S}_0(R)| \leqslant 2$.

Proof
By Theorem 3.4 and the proof of Proposition 4.3, we need only to show that $[M_1]$, $[M_2]$, $[M_3]$, $[M_4]$ in Proposition 4.3 must be trivial semidualizing modules.

CASE 1
Let us first consider

$$[M_3] = \lambda_1[\mathfrak{q}_1] + \sum_{j=1}^{k} \lambda_1[\mathfrak{p}_j] = -\lambda_1[\mathfrak{q}_1'] \qquad \text{and}$$
$$[M_4] = \lambda_{h+1}[\mathfrak{q}_{h+1}] = [\omega_R] - [M_3],$$

where $\lambda_1 = \delta_1 = \delta_2 = \cdots = \delta_k$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_h = 0$. In this case, $\lambda_1 = a_{h+1} + b_{h+1} - c_k - d_k < a_{h+1} + b_{h+1} - a_1 - b_1 = \lambda_{h+1}$.

CASE 1.1
Suppose that $\lambda_1 > 0$. Write $M_3 = \mathfrak{q}_1^{\lambda_1} \cap \mathfrak{p}_1^{\lambda_1} \cap \cdots \cap \mathfrak{p}_k^{\lambda_1}$ and $M_4 = \mathfrak{q}_{h+1}^{\lambda_{h+1}}$. Under the multiplication map of ideals $\mu\colon M_3 \otimes_R M_4 \to M_3M_4$, we have

$$\begin{aligned}
&\mu(x_{a_0b_1}^{\lambda_1-1} x_{a_0b_0} x_{a_hb_{h+1}} \otimes x_{a_hb_1}^{\lambda_{h+1}}) \\
&\quad = x_{a_hb_1}^{\lambda_{h+1}-1} x_{a_0b_1}^{\lambda_1-1} x_{a_hb_1} x_{a_0b_0} x_{a_hb_{h+1}} \\
&\quad = x_{a_hb_1}^{\lambda_{h+1}-1} x_{a_0b_1}^{\lambda_1} x_{a_hb_0} x_{a_hb_{h+1}} \qquad \text{since } x_{a_hb_1}x_{a_0b_0} = x_{a_hb_0}x_{a_0b_1} \\
&\quad = \mu(x_{a_0b_1}^{\lambda_1} \otimes x_{a_hb_1}^{\lambda_{h+1}-1} x_{a_hb_0} x_{a_hb_{h+1}}).
\end{aligned}$$

Hence, μ is not injective, contradicting Fact 2.2.

CASE 1.2
Suppose that $\lambda_{h+1} > 0$ and $\lambda_1 < 0$. Let $M_3 = {\mathfrak{q}_1'}^{-\lambda_1}$, $M_4 = \mathfrak{q}_{h+1}^{\lambda_{h+1}}$, and $\omega_R = {\mathfrak{q}_1'}^{-\lambda_1} \cap \mathfrak{q}_{h+1}^{\lambda_{h+1}}$. As in Case 3 of Theorem 3.4, we get a contradiction since the function

$$\operatorname{lcm}\colon \operatorname{mingen}((\mathfrak{q}_1')^{|\lambda_1|}) \times \operatorname{mingen}(\mathfrak{q}_{h+1}^{\lambda_{h+1}}) \to (\mathfrak{q}_1')^{|\lambda_1|} \cap \mathfrak{q}_{h+1}^{\lambda_{h+1}}$$

does not give a bijection of minimal generating sets.

CASE 1.3
Suppose that $\lambda_{h+1} < 0$. We antitranspose Y to get

$$\begin{aligned}
[\omega_{\tilde{R}}] &= (a_0 + b_0 - c_1 - d_1)[\tilde{\mathfrak{q}}_1] + (c_k + d_k - a_{h+1} - b_{h+1})[\tilde{\mathfrak{q}}_{k+1}] \\
&\qquad + (a_1 + b_1 - a_{h+1} - b_{h+1})[\tilde{\mathfrak{p}}_1] + \cdots + (a_h + b_h - a_{h+1} - b_{h+1})[\tilde{\mathfrak{p}}_h] \\
&= -\lambda_1[\tilde{\mathfrak{q}}_{k+1}] - \lambda_{h+1}\Big([\tilde{\mathfrak{q}}_1] + \sum_{j=1}^{h} [\tilde{\mathfrak{p}}_j]\Big).
\end{aligned}$$

We may then use Case 1.1 to reach our contradiction.

CASE 2

Now we consider the candidates $[M_1]$, $[M_2]$, where, as a result of the hypotheses, $\eta_1 = 1$, $\eta_2 = h$, $\kappa_1 = 1$, and $\kappa_2 = k$. Thus, $[M_1] = \sum_{i=1}^{h+1} \lambda_i [\mathfrak{q}_i]$ and $[M_2] = [\omega_R] - [M_1] = \sum_{j=1}^{k} \delta_j [\mathfrak{p}_j]$, where $\sum_{i=1}^{h+1} \lambda_i = 0$; i.e., $a_0 + b_0 = a_{h+1} + b_{h+1}$. We note that $a_i + b_i < c_j + d_j$ for all $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant k$.

CASE 2.1

Suppose that $c_j + d_j \leqslant a_{h+1} + b_{h+1}$ for all $1 \leqslant j \leqslant k$. Then $\delta_j \geqslant 0$ for all $1 \leqslant j \leqslant k$ and $\lambda_{h+1} > 0$. Let us write

$$\begin{aligned}
[M_1] &= \sum_{\lambda_i > 0} \lambda_i [\mathfrak{q}_i] + \sum_{\lambda_i < 0} -|\lambda_i| [\mathfrak{q}_i] \\
&= \sum_{\lambda_i > 0} \lambda_i [\mathfrak{q}_i] + \sum_{\lambda_i < 0} |\lambda_i| \Big([\mathfrak{q}_i'] + \sum_{j=1}^{k} [\mathfrak{p}_j] \Big) \\
&= \sum_{\lambda_i > 0} \lambda_i [\mathfrak{q}_i] + \sum_{\lambda_i < 0} |\lambda_i| [\mathfrak{q}_i'] + \sum_{j=1}^{k} \sum_{\lambda_i < 0} |\lambda_i| [\mathfrak{p}_j].
\end{aligned}$$

We let

$$\begin{aligned}
M_1 &= \bigcap_{\lambda_i > 0} \mathfrak{q}_i^{\lambda_i} \cap \bigcap_{\lambda_i < 0} (\mathfrak{q}_i')^{|\lambda_i|} \cap \bigcap_{j=1}^{k} \mathfrak{p}_j^{\sum_{\lambda_i < 0} |\lambda_i|} \\
M_2 &= \bigcap_{j=1}^{k} \mathfrak{p}_j^{\delta_j}.
\end{aligned}$$

Let $r = \max\{\delta_j \mid 1 \leqslant j \leqslant k\}$. Under the multiplication map $\mu \colon M_1 \otimes_R M_2 \to M_1 M_2$ we have

$$\mu\Big(x_{a_h b_{h+1}}^{\lambda_{h+1}} \prod_{i=1}^{h} x_{a_{i-1} b_i}^{|\lambda_i|} \otimes x_{a_h b_1}^{r} \Big) = \mu\Big(x_{a_h b_{h+1}}^{\lambda_{h+1}-1} x_{a_h b_1} \prod_{i=1}^{h} x_{a_{i-1} b_i}^{|\lambda_i|} \otimes x_{a_h b_1}^{r-1} x_{a_h b_{h+1}} \Big).$$

Hence, μ is not injective, contradicting Fact 2.2.

CASE 2.2

Suppose that we have $c_j + d_j > a_{h+1} + b_{h+1}$ for some $1 \leqslant j \leqslant k$ and $a_i + b_i < a_0 + b_0 = a_{h+1} + b_{h+1}$ for all $1 \leqslant i \leqslant h$. Then $\lambda_1 < 0$ and $\lambda_h > 0$. Let us write $\delta_{j_0} = \min\{\delta_j \mid 1 \leqslant j \leqslant k\}$, and

$$\begin{aligned}
[M_2] &= -|\delta_{j_0}| [\mathfrak{p}_{j_0}] + \sum_{j \neq j_0} \delta_j [\mathfrak{p}_j] \\
&= |\delta_{j_0}| \Big([\mathfrak{q}_1] + [\mathfrak{q}_1'] + \sum_{j \neq j_0} [\mathfrak{p}_j] \Big) + \sum_{j \neq j_0} \delta_j [\mathfrak{p}_j] \\
&= |\delta_{j_0}| [\mathfrak{q}_1] + |\delta_{j_0}| [\mathfrak{q}_1'] + \sum_{j \neq j_0} (\delta_j - \delta_{j_0}) [\mathfrak{p}_j].
\end{aligned}$$

As in Case 2.1, we let

$$M_1 = \bigcap_{\lambda_i>0} \mathfrak{q}_i^{\lambda_i} \cap \bigcap_{\lambda_i<0} (\mathfrak{q}'_i)^{|\lambda_i|} \cap \bigcap_{j=1}^{k} \mathfrak{p}_j^{\sum_{\lambda_i<0}|\lambda_i|}$$

$$M_2 = \mathfrak{q}_1^{|\delta_{j_0}|} \cap (\mathfrak{q}'_1)^{|\delta_{j_0}|} \cap \bigcap_{j\neq j_0} \mathfrak{p}_j^{\delta_j+|\delta_{j_0}|}.$$

Let $r = \max(\{0\} \cup \{\delta_j + 1 \mid 1 \leqslant j \leqslant k, j \neq j_0\})$. Under the multiplication map $\mu\colon M_1 \otimes_R M_2 \to M_1 M_2$, we have

$$\begin{aligned}
&\mu\Big(x_{a_0b_1}^{|\lambda_1|} x_{a_hb_{h+1}}^{\lambda_{h+1}} \prod_{i=2}^{h} x_{a_{i-1}b_i}^{|\lambda_i|} \otimes x_{a_0b_1}^{|\delta_{j_0}|-1} x_{a_hb_{h+1}}^{r} x_{a_{h+1}b_1} x_{a_0b_0}\Big)\\
&\quad = x_{a_0b_1}^{|\lambda_1|+|\delta_{j_0}|-1} x_{a_hb_{h+1}}^{\lambda_{h+1}+r-1} (x_{a_hb_{h+1}} x_{a_{h+1}b_1}) x_{a_0b_0} \prod_{i=2}^{h} x_{a_{i-1}b_i}^{|\lambda_i|}\\
&\quad = x_{a_0b_1}^{|\lambda_1|+|\delta_{j_0}|-1} x_{a_hb_{h+1}}^{\lambda_{h+1}+r-1} (x_{a_hb_1} x_{a_{h+1}b_{h+1}}) x_{a_0b_0} \prod_{i=2}^{h} x_{a_{i-1}b_i}^{|\lambda_i|}\\
&\quad = \mu\Big(x_{a_0b_1}^{|\lambda_1|-1} x_{a_hb_{h+1}}^{\lambda_{h+1}-1} x_{a_hb_1} \prod_{i=2}^{h} x_{a_{i-1}b_i}^{|\lambda_i|} \otimes x_{a_0b_1}^{|\delta_{j_0}|} x_{a_hb_{h+1}}^{r} x_{a_{h+1}b_{h+1}} x_{a_0b_0}\Big).
\end{aligned}$$

Again μ is not injective, contradicting Fact 2.2.

CASE 2.3

Suppose that $a_i + b_i \geqslant a_0 + b_0 = a_{h+1} + b_{h+1}$ for some $1 \leqslant i \leqslant h$, so that $c_j + d_j > a_{h+1} + b_{h+1}$ for all $1 \leqslant j \leqslant k$. We can then antitranspose Y and reduce to Case 2.1 or 2.2. □

DEFINITION 4.6

Let Y be a two-sided ladder. We say that Y is a *spine* if

- $h = k$, $a_1 < c_1 < a_2 < c_2 < \cdots < a_h < c_h$ and $b_1 > d_1 > b_2 > d_2 > \cdots > b_h > d_h$; or
- $h = k$, $c_1 < a_1 < c_2 < a_2 < \cdots < c_h < a_h$ and $d_1 > b_1 > d_2 > b_2 > \cdots > d_h > b_h$; or
- $h = k + 1$, $a_1 < c_1 < a_2 < c_2 < \cdots < a_k < c_k < a_{k+1}$ and $b_1 > d_1 > b_2 > d_2 > \cdots > b_k > d_k > b_{k+1}$; or
- $k = h + 1$, $c_1 < a_1 < c_2 < a_2 < \cdots < c_h < a_h < c_{h+1}$ and $d_1 > b_1 > d_2 > b_2 > \cdots > d_h > b_h > d_{h+1}$.

DEFINITION 4.7

Let Y be a two-sided connected 2-connected ladder such that $(a_i, b_i) \nleqslant (c_j, d_j)$ for all $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant k$. We define the *spine* $\tilde{Y}$ *of* Y inductively as follows. Assume that $a_1 < c_1$. If $a_1 > c_1$, the definition is similar. Start with $u = 1$ and repeat the following steps:

(1) If (a_u, b_u) is not an inside corner, then stop. Otherwise, suppose that $a_u < a_{u+1} < \cdots < a_v < c_u \leqslant a_{v+1}$. Delete the indeterminates in the entries (e, f), where $e < a_v$ and $f < b_u$. Update the corner (a_u, b_u) with the values (a_v, b_u). Relabel the remaining corners according to our conventions.

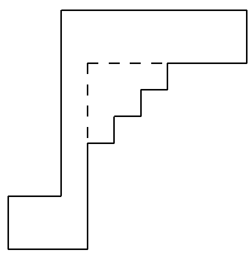
(2) If (c_u, d_u) is not an inside corner, then stop. Otherwise, suppose that $c_u < c_{u+1} < \cdots < c_v < a_{u+1} \leqslant c_{v+1}$. Delete the indeterminates in the entries (e, f), where $e > c_u$ and $f > d_v$. Update the corner (c_u, d_u) with the values (c_u, d_v). Relabel the remaining corners according to our conventions.

(3) Update the value of u to $u + 1$ and repeat.

We let $\tilde{Y}$ be the ladder obtained when the induction stops.

EXAMPLE 4.8

Shown below is a two-sided ladder where $a_1 > c_1$, with $h = 1, k = 4$, and its associated spine, which has only one upper inside corner (at (c_1, d_4)):



THEOREM 4.9 (Thin Ladder Theorem)

Let Y be a two-sided 2-connected ladder, with $h \geqslant 1$ lower inside corners and $k \geqslant 1$ upper inside corners, such that $(a_i, b_i) \not\leqslant (c_j, d_j)$ for all $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant k$. Let $R = R_2(Y)$. Then $|\mathfrak{S}_0(R)| \leqslant 2$.

Proof

We prove by induction on $h + k$. By Proposition 4.3, we need only to show that $[M_5]$, $[M_6]$ must be trivial semidualizing modules, where

$$[M_5] = \lambda_1[\mathfrak{q}_1] \qquad \text{and}$$

$$[M_6] = \lambda_{h+1}[\mathfrak{q}_{h+1}] + \sum_{j=\kappa_1}^{k} \lambda_{h+1}[\mathfrak{p}_j] = -\lambda_{h+1}[\mathfrak{q}'_{h+1}] = [\omega_R] - [M_5],$$

and all inside corners lie on the same antidiagonal.

It suffices to show that $\lambda_1\lambda_{h+1} \neq 0$ leads to a contradiction. By antitransposing Y, we may assume that $a_1 > c_1$. If $\lambda_1 > 0$, then $[M_5] = [\mathfrak{q}_1^{\lambda_1}]$, and we will let $M_5 =$

$\mathfrak{q}_1^{\lambda_1} = (x_{a_0b_1}, x_{a_0b_1+1}, \ldots, x_{a_0b_0})^{\lambda_1}$. If $\lambda_1 < 0$, then

$$[M_5] = -\lambda_1\Big([\mathfrak{q}_1'] + \sum_{j=1}^{\kappa_2}[\mathfrak{p}_j]\Big) = |\lambda_1|\big[(x_{a_0b_1}, x_{a_0+1b_1}, \ldots, x_{c_1b_1})\big].$$

We then let $M_5 = (x_{a_0b_0}, x_{a_0+1b_0}, \ldots, x_{c_1b_0})^{|\lambda_1|}$. Similarly, if $\lambda_{h+1} > 0$, then we let $M_6 = (x_{a_{h+1}b_{h+1}}, x_{a_{h+1}b_{h+1}+1}, \ldots, x_{a_{h+1}d_k})^{\lambda_{h+1}}$, and if $\lambda_{h+1} < 0$, then finally we let $M_6 = (x_{a_hb_{h+1}}, x_{a_h+1b_{h+1}}, \ldots, x_{a_{h+1}b_{h+1}})^{|\lambda_{h+1}|}$.

Consider the case when $\lambda_1, \lambda_{h+1} < 0$. Let $\tilde{Y}$ be the spine of Y. We construct part of a minimal free resolution of M_5 over $\tilde{R} = R_2(\tilde{Y})$ given by

$$0 \xleftarrow{\partial_0} \tilde{R}^{\beta_0} \xleftarrow{\partial_1} \tilde{R}^{\beta_1} \xleftarrow{\tilde{\partial}_2} \tilde{R}^{\tilde{\beta}_2} \xleftarrow{\tilde{\partial}_3} \cdots \xleftarrow{\tilde{\partial}_{2\tilde{h}-1}} \tilde{R}^{\tilde{\beta}_{2\tilde{h}-1}} \xleftarrow{\tilde{\partial}_{2\tilde{h}}} \tilde{R}^{\tilde{\beta}_{2\tilde{h}}} \xleftarrow{\tilde{\partial}_{2\tilde{h}+1}} \cdots,$$

where

$$\partial_0 = \begin{pmatrix} x_{a_0b_0}^{|\lambda_1|} & x_{a_0b_0}^{|\lambda_1|-1}x_{a_0+1b_0} & \cdots \end{pmatrix},$$

$$\partial_1 = \begin{pmatrix} x_{a_0+1\tilde{d}_1-1} & -x_{a_0+1\tilde{d}_1} & \cdots \\ -x_{a_0\tilde{d}_1-1} & x_{a_0\tilde{d}_1} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \end{pmatrix},$$

$$\tilde{\partial}_2 = \begin{pmatrix} x_{\tilde{a}_1\tilde{d}_1} & x_{\tilde{a}_1+1\tilde{d}_1} & \cdots \\ x_{\tilde{a}_1\tilde{d}_1-1} & x_{\tilde{a}_1+1\tilde{d}_1-1} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \end{pmatrix}, \ldots$$

$$\tilde{\partial}_{2\tilde{h}-1} = \begin{pmatrix} x_{\tilde{a}_{\tilde{h}-1}+1\tilde{d}_{\tilde{h}}-1} & -x_{\tilde{a}_{\tilde{h}-1}+1\tilde{d}_{\tilde{h}}} & \cdots \\ -x_{\tilde{a}_{\tilde{h}-1}\tilde{d}_{\tilde{h}}-1} & x_{\tilde{a}_{\tilde{h}-1}\tilde{d}_{\tilde{h}}} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \end{pmatrix},$$

$$\tilde{\partial}_{2\tilde{h}} = \begin{pmatrix} x_{\tilde{a}_{\tilde{h}}\tilde{d}_{\tilde{h}}} & x_{\tilde{a}_{\tilde{h}}+1\tilde{d}_{\tilde{h}}} & \cdots \\ x_{\tilde{a}_{\tilde{h}}\tilde{d}_{\tilde{h}}-1} & x_{\tilde{a}_{\tilde{h}}+1\tilde{d}_{\tilde{h}}-1} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \end{pmatrix} = \begin{pmatrix} x_{a_h\tilde{d}_{\tilde{h}}} & x_{a_h+1\tilde{d}_{\tilde{h}}} & \cdots \\ x_{a_h\tilde{d}_{\tilde{h}}-1} & x_{a_h+1\tilde{d}_{\tilde{h}}-1} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \end{pmatrix}, \ldots$$

This minimal free resolution forms part of a minimal free resolution of M_5 over $R = R_2(Y)$ given by

$$F_\bullet = 0 \xleftarrow{\partial_0} R^{\beta_0} \xleftarrow{\partial_1} R^{\beta_1} \xleftarrow{\partial_2} R^{\beta_2} \xleftarrow{\partial_3} \cdots \xleftarrow{\partial_{2\tilde{h}-1}} R^{\beta_{2\tilde{h}-1}} \xleftarrow{\partial_{2\tilde{h}}} R^{\beta_{2\tilde{h}}} \xleftarrow{\partial_{2\tilde{h}+1}} \cdots,$$

with $\tilde{\partial}_2, \tilde{\partial}_3, \ldots$ being represented by upper left submatrices of the matrices representing $\partial_2, \partial_3, \ldots$. In $F_\bullet \otimes M_6$, we have

$$\mathbf{x} = \begin{pmatrix} x_{a_h+1 b_{h+1}}^{|\lambda_{h+1}|} & -x_{a_h+1 b_{h+1}}^{|\lambda_{h+1}|-1} x_{a_h b_{h+1}} & 0 & \cdots & 0 \end{pmatrix}^T \in \operatorname{Ker}(\partial_{2\tilde{h}}).$$

However, $\mathbf{x} \notin \operatorname{Im}(\partial_{2\tilde{h}+1})$ since $F_\bullet$ is a minimal resolution. Hence, $\operatorname{Tor}^R_{2\tilde{h}}(M_5, M_6) \neq 0$, contradicting Fact 2.3.

Now suppose that $\kappa_1 \neq k$. If $\lambda_1, \lambda_{h+1} > 0$, then we also use $\operatorname{Tor}^R_{2\tilde{h}}(M_5, M_6)$ to reach a contradiction. If $\lambda_1 > 0$ and $\lambda_{h+1} < 0$, then we use $\operatorname{Tor}^R_{2\tilde{h}-1}(M_5, M_6)$; and if $\lambda_1 < 0$ and $\lambda_{h+1} > 0$, then we use $\operatorname{Tor}^R_{2\tilde{h}+1}(M_5, M_6)$.

Finally, suppose that $\kappa_1 = k$. If $\lambda_1 \lambda_{h+1} < 0$, then we use $\operatorname{Tor}^R_{2\tilde{h}-1}(M_5, M_6)$; and if $\lambda_1, \lambda_{h+1} > 0$, then we use $\operatorname{Tor}^R_{2\tilde{h}-2}(M_5, M_6)$ to reach a contradiction. □

THEOREM 4.10 (Two-Sided Ladder Theorem)
Let Y be a 2-connected ladder, with h lower inside corners and k upper inside corners, such that $(a_i, b_i) \neq (c_j, d_j)$ for all $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant k$. Then $|\mathfrak{S}_0(R_2(Y))| \leqslant 2$.

Proof
We will argue by induction on $h + k$. By Theorem 3.4, we may assume that $h, k > 0$. The case $h = k = 1$ is given by Theorems 4.5 and 4.9. In the induction step, by Proposition 4.3 we need only to show that $[M_1]$, $[M_2]$ must be trivial semidualizing modules, where

$$[M_1] = \sum_{i=\eta_1}^{\eta_2+1} \lambda_i [\mathfrak{q}_i] \qquad \text{and} \qquad [M_2] = \sum_{j=\kappa_1}^{\kappa_2} \delta_j [\mathfrak{p}_j] = [\omega_R] - [M_1].$$

$\eta_1 \leqslant \eta_2$, $\kappa_1 \leqslant \kappa_2$, η_1, κ_1 are not both 1 (equivalently, we cannot have both $\eta_2 = h$ and $\kappa_2 = k$), and the corners (a_0, b_0), (a_{h+1}, b_{h+1}) together with all inside corners, except (a_i, b_i), (c_j, d_j) for $\eta_1 \leqslant i \leqslant \eta_2$ and $\kappa_1 \leqslant j \leqslant \kappa_2$, all lie on the same antidiagonal.

Note that we cannot have both $\eta_2 \neq h$ and $\kappa_2 \neq k$. Otherwise, we would have $a_h \leqslant c_{\kappa_1} \leqslant c_{\kappa_2} < c_k$ and $b_h < b_{\eta_2} \leqslant b_{\eta_1} \leqslant d_k$, a contradiction since (a_h, b_h) and (c_k, d_k) should lie on the same antidiagonal.

By antitransposing Y, we may assume that $\eta_2 \neq h$ and $\kappa_2 = k$. In this case, we have $a_i \leqslant a_{\eta_2} \leqslant c_1 \leqslant c_k$ for all $1 \leqslant i \leqslant \eta_2$, and $b_i \leqslant b_1 \leqslant d_k \leqslant d_1$ for all $i \geqslant 1$. So $(a_i, b_i) \lneqq (c_j, d_j)$ for all $1 \leqslant i \leqslant \eta_2$ and $1 \leqslant j \leqslant k$.

We also note that for $\kappa_1 \leqslant j \leqslant \kappa_2 = k$, we have $c_j \geqslant a_h$ and $d_j \geqslant d_k \geqslant b_1 > b_h$. Hence, $c_j + d_j > a_h + b_h = a_{h+1} + b_{h+1}$ for all $\kappa_1 \leqslant j \leqslant k$.

Now suppose that $a_i + b_i \geqslant a_0 + b_0 = a_{h+1} + b_{h+1}$, where $1 \leqslant i \leqslant \eta_2$. Since $\eta_2 \neq h$, i.e., $\kappa_1 \neq 1$, the corners (c_1, d_1), (a_0, b_0) and (a_{h+1}, b_{h+1}) lie on the same antidiagonal. This is a contradiction since $(a_i, b_i) \lneqq (c_1, d_1)$. Hence, $a_i + b_i < a_0 + b_0 = a_{h+1} + b_{h+1}$ for all $1 \leqslant i \leqslant \eta_2$.

Finally, let $\tilde{Y}$ be the ladder obtained by deleting from Y columns $b_{h+1}, b_{h+1} + 1, \ldots, b_{\eta_2+1} - 1$. We can then use arguments similar to those in Theorem 4.5, Case 2.2 on $\tilde{Y}$ to finish the induction. □

We will generalize Theorem 3.6 and Theorem 4.10 in [18]. Here we end with some examples to illustrate our results and point to our future work.

EXAMPLE 4.11

We consider $R = R_t(-)$ for the ladders shown earlier:

(1) If (L) is the ladder on page 168, then $\mathfrak{S}_0(R_2(L)) = \{[R], [\omega_R]\}$, by Theorem 4.10 and [5, Proposition 2.5];
(2) If O is the one-sided ladder on page 166, then $\mathfrak{S}_0(R_3(O)) = \{[R]\}$ and $\mathfrak{S}_0(R_2(O)) = \{[R], [\omega_R]\}$, by Theorem 3.4 and [5, Example 4.10];
(3) For the ladder T with a coincidental inside corner on page 166, we show in [18] that $\mathfrak{S}_0(R_2(T)) = \{[R], [\omega_R], [(x_{12}, x_{13})], [(x_{31}, x_{32})]\}$.

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