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ON THE STRUCTURE OF S_2-IFICATIONS OF COMPLETE LOCAL RINGS

SEAN SATHER-WAGSTAFF AND SANDRA SPIROFF

ABSTRACT. Motivated by work of Hochster and Huneke, we investigate several constructions related to the S_2-ification T of a complete equidimensional local ring R: the canonical module, the top local cohomology module, topological spaces of the form $\operatorname{Spec}(R) - V(J)$, and the (finite simple) graph Γ_R with vertex set $\operatorname{Min}(R)$ defined by Hochster and Huneke. We generalize one of their results by showing, e.g., that the number of connected components of Γ_R is equal to the maximum number of connected components of $\operatorname{Spec}(R) - V(J)$ for all J of height 2. We further investigate this graph by exhibiting a technique for showing that certain graphs G can be realized in the form Γ_R.

1. Introduction.

Remark 1.1. Throughout this paper, the term *ring* is short for "commutative noetherian ring," and *graph* is short for "finite simple (undirected) graph." In addition, k will be a field, and $(R, \mathfrak{m}, \mathsf{k})$ will be a local ring.

This project was motivated by [**7**], which explores the relation between indecomposability of canonical and local cohomology modules and connectedness properties of $\operatorname{Spec}(R)$. These ideas originate with Faltings [**4**] and Hartshorne [**6**]. Also, see Eghbali and Schenzel [**3**] and Schenzel [**12**].

Our interest in this subject comes from the connection with S_2-ifications of complete, equidimensional, local rings, where by *equidimensional* we mean that $\dim(R/\mathfrak{p}) = \dim(R)$ for every minimal prime

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$\mathfrak{p}$ of R. (See Section 2 for S_2-ification definition.) Specifically, in our paper **[11]**, we use **[7,** Proposition (3.9)] to show that a certain integral closure must be local. The utility of this construction has led us to investigate its properties more carefully. In this paper, we focus on the following construction (appropriately called the "Hochster-Huneke graph" of R by Zhang **[14]**) and the subsequent result.

Definition 1.2 ([**7,** Definition (3.4)]). Assume that R is equidimensional. We denote by Γ_R the graph whose vertices are the minimal primes of R, and whose edges are determined by the following rule: if $\mathfrak{p}, \mathfrak{q}$ are distinct minimal primes of R, then $\mathfrak{p}$ and $\mathfrak{q}$ are adjacent in Γ_R if and only if $\mathrm{ht}_R(\mathfrak{p}+\mathfrak{q}) = 1$.

Remark 1.3 ([**7,** Theorem (3.6)]). If R is complete and equidimensional, then the following conditions are equivalent:

(a) the local cohomology module $H^{\dim R}_{\mathfrak{m}}(R)$ is indecomposable;
(b) the canonical module of R is indecomposable;
(c) the S_2-ification of R is local;
(d) for every ideal J of height at least two, $\mathrm{Spec}(R) - V(J)$ is connected;
(e) the graph Γ_R from Definition 1.2 is connected.

The main result of the current paper is a generalization of this fact, which requires some notation and discussion.

Remark 1.4. Assume that R is complete. The Krull-Remak-Schmidt theorem states that a finitely generated R-module decomposes uniquely as a direct sum of indecomposable R-modules. By Matlis duality, the same is true for artinian R-modules. For an R-module M which is either finitely generated or artinian, let $\zeta_R(M)$ denote the number of summands in a direct sum decomposition of M by indecomposable R-modules. For a topological space or graph X, let $\beta(X)$ denote the number of connected components of X. For a ring S, let m-$\mathrm{Spec}(S)$ denote the set of its maximal ideals.

Here, we provide our generalization of Remark 1.3, which extends a case of a result of Eghbali and Schenzel [**3,** Theorem 5.5]. The proof of Theorem 1.5 is given in Section 2.

Theorem 1.5. *If R is complete and equidimensional, then the following quantities are equal.*

(a) $\zeta_R(\mathrm{H}_{\mathfrak{m}}^{\dim R}(R))$;
(b) $\zeta_R(\omega_R)$, *where ω_R is a canonical module for R*;
(c) $|m\!-\operatorname{Spec}(T)|$, *where T is the S_2-ification of R*;
(d) $\max\{\beta(\operatorname{Spec}(R) - V(J)) \mid J$ *is an ideal of R such that* $\mathrm{ht}_R(J) \geq 2\}$;
(e) $\max\{\beta(\operatorname{Spec}(R) - V(J)) \mid J$ *is an ideal of R such that* $\mathrm{ht}_R(J) = 2\}$;
(f) $\beta(\Gamma_R)$.

It is worth noting that the quantity $\beta(\Gamma_R)$ arises in other contexts. For instance, in [**9, 14**] it is shown that, if R is the completion of the strict hensilization of the completion of an equicharacteristic local ring A, then $\beta(\Gamma_R)$ equals the top "Lyubeznik number" $\lambda_{d,d}(A)$.

Recently, there has been greater interest in understanding more about the graph Γ_R. For example, Holmes [**8**] studied the diameter of Γ_R. Along this line, it is natural to ask whether or not an arbitrary graph G can be realized as Γ_R for some complete local equidimensional ring R. Toward this end, we introduce the notion of an *admissible labeling* for G and prove the following.

Theorem 1.6. *Let G be a graph.*

(a) *If G admits an admissible labeling, then there is a complete local equidimensional ring R such that Γ_R is graph-isomorphic to G. Moreover, the ring R is of the form $\mathsf{k}[\![X_1, \ldots, X_n]\!]/I$, where I is generated by square-free monomials.*

(b) *Conversely, if I is a ideal of $\mathsf{k}[\![X_1, \ldots, X_n]\!]$ generated by monomials such that the quotient $R = \mathsf{k}[\![X_1, \ldots, X_n]\!]/I$ is equidimensional, then the graph Γ_R admits an admissible labeling.*

Admissible labelings and the proof of this result are the subject of Section 3. In particular, see Definition 3.1 and the proof of Theorem 1.6. Note that, after the initial posting of this result, a complete answer to this question was given by Benedetti, Bolognese and Varbaro [**1,** Corollary 3.6] using geometric techniques, in contrast with our combinatorial techniques. See the end of Section 3 for more details.

2. Connected components, maximal ideals and indecomposable summands.

Remark 2.1. Throughout this section, we assume that R, in addition to being local, is complete and equidimensional. Let T be the S_2-ification (see Definition 2.2 or [**7,** Section 2]) of R and ω_R the canonical module of R, whence $T \cong \operatorname{Hom}_R(\omega_R, \omega_R)$. Recall that the symbols ζ and β are as described in Remark 1.4.

Definition 2.2 ([**7,** Discussion (2.3)])**.** If R has no embedded associated primes, then an R-subalgebra T of the total ring of quotients of R is an S_2*-ification* of R if:

(i) T is module finite over R;
(ii) T satisfies the Serre condition (S_2) over R; and
(iii) $\operatorname{Coker}(R \to T)$ has no prime ideal of R of height less than two in its support.

When R is equidimensional, possibly with embedded associated primes, then, by an S_2*-ification* of R, we mean an S_2-ification of $R/j(R)$, where $j(R)$ is the largest ideal, which is a submodule of R of dimension smaller than that of R.

This section is devoted to establishing the equivalent conditions in Theorem 1.5. Note that, if $\dim(R) \leq 1$, then the quantities (a)–(f) in Theorem 1.5 are all 1. Thus, we may take $\dim(R) \geq 2$, although that assumption is unnecessary unless expressly stated. Some of the equalities are consequences of results found in the existing literature. To wit:

Proposition 2.3 ([**3**])**.** *Under the assumptions in Remark* 2.1*, if* $\dim(R) \geq 2$*, then*

$$\beta(\Gamma_R) = \zeta_R(\mathrm{H}_{\mathfrak{m}}^{\dim R}(R)) = \zeta_R(\omega_R) = |m\text{-}\operatorname{Spec}(T)|.$$

Sketch of the proof. If any of the values in question is equal to one, then they are all equal to one by Remark 1.3; thus, suppose that each is at least two. Then, the statement follows as the special case $I = \mathfrak{m}$, taking $Q = (0)$ of [**3,** Theorem 5.5]. We provide an outline of the beginning of the argument for completeness in this particular case and to set the notation used in the sequel.

Write $\Gamma_R = \Gamma_1 \bigsqcup \cdots \bigsqcup \Gamma_t$, where the Γ_j are connected components of the graph Γ_R and $t \geq 2$. For $j = 1, \dots t$, let $V_j = \{\mathfrak{p}_{j1}, \mathfrak{p}_{j2}, \dots, \mathfrak{p}_{ja_j}\}$ be the vertex set of Γ_j, that is, the set of (distinct) minimal primes of R in the component Γ_j. In particular, we assume that $a_j = |V_j|$ and $V_i \bigcap V_j = \emptyset$ for $i \neq j$. For each j, let I_j denote the intersection of all $\mathfrak{p}_{ji}$-primary components of a minimal primary decomposition of the zero ideal (0) of R, i.e., $I_j = \bigcap_{i=1}^{a_j} \mathfrak{q}_{ji}$, where $\mathfrak{q}_{ji}$ is $\mathfrak{p}_{ji}$-primary. Set $J_k = \bigcap_{j=1}^{k} I_j$. For $k = 2, \dots, t$, there is an exact sequence

$$0 \longrightarrow R/J_k \longrightarrow R/J_{k-1} \bigoplus R/I_k \longrightarrow R/(J_{k-1} + I_k) \longrightarrow 0,$$

from which there is a long exact sequence of local cohomology modules:

$$\begin{aligned}\cdots \mathrm{H}_{\mathfrak{m}}^{\dim R-1}(R/(J_{k-1}+I_k)) &\longrightarrow \mathrm{H}_{\mathfrak{m}}^{\dim R}(R/J_k) \\ &\longrightarrow \mathrm{H}_{\mathfrak{m}}^{\dim R}(R/J_{k-1} \oplus R/I_k) \longrightarrow \mathrm{H}_{\mathfrak{m}}^{\dim R}(R/(J_{k-1}+I_k)) \cdots .\end{aligned}$$

Let P be a prime ideal of R that contains $J_{k-1} + I_k$. Since $P \supseteq I_j$ for some $j < k$, it follows that $P \supseteq \mathfrak{p}_{ji}$ for some i. Likewise, P contains $\mathfrak{p}_{kl}$ for some l, i.e., $P \supseteq \mathfrak{p}_{kl} + \mathfrak{p}_{ji}$. Since Γ_R is disconnected and $j \neq k$, we have $\mathrm{ht}_R(\mathfrak{p}_{kl} + \mathfrak{p}_{ji}) \geq 2$. It follows that $\dim R/(J_{k-1}, I_k) < \dim R - 1$. Consequently, the two modules, in the long exact sequence shown above, bookending the middle pair, are both zero. As a result, for each $k = 2, \dots, n$,

$$\mathrm{H}_{\mathfrak{m}}^{\dim R}(R/J_k) \cong \mathrm{H}_{\mathfrak{m}}^{\dim R}(R/J_{k-1}) \bigoplus \mathrm{H}_{\mathfrak{m}}^{\dim R}(R/I_k).$$

Since $J_1 = I_1$, $J_k = \bigcap_{j=1}^{k} I_j$ and $J_t = (0)$, we have

$$\mathrm{H}_{\mathfrak{m}}^{\dim R}(R) \cong \bigoplus_{k=1}^{t} \mathrm{H}_{\mathfrak{m}}^{\dim R}(R/I_k),$$

so $t = \zeta_R(\mathrm{H}_{\mathfrak{m}}^{\dim R}(R))$. By Matlis duality, a canonical module for R is $\omega_R \cong \mathrm{H}_{\mathfrak{m}}^{\dim R}(R)^{\vee}$, and hence, $t = \zeta_R(\omega_R)$. The remainder of the proof follows as in that of [**3,** Theorem 5.5]. □

Thus, it remains to show the equivalence of (d) and (e) in Theorem 1.5 with the other quantities.

Proposition 2.4. *Under the assumptions of Remark* 2.1*, if J is an ideal of R such that* $\mathrm{ht}_R(J) \geq 2$*, then* $\beta(\mathrm{Spec}(R) - V(J)) \leq |m\text{-}\mathrm{Spec}(T)|$.

Proof. Set $t := |m\text{-}\mathrm{Spec}(T)|$. Since $t = 1$ if and only if $\mathrm{Spec}(R) - V(J)$ is connected for every ideal J of R such that $\mathrm{ht}_R(J) \geq 2$ by Remark 1.3, assume that $t \geq 2$. Let P be a prime ideal of T such that $P \supseteq JT$. Then, $\mathrm{ht}_T(P) \geq \mathrm{ht}_T(JT) = \mathrm{ht}_R(J) \geq 2$, by [**7,** Proposition (3.5)(b)]. Since T decomposes as a product of local rings $T = T_1 \times \cdots \times T_t$ by [**7,** Remark (2.2)(k)], there exist unique i and $P_i \in \mathrm{Spec}(T_i)$ such that

$$P = T_1 \times \cdots \times T_{i-1} \times P_i \times T_{i+1} \times \cdots \times T_t.$$

In other words, there is a containment-respecting bijection

$$\mathrm{Spec}(T) \rightleftarrows \bigsqcup_{i=1}^{t} \mathrm{Spec}(T_i).$$

It is straightforward to show that, under this bijection, $P \in V(JT)$ if and only if $P_i \in V(JT_i)$, that is, there is another containment-respecting bijection

$$\mathrm{Spec}(T) - V(JT) \rightleftarrows \bigsqcup_{i=1}^{t} (\mathrm{Spec}(T_i) - V(JT_i)).$$

It follows that these bijections are homeomorphisms for the topologies induced by the Zariski topologies.

Next, if $P_i \in V(JT_i)$, then, taking P as above, we have $P \supseteq JT$, and $\mathrm{ht}_{T_i}(P_i) = \mathrm{ht}_T(P) \geq 2$. Consequently, $\mathrm{ht}_{T_i}(JT_i) \geq 2$. Moreover, for each i, the set $\mathrm{Spec}(T_i) - V(JT_i)$ is non-empty since $\mathrm{ht}_{T_i}(JT_i) \geq 2$. The implication of this is that each space $\mathrm{Spec}(T_i) - V(JT_i)$ is connected. In order to be specific, the ring T_i satisfies the assumptions of Remark 2.1 as well as the (S_2) condition; thus, T_i is its own S_2-ification, and, since it is local, the equivalent conditions of Remark 1.3 apply. From this, we conclude that $\beta(\mathrm{Spec}(T) - V(JT)) = t$.

We claim that $\beta(\mathrm{Spec}(R) - V(J)) \leq t$. Recall that, if $X \to Y$ is a continuous and surjective map of topological spaces, then $\beta(X) \geq \beta(Y)$. Apply this to the map

$$f\colon \mathrm{Spec}(T) - V(JT) \longrightarrow \mathrm{Spec}(R) - V(J),$$

which is induced from the map

$$F\colon \operatorname{Spec}(T) \longrightarrow \operatorname{Spec}(R).$$

The latter map is given by contraction and is onto since $R \to T$ is an integral extension. The map f is well-defined, due to the fact that $F^{-1}(V(J)) = V(JT)$, as well as surjective since F is surjective. $\square$

The next result is the last component necessary for establishing the equivalent conditions in Theorem 1.5.

Proposition 2.5. *Under the assumptions of Remark* 2.1*, if* $\dim(R) \geq 2$*, then there is an ideal* I *of* R *such that* $\operatorname{ht}_R(I) = 2$ *and* $\beta(\operatorname{Spec}(R) - V(I)) = \beta(\Gamma_R)$.

Proof. Set $t = \beta(\Gamma_R)$, and, as above, assume that $t \geq 2$. As (with the same notation) in the proof of Proposition 2.3, write

$$\Gamma_R = \Gamma_1 \bigsqcup \cdots \bigsqcup \Gamma_t,$$

where the Γ_i are connected components of the graph Γ_R. For all pairs of distinct integers $i, j \in \{1, \ldots, t\}$, set

$$\mathfrak{r}_i = \bigcap_{j=1}^{a_i} \mathfrak{p}_{ij}, \qquad \mathfrak{s}_{ij} = \mathfrak{r}_i + \mathfrak{r}_j, \qquad J = \bigcap_{i \neq j} \mathfrak{s}_{ij}.$$

Set $V^\circ(\mathfrak{r}_i)$ equal to the set of primes of R containing $\mathfrak{r}_i$ but not J, i.e.,

$$V^\circ(\mathfrak{r}_i) = V(\mathfrak{r}_i) - V(J) = (\operatorname{Spec}(R) - V(J)) \bigcap V(\mathfrak{r}_i).$$

Since $\bigcap_i \mathfrak{r}_i$ is the intersection of all the minimal primes of R, we have

$$\operatorname{Spec}(R) - V(J) = V^\circ(\mathfrak{r}_1) \bigcup \cdots \bigcup V^\circ(\mathfrak{r}_t).$$

If $P \in V^\circ(\mathfrak{r}_i) \bigcap V^\circ(\mathfrak{r}_j)$ for some $i \neq j$, then $P \supseteq \mathfrak{s}_{ij} \supseteq J$, a contradiction to the definition of $V^\circ(\mathfrak{r}_i)$. Therefore, the union is disjoint.

Next, we note that $\operatorname{ht}_R(J) \geq 2$ since $\operatorname{ht}_R(J) = \min\{\operatorname{ht}_R(\mathfrak{s}_{ij}) \mid 1 \leq i < j \leq t\}$, and, if $\operatorname{ht}_R(\mathfrak{s}_{ij}) \leq 1$, then it follows that $\operatorname{ht}_R(\mathfrak{p}_{ik} + \mathfrak{p}_{jl}) \leq 1$ for some minimal primes of R in the disjoint components V_i and V_j, respectively, a contradiction. Each $V^\circ(\mathfrak{r}_i)$ is closed in $\operatorname{Spec}(R) - V(J)$ by definition of the topology of $\operatorname{Spec}(R) - V(J)$, which is induced by the

Zariski topology on $\mathrm{Spec}(R)$. In addition, $V^\circ(\mathfrak{r}_i)$ is non-empty since $\mathfrak{p}_{im} \in V^\circ(\mathfrak{r}_i)$ for $m = 1, \ldots, a_i$.

If $\mathrm{ht}_R(J) = 2$, then J has the desired properties, i.e., take $I = J$. Thus, assume that $\mathrm{ht}_R(\mathfrak{s}_{ij}) \geq 3$ for each pair $i \neq j$. Write

$$\sqrt{\mathfrak{s}_{12}} = \bigcap_{\ell=1}^{m} Q_\ell,$$

where each Q_ℓ is a prime such that $\mathrm{ht}_R(Q_\ell) \geq 3$. Let $\mathfrak{q}$ be a height two prime in Q_1 that contains a minimal prime, say $\mathfrak{p}_{1j}$. Then, $\mathfrak{r}_1 \subseteq \mathfrak{q}$. Consider the ideals

$$\mathfrak{t}_{12} = \mathfrak{q} \bigcap \left(\bigcap_{\ell=2}^{m} Q_\ell \right)$$

and

$$J^* = \mathfrak{t}_{12} \bigcap \left(\bigcap_{\{i,j\} \neq \{1,2\}} \mathfrak{s}_{ij} \right).$$

Using $V^*(\mathfrak{r}_i) = (\mathrm{Spec}(R) - V(J^*)) \bigcap V(\mathfrak{r}_i)$ as in the previous paragraph, we see that, by construction, the height of J^* is exactly two. In this case, J^* has the desired properties; take $I = J^*$. $\square$

This completes the proof of Theorem 1.5.

The result below is a corollary to Proposition 2.5 in the case where $\dim(R) = 2$. In the statement, we have the *Lyubeznik number*

$$\lambda_{0,1}(A) = \dim_{\mathsf{k}}(\mathrm{Hom}_Q(\mathsf{k}, \mathrm{H}_I^{n-1}(Q))),$$

where Q is an n-dimensional regular local ring and I is an ideal of Q such that $A \cong Q/I$. See **[13, 14]** for more details.

Corollary 2.6. *Under the assumptions of Remark* 2.1*, if* $\dim(R) = 2$*, then* $\beta(\mathrm{Spec}^\circ(R)) = \beta(\Gamma_R)$*, where* $\mathrm{Spec}^\circ(R) = \mathrm{Spec}(R) - \{\mathfrak{m}\}$ *is the punctured spectrum of R. Moreover, if R is the completion of the strict Hensilization of a complete equicharacteristic local ring A, then* $\beta(\Gamma_R) = \lambda_{0,1}(A) + 1$.

Proof. Since $\dim(R) = 2$, the condition $\mathrm{ht}_R(I) = 2$ is equivalent to $\sqrt{I} = \mathfrak{m}$. The second statement follows from **[13,** Proposition 3.1] and **[14]**. $\square$

3. Graph labeling and realizing graphs as Γ_R. We introduce labeling for graphs G which provides a method for constructing rings R such that Γ_R is graph-isomorphic to G. Intuitively, the labeling works as follows. Each vertex in the vertex set V of G is assigned a distinct address consisting of s distinct numbers, from a set of size n, such that two vertices are adjacent if and only if their addresses differ by exactly one number. Set $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{s}$ denote the set of subsets of $[n]$ with cardinality s. More precisely, we have the following:

Definition 3.1. An *admissible labeling* of G is an injective function $\phi\colon V \hookrightarrow \binom{[n]}{s}$, for some choice of n and s, satisfying the following conditions:

(1) $\phi(v_1) \bigcup \cdots \bigcup \phi(v_d) = [n]$, where $V = \{v_1, \ldots, v_d\}$; and
(2) for all vertices v and w, we have v adjacent to w in G if and only if $|\phi(v) \bigcap \phi(w)| = s-1$, that is, if and only if $|\phi(v) \bigcup \phi(w)| = s+1$.

Remark 3.2. The graphs that admit admissible labelings are precisely the graphs isomorphic to induced subgraphs of Johnson graphs studied in [**10**].

As the terminology suggests, we visualize admissible labelings by placing labels on the vertices of a graph, as in Figure 1.

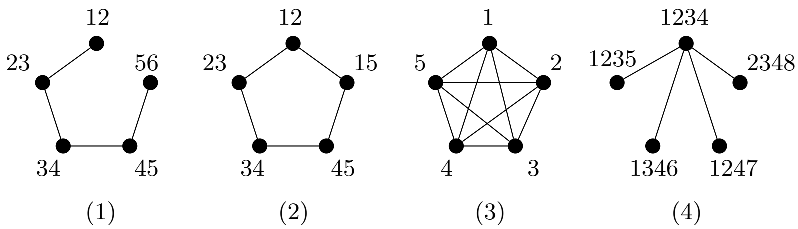


FIGURE 1.

Lemma 3.3.

(1) *If G has an admissible labeling $\phi\colon V \hookrightarrow \binom{[n]}{s}$, then each induced subgraph G' has an admissible labeling $\phi'\colon V' \hookrightarrow \binom{[n']}{s}$ for some $n' \leq n$.*

(2) *G has an admissible labeling $\phi\colon V \hookrightarrow \binom{[n]}{1}$ if and only if G is complete.*

Proof.

(1) Let V' be the vertex set for G', and reorder the elements of $[n]$ to assume that $\bigcup_{v\in V'}\phi(v)$ is of the form $[n']$ for some $n'\leq n$. Define $\phi'\colon V'\hookrightarrow\binom{[n']}{s}$ by the formula $\phi'(v):=\phi(v)$. Since two vertices in V' are adjacent in G' if and only if they are adjacent in G, it readily follows by definition that ϕ' is an admissible labeling of G'.

(2) The proof is straightforward. □

Proof of Theorem 1.6. Set $Q=\mathsf{k}[\![X_1,\dots,X_n]\!]$. For each subset

$$\mathcal{A}=\{i_1,\dots,i_s\}\quad\text{in }\binom{[n]}{s},$$

set $P_{\mathcal{A}}=(X_{i_1},\dots,X_{i_s})Q$.

(a) Let $\phi\colon V\hookrightarrow\binom{[n]}{s}$ be an admissible labeling of G. Define

$$I=\bigcap_{v\in V}P_{\phi(v)},$$

and set $R=Q/I$. For instance, denote graph (1) from Figure 1 as $G_{(1)}$; then, the vertices of $G_{(1)}$ are labeled 12, 23, 34, 45, 56. Thus, the ideal I in this case is

(3.1)

$$I=(X_1,X_2)Q\bigcap(X_2,X_3)Q\bigcap(X_3,X_4)Q\bigcap(X_4,X_5)Q\bigcap(X_5,X_6)Q.$$

In general, since I is defined as an intersection of square-free monomial ideals $P_{\mathcal{A}}$ of Q, it follows that I is also a square-free monomial ideal of Q. Since the prime ideals $P_{\mathcal{A}}$ in this intersection are all generated by the same number of variables, namely, s, there is no non-trivial containment between these primes; hence, the minimal primes of R are exactly the ideals of the form $P_{\phi(v)}R$ with $v\in V$. Moreover, this implies that R is equidimensional with $\dim(R)=n-s$.

In order to show that Γ_R is isomorphic to G, further note that the sum of two minimal primes

$$\mathfrak{p}=(X_{i_1},\dots,X_{i_s})R$$

and

$$\mathfrak{q}=(X_{j_1},\dots,X_{j_s})R$$

has height one if and only if the sets $\{i_1, \ldots, i_s\}$ and $\{j_1, \ldots, j_s\}$ differ by exactly one element. For instance, using the ideal (3.1), the sum $(X_1, X_2)R + (X_2, X_3)R = (X_1, X_2, X_3)R$ has height one in $R = Q/I$; thus, the vertices $(X_1, X_2)R$ and $(X_2, X_3)R$ of Γ_R are adjacent in Γ_R, just as the vertices of $G_{(1)}$ labeled 12 and 23 are adjacent in $G_{(1)}$. On the other hand, the sum

$$(X_1, X_2)R + (X_3, X_4)R = (X_1, X_2, X_3, X_4)R$$

has height two in R; thus, the vertices $(X_1, X_2)R$ and $(X_3, X_4)R$ are not adjacent in Γ_R, just as the vertices labeled 12 and 34 are not adjacent in $G_{(1)}$.

(b) Conversely, let I be a monomial ideal of Q such that the quotient $R = Q/I$ is equidimensional of dimension $n - s$. Then, the minimal primes of R are all of the form

$$(X_{i_1}, \ldots, X_{i_s})R = P_{\mathcal{A}}R,$$

where $\mathcal{A} = \{i_1, \ldots, i_s\} \in \binom{[n]}{s}$. As in the proof of (a), the sum of two such primes

$$\mathfrak{p} = (X_{i_1}, \ldots, X_{i_s})R$$

and

$$\mathfrak{q} = (X_{j_1}, \ldots, X_{j_s})R$$

has height one if and only if the subscript sets $\{i_1, \ldots, i_s\}$ and $\{j_1, \ldots, j_s\}$ differ by exactly one element. From the definition of an admissible labeling, it follows that the subscript sets define an admissible labeling of Γ_R. Specifically, the function $\phi\colon V \hookrightarrow \binom{[n]}{s}$ is defined as

$$\phi((X_{i_1}, \ldots, X_{i_s})R) = \{i_1, \ldots, i_s\}. \qquad \square$$

Example 3.4. The constructive proof of Theorem 1.6 shows how to obtain rings with Hochster-Huneke graphs isomorphic to those given in Figure 1. Moreover, given a positive integer n:

(1) for $n \geq 3$, for the ring

$$R = \frac{\mathsf{k}[\![X_1, \ldots, X_n]\!]}{(X_1, X_2) \bigcap (X_2, X_3) \bigcap \cdots \bigcap (X_{n-1}, X_n) \bigcap (X_n, X_1)},$$

the graph Γ_R is C_n, the cycle graph with n vertices, which has girth n and diameter $\lfloor n/2 \rfloor$.

(2) For $R = \mathsf{k}[\![X_1, \ldots, X_n]\!]/(X_1 \cdots X_n)$, we have $\Gamma_R = K_n$, the complete graph on n vertices.

The path P_n may be similarly realized, as well as the specific star graph Figure 1 (4); for more general star graphs, see Proposition 3.7.

The next two results give bounds on the numbers n, $d = |V|$ and s from Definition 3.1. Prior to the proofs, we recall that, for a connected graph G with vertex set V, a *spanning tree* of G is a tree T that is a subgraph of G with vertex set V, and that every connected graph has a spanning tree.

Proposition 3.5. *Let G be a graph with $d = |V|$ and an admissible labeling $\phi\colon V \hookrightarrow \binom{[n]}{s}$. Let $v_1, \ldots, v_m$ be vertices in G such that the subgraph of G induced by $v_1, \ldots, v_m$ is connected. Then, $|\phi(v_1) \bigcup \cdots \bigcup \phi(v_m)| \leq s + m - 1$. In particular, if G is connected, then $n \leq s + d - 1$.*

Proof. Induct on m. If $m = 1$, then $|\phi(v_1)| = s = s + 1 - 1$, and the base case is established. Now, assume the claim is true for lists of m vertices, and consider vertices $v_1, \ldots, v_m, v_{m+1}$ such that the induced subgraph G' of G is connected. Let T be a spanning tree of G'. Since T is a tree, we may reorder the vertices (if necessary) in order to assume that the subgraph of T induced by $v_1, \ldots, v_m$ is also connected, and v_{m+1} is adjacent to v_1 in T. (For instance, let v_{m+1} be a leaf adjacent to v_1.)

The inclusion-exclusion principle implies that

$$\left|\phi(v_1) \bigcup \cdots \bigcup \phi(v_{m+1})\right| = \left|\phi(v_1) \bigcup \cdots \bigcup \phi(v_m)\right| + |\phi(v_{m+1})| - \left|\left(\phi(v_1) \bigcup \cdots \bigcup \phi(v_m)\right) \bigcap \phi(v_{m+1})\right|.$$

By the induction hypothesis, the first term on the right-hand side of this equation is less than or equal to $s + m - 1$. The second term on the right may be rewritten as

$$\left|\left(\phi(v_1) \bigcap \phi(v_{m+1})\right) \bigcup \cdots \bigcup \left(\phi(v_m) \bigcap \phi(v_{m+1})\right)\right|,$$

which is greater than or equal to $s - 1$ since v_{m+1} is adjacent to v_1, and hence, $|\phi(v_1) \bigcap \phi(v_{m+1})| = s - 1$. Therefore, as desired, we have

$$\left|\phi(v_1) \bigcup \cdots \bigcup \phi(v_{m+1})\right| \leq (s+m-1) + s - (s-1) = s+(m+1)-1.$$

The last statement is from the equality $\phi(v_1) \bigcup \cdots \bigcup \phi(v_d) = [n]$. $\square$

Proposition 3.6. *Let G be a graph with $d = |V|$ and an admissible labeling $\phi\colon V \hookrightarrow \binom{[n]}{s}$. Let $v_1, \ldots, v_m$ be vertices in G such that the subgraph of G induced by $v_1, \ldots, v_m$ is connected. Then,*

$$\left|\phi(v_1)\bigcap\cdots\bigcap\phi(v_m)\right| \geq s - m + 1.$$

In particular, if G is connected, then

$$\left|\phi(v_1)\bigcap\cdots\bigcap\phi(v_d)\right| \geq s - d + 1.$$

Proof. Induct on m. If $m = 1$, then $|\phi(v_1)| = s = s - 1 + 1$, and the base case is established. Now assume the claim is true for lists of m vertices, and consider vertices $v_1, \ldots, v_m, v_{m+1}$ such that the induced subgraph G' of G is connected. Let T be a spanning tree of G', and, as in the previous proof, assume that the subgraph of T induced by $v_1, \ldots, v_m$ is also connected and v_{m+1} is adjacent to v_1 in T. The inclusion-exclusion principle yields:

$$\begin{aligned}(3.2)\quad &\left|\left(\phi(v_1)\bigcap\cdots\bigcap\phi(v_m)\right)\bigcup\phi(v_{m+1})\right| \\ &\qquad = \left|\phi(v_1)\bigcap\cdots\bigcap\phi(v_m)\right| + |\phi(v_{m+1})| \\ &\qquad\qquad - \left|\left(\phi(v_1)\bigcap\cdots\bigcap\phi(v_m)\right)\bigcap\phi(v_{m+1})\right|.\end{aligned}$$

By the induction hypothesis, the first term on the right-hand side of this equation is greater than or equal to $s - m + 1$. The left-hand side may be rewritten as:

$$\begin{aligned}&\left|\left(\phi(v_1)\bigcup\phi(v_{m+1})\right)\bigcap\cdots\bigcap\left(\phi(v_m)\bigcup\phi(v_{m+1})\right)\right| \\ &\qquad \leq \left|\phi(v_1)\bigcup\phi(v_{m+1})\right| = s + 1.\end{aligned}$$

Therefore, equation (3.2) implies that

$$\begin{aligned}s + 1 &\geq \left|\left(\phi(v_1)\bigcap\cdots\bigcap\phi(v_m)\right)\bigcup\phi(v_{m+1})\right| \\ &\geq (s - m + 1) + s - \left|\phi(v_1)\bigcap\cdots\bigcap\phi(v_m)\bigcap\phi(v_{m+1})\right|,\end{aligned}$$

and thus, $|\phi(v_1)\bigcap\cdots\bigcap\phi(v_m)\bigcap\phi(v_{m+1})| \geq s - (m + 1) + 1$. □

Our next result shows that the bounds from Propositions 3.5 and 3.6 are sharp; the star graph in Figure 1 provides a specific example.

Proposition 3.7. *Let G be the star graph on $d \geq 2$ vertices, i.e., the complete bipartite graph $K_{1,d-1}$. Then, G has an admissible labeling*

$$\phi\colon V \longrightarrow \binom{[2(d-1)]}{d-1}$$

such that $\phi(v_1)\bigcap\cdots\bigcap\phi(v_d)=\emptyset$. Furthermore, any admissible labeling $\psi\colon V \hookrightarrow \binom{[n]}{s}$ of G has $s \geq d-1$, $n = s+d-1 \geq 2(d-1)$ and $|\phi(v_1)\bigcap\cdots\bigcap\phi(v_d)| = s-d+1$.

Proof. By definition, G has a vertex v_d with degree $d-1$, and all other vertices $v_1,\ldots,v_{d-1}$ have degree one. (Note that v_d is uniquely determined unless $d=2$.) Define

$$\phi\colon V \longrightarrow \binom{[2(d-1)]}{d-1}$$

as follows:

$$\phi(v_d) = \{1,\ldots,d-1\}$$

and

$$\phi(v_i) = \{1,\ldots,d-1\} - \{i\}\bigcup\{d-1+i\}$$

for $i=1,\ldots,d-1$. It is straightforward to verify that, for $i<j<d$, we have

$$\phi(v_i)\bigcap\phi(v_d) = \{1,\ldots,d-1\} - \{i\},$$

and similarly,

$$\phi(v_i)\bigcap\phi(v_j) = \{1,\ldots,d-1\} - \{i,j\}.$$

Moreover,

$$\phi(v_1)\bigcup\cdots\bigcup\phi(v_d) = \{1,\ldots,2(d-1)\} = [2(d-1)];$$

thus, ϕ is an admissible labeling of G. From the explicit description of ϕ, it may be verified that $\phi(v_1)\bigcap\cdots\bigcap\phi(v_d)=\emptyset$.

Now, suppose that $\psi\colon V \hookrightarrow \binom{[n]}{s}$ is an admissible labeling. We claim that the elements of $[n]$ can be reordered such that

$$\psi(v_i) = (\psi(v_d) - \{i\})\bigcup\{s+i\}$$

for $i=1,\ldots,d-1$. In order to prove this, begin by reordering the elements of $[n]$ to assume that $\psi(v_d)=\{1,\ldots,s\}$. Consider the edge

$v_1 v_d$. Since $|\psi(v_1) \bigcap \psi(v_d)| = s - 1$, it follows that

$$\psi(v_1) = (\psi(v_d) - \{a\}) \bigcup \{b\}$$

for some $a \in [s]$ and some $b \in [n] - [s]$. Thus, the elements of $[n]$ can be reordered to assume that

$$\psi(v_1) = (\psi(v_d) - \{1\}) \bigcup \{s + 1\} = \{2, \ldots, s, s + 1\}.$$

Next, consider the edge $v_2 v_d$. As with the previous edge, $\psi(v_2) = (\psi(v_d) - \{p\}) \bigcup \{q\}$ for some $p \in [s]$ and some $q \in [n] - [s]$. If $p = 1$, then $2, \ldots, s \in \psi(v_1) \bigcap \psi(v_2)$; however, v_1 is not adjacent to v_2. Therefore, $|\psi(v_1) \bigcap \psi(v_2)| \leq s - 2$, a contradiction. It follows that $2 \leq p \leq s$; hence, the set $\{2, \ldots, s\}$ can be reordered to assume that $p = 2$. Similarly, $q > s + 1$; thus, the set $\{s + 2, \ldots, n\}$ can be reordered to assume that $q = s + 2$. Continue in this manner for the edges $v_i v_d$ with $i = 3, \ldots, d - 1$ to complete the proof of the claim.

Consequently, we must have $1, \ldots, d - 1 \in [s]$. It follows that $s \geq d - 1$, establishing the first conclusion of our result. For the second conclusion, note that the sets $\psi(v_d), \psi(v_1), \psi(v_2), \ldots,$ and $\psi(v_{d-1})$ are $\{1, \ldots, s\}$, $\{2, \ldots, s, s + 1\}$, $\{1, 3, \ldots, s, s + 2\}$, $\ldots$, and $\{1, 2, \ldots, d - 2, d, \ldots, s, s + d - 1\}$, respectively. From this description, the largest integer which occurs in any set $\psi(v_p)$ is $s + d - 1$. Since $\bigcup_p \psi(v_p) = [n]$, it follows that the largest number n in this set is $s + d - 1$. For the final conclusion, use the preceding description to observe that

$$\phi(v_1) \bigcap \cdots \bigcap \phi(v_d) = \{d, \ldots, s\},$$

which has cardinality $s - d + 1$, as desired. $\square$

We conclude with some connections between our results and those from **[1]**. Some classes of graphs that have admissible labelings are cycles and complete graphs (by Example 3.4), totally disconnected graphs (straightforward), and trees (by induction on the number of vertices). On the other hand, not every graph has an admissible labeling as is shown in Figure 2. Before this, we provide an explicit characterization of the graphs that have admissible labelings.

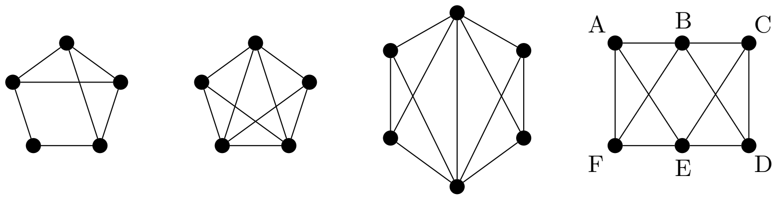
Definition 3.8. Let Δ be a pure simplicial complex. The *dual graph* of Δ has vertices equal to the facets $F_1, \ldots, F_d$ of Δ such that distinct F_i and F_j are adjacent in the dual graph, provided that their intersection contains a face of one smaller dimension.

Theorem 3.9. *A graph G has an admissible labeling if and only if it is isomorphic to the dual graph of a pure simplicial complex.*

Proof. For the forward implication, assume that G has an admissible labeling. Theorem 1.6 (a) implies that there is an equidimensional ideal I of $S = \mathsf{k}[\![X_1, \ldots, X_n]\!]$ generated by square-free monomials such that the quotient $R = S/I$ has a Hochster-Huneke graph isomorphic to G. The standard correspondence between square-free monomial ideals and simplicial complexes provides a pure simplicial complex Δ such that I is the Stanley-Reisner ideal, of Δ in S. From [**2,** Lemma 2.7], we conclude that the dual graph of Δ is isomorphic to $\Gamma_R \cong G$.

Conversely, assume that G is isomorphic to the dual graph of a pure simplicial complex Δ on a vertex set with n elements. Set $S = \mathsf{k}[\![X_1, \ldots, X_n]\!]$, let I be the Stanley-Reisner ideal of Δ in S and set $R = S/I$. Again, by [**2,** Lemma 2.7], we conclude that the dual graph G of Δ is isomorphic to Γ_R; therefore, Theorem 1.6 (b) implies that G has an admissible labeling. $\square$

The graphs [**1,** Proposition 3.5] in Figure 2 do not have admissible labelings.



G_1 G_2 G_3 G_4

FIGURE 2.

We show that G_4 does not admit a labeling since a similar argument works for the other graphs. From Proposition 3.6, assume that the size s of the labelings is at most five. Suppose that there is an admissible labeling ψ with $s = 5$ since any redundancy can eventually be deleted.

Let $\psi(A) = \{1, 2, 3, 4, 5\}$ and $\psi(F) = \{1, 2, 3, 4, 6\}$. There are essentially two choices of label for B:

$$\{1, 2, 3, 4, 7\} \quad \text{or} \quad \{a, b, c, 5, 6\},$$

where $a, b, c \in \{1, 2, 3, 4\}$. Moreover, the same two types of choices also exist for E. Thus, there are two cases to consider, namely, whether B and E take the same type of label or different types. However, in the former case, the labels for B and E will differ by only a single digit, a contradiction to the fact that the vertices are not adjacent. Thus, B and E must have labels of different types. Without loss of generality, assume that $\psi(B) = \{1, 2, 3, 4, 7\}$ and that $\psi(E)$ contains the digits 5 and 6. By symmetry of the choices regarding subsets of $\{1, 2, 3, 4\}$, we may assume that $\psi(E) = \{1, 2, 3, 5, 6\}$. We obtain a contradiction via the vertex C.

The label $\psi(C)$ must include the digits 6 and 7 for, if the label for C includes any 4-tuple from the set $\{1, 2, 3, 4, 5\}$, then C would be adjacent to A. However, the remaining three digits of the label must come from the set $\{1, 2, 3\}$ since $\psi(C)$ must share exactly four digits with both $\psi(B)$ and $\psi(E)$. Note that each of the four possibilities of $\{1, 2, 3, 6, 7\}$, $\{1, 2, 4, 6, 7\}$, $\{1, 3, 4, 6, 7\}$ and $\{2, 3, 4, 6, 7\}$ for $\psi(C)$ forces an edge between F and C, a contradiction.

Remark 3.10. The result [**1,** Corollary 3.6] shows that every graph is the Hochster-Huneke graph of a complete equidimensional local ring. This is proven by using the *dual graph* of a complex projective curve C, which has vertices equal to the irreducible components $C_1, \dots, C_d$ of C such that distinct C_i and C_j are adjacent in the dual graph, provided that their intersection is non-trivial. Using projective line configurations and blowing-up, the authors showed that, for every connected graph G, there is a projective curve C with dual graph isomorphic to G; furthermore, localizing the homogeneous coordinate ring of C at the irrelevant maximal ideal and then completing yields a local ring R such that $\Gamma_R \cong G$.

Given that every graph can be realized from this process, it is natural to ask which complex projective curves yield graphs with admissible labelings. We answer this question in our final result below.

Definition 3.11. Let C be a complex projective curve with irreducible components $C_1, \dots, C_d$. An *admissible labeling* of C is an injective function

$$\phi\colon \{C_1, \dots, C_d\} \hookrightarrow \binom{[n]}{s},$$

for some choice of n and s, satisfying the following conditions:

(1) $\phi(C_1) \bigcup \cdots \bigcup \phi(C_d) = [n]$; and

(2) two distinct components C_i and C_j intersect if and only if

$$|\phi(C_i) \bigcap \phi(C_j)| = s - 1,$$

that is, if and only if

$$|\phi(C_i) \bigcup \phi(C_j)| = s + 1.$$

(Compare this with Definition 3.1.)

Proposition 3.12. *Let C be a complex projective curve with dual graph G. Then, C has an admissible labeling if and only if G has an admissible labeling.*

Proof. The proof follows directly from the relevant definitions that an admissible labeling of C yields an admissible labeling of G, and vice versa. $\square$

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REFERENCES

1. B. Benedetti, B. Bolognese and M. Varbaro, *Regulating Hartshorn's connectedness theorem*, J. Alg. Comb. **46** (2017), 33–50.

2. B. Benedetti and M. Varbaro, *On the dual graph of Cohen-Macaulay algebras*, Int. Math. Res. Not. **2015** (2015), 8085–8115.

3. M. Eghbali and P. Schenzel, *On an endomorphism ring of local cohomology*, Comm. Algebra **40** (2012), 4295–4305.

4. G. Faltings, *Some theorems about formal functions*, Publ. Res. Inst. Math. Sci. **16** (1980), 721–737.

5. F. Harary, *Graph theory*, Addison-Wesley Publishing Co., Reading, MA, 1969.

6. R. Hartshorne, *Complete intersections and connectedness*, Amer. J. Math. **84** (1962), 497–508.

7. M. Hochster and C. Huneke, *Indecomposable canonical modules and connectedness*, Contemp. Math. **159** (1992), 197–208.

8. B. Holmes, *On simplicial complexes with Serre property S_2*, in preparation.

9. G. Lyubeznik, *On some local cohomology invariants of local rings*, Math. Z. **254** (2006), 627–640.

10. R. Naimi and J. Shaw, *Induced subgraphs of Johnson graphs*, Involve **5** (2012), 25–37.

11. S. Sather-Wagstaff and S. Spiroff, *Torsion in kernels of induced maps on divisor class groups*, J. Alg. Appl. **14** (2015), 1500123, 23 pages.

12. P. Schenzel, *On connectedness and indecomposibility of local cohomology modules*, Manuscr. Math. **128** (2009), 315–327.

13. U. Walther, *On the Lyubeznik numbers of a local ring*, Proc. Amer. Math. Soc. **129** (2001), 1631–1634.

14. W. Zhang, *On the highest Lyubeznik number of a local ring*, Compos. Math. **143** (2007), 82–88.

Clemson University, Department of Mathematical Sciences, O-110 Martin Hall, Box 340975, Clemson, SC 29634
Email address: **ssather@clemson.edu**

University of Mississippi, Department of Mathematics, 305 Hume Hall, P.O.Box 1848, University, MS 38677
Email address: **spiroff@olemiss.edu**