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Ladder determinantal rings over normal domains

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ABSTRACT

We explicitly describe the divisor class groups and semidualizing modules for ladder determinantal rings with coefficients in an arbitrary normal domain for arbitrary ladders, not necessarily connected, and all sizes of minors.

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1. Introduction

Throughout this paper, let k be a field, and let $t \geq 2$ be an integer. All rings are commutative with 1. In this section, let A be a normal domain, i.e., a noetherian, integrally closed domain.

This paper investigates divisor class groups and semidualizing modules for ladder determinantal rings over A . Ladder determinantal rings generalize the more classical determinantal rings that are central to the study of Grassmannian and Schubert varieties [3], and they are useful for investigating Young tableaux [1]. These rings feature in a number of publications, e.g., [4, 5, 12, 13]. We recall that a subset Y of an $m \times n$ matrix $X = (X_{ij})$ of indeterminates is called a **ladder** if whenever the main diagonal of a minor of X is in Y then the minor is in Y . To avoid trivialities, we assume that Y contains X_{1n} and X_{m1} , i.e., that X is the smallest matrix containing Y , and that every row and column of Y is non-empty. Let $I_t(Y)$ be the ideal of the polynomial ring $A[Y]$ generated by the $t \times t$ minors lying entirely in Y . Then $A_t(Y) := A[Y]/I_t(Y)$ is a **ladder determinantal ring** of t -minors. (Note that this notation differs slightly from that in [4, 5, 12, 13]. We require the extra flexibility afforded by this notation since we analyze the ladder construction over different coefficient rings.)

In the main result of this paper, **Theorem 4.16**, we determine how many non-isomorphic semidualizing modules the ring $A_t(Y)$ has, where A is not necessarily a field, and regardless of the size of the t -minors or any connectedness conditions on Y . Recall that for a commutative noetherian ring R , a finitely generated R -module C is **semidualizing** if $\text{Hom}_R(C, C) \cong R$ and $\text{Ext}_R^i(C, C) = 0$ for all $i \geq 1$. The set of isomorphism classes of semidualizing R -modules is denoted $\mathfrak{S}_0(R)$. One reason to want to understand semidualizing modules is that they provide

nice dualities. For instance, the free R -module of rank 1 is semidualizing, giving rise to the classical duality $(-)^* := \text{Hom}_R(-, R)$ which is crucial, e.g., for Auslander and Bridger's G-dimension [2]. For another example, if R is Cohen-Macaulay and either complete local or standard graded, then the canonical module ω of R is semidualizing, yielding Grothendieck's [9] local duality $(-)^{\dagger} := \text{Hom}_R(-, \omega)$.

Main Result [See [Theorem 4.16](#)]. Let A be a normal domain with field of fractions K , and Y a ladder of variables, not necessarily path-connected or t -connected. Let D be a subring of A that is a principal ideal domain, and let L denote the field of fractions of D . The natural maps below are bijections, for some $f \in A_t(Y)$:

$$\mathfrak{S}_0(A) \times \mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(A_t(Y)) \rightarrow \mathfrak{S}_0(A_t(Y)_f) \times \mathfrak{S}_0(K_t(Y)).$$

In particular, (using notation from [Section 3](#)),

$$\begin{aligned} \mathfrak{S}_0(A_t(Y)) &\approx \mathfrak{S}_0(A_t(Y)_f) \times \mathfrak{S}_0(K_t(Y)) \approx \mathfrak{S}_0(A) \times \mathfrak{S}_0(K_t(Y)) \\ &\approx \mathfrak{S}_0(A) \times \prod_{i=1}^s \prod_{j=0}^{k_i^*} \mathfrak{S}_0(\mathfrak{k}_t(Y_{i,j})) \approx \mathfrak{S}_0(A) \times \{0, 1\}^e. \end{aligned}$$

The key tool for our proof of this result is the divisor class group $\text{Cl}(A_t(Y))$.

Main Tool [See [Theorem 3.7](#)]. Under the same assumptions as above, the natural maps below are bijections

$$\text{Cl}(A) \times \text{Cl}(D_t(Y)) \rightarrow \text{Cl}(A_t(Y)) \rightarrow \text{Cl}(A_t(Y)_f) \times \text{Cl}(K_t(Y)).$$

In particular, (using the fact that $\text{Cl}(D_t(Y)) \cong \text{Cl}(K_t(Y))$),

$$\text{Cl}(A_t(Y)) \cong \text{Cl}(A_t(Y)_f) \times \text{Cl}(K_t(Y)) \cong \text{Cl}(A) \times \text{Cl}(K_t(Y)).$$

The contents of the paper is organized as follows. Preliminaries, [Section 2](#), contains technical results about ladders and ladder determinantal rings, including two isomorphisms for localizations for use in the other sections. In [Section 4](#) the main result is established. [Section 3](#) analyzes divisor class groups of ladder determinantal rings over normal domains, the key tool for the proof of [Theorem 4.16](#); see [Corollary 3.5](#) and [Theorem 3.7](#) for the main conclusions about divisor class groups.

We conclude this Introduction with a few facts for use throughout the paper.

Remark 1.1. If R is an integral domain, then R either contains a field or an isomorphic copy of \mathbb{Z} . In particular, R contains a subring D that is a principal ideal domain. (In this paper, we consider fields to be principal ideal domains.) With this set-up, every torsion-free R -module is torsion-free over D , hence flat over D ; in particular, R is flat over D , as is every reflexive R -module.

In this paper, we use the description of the **divisor class group** of our normal domain A as the set of isomorphism classes of finitely generated rank-1 reflexive A -modules with operations $[a] + [b] = [(a \otimes_A b)^{**}]$ and $[a] - [b] = [\text{Hom}_A(b, a)]$, and with additive identity $[A]$.

Fact 1.2 ([10, Proposition 3.4]). If N is a semidualizing A -module, then N is reflexive of rank 1. In particular, we have $\mathfrak{S}_0(A) \subseteq \text{Cl}(A)$.

Example 1.3. Let Y be a ladder of variables, and let D be a principal ideal domain contained in A as a subring; see [Remark 1.1](#). (This construction holds for any integral domain, but our application of the example will be when the ring is normal.) Then the natural inclusions make the following diagram commute and we have $A \otimes_D D_t(Y) \cong A_t(Y)$.

$$\begin{array}{ccc}
 D & \longrightarrow & A \\
 \downarrow & & \downarrow \\
 D_t(Y) & \longrightarrow & A_t(Y)
 \end{array}$$

Thus, [11, Proposition 3.5] yields a well-defined, relation-respecting map $\mathfrak{S}_0(A) \times \mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(A_t(Y))$ given by $([C_1], [C_2]) \mapsto [C_1 \otimes_D C_2]$. The relation \leq on $\mathfrak{S}_0(A)$ is defined via total reflexivity, as described in [8] and [11, Definition 1.5], and similarly for $\mathfrak{S}_0(D_t(Y))$ and $\mathfrak{S}_0(A_t(Y))$, while $\mathfrak{S}_0(A) \times \mathfrak{S}_0(D_t(Y))$ uses the product relation. Thus, if $[M_1] \in \mathfrak{S}_0(A)$ and $[M_2] \in \mathfrak{S}_0(D_t(Y))$ such that $[C_1] \leq [M_1]$ and $[C_2] \leq [M_2]$, then $[C_1 \otimes_D C_2] \leq [M_1 \otimes_D M_2]$. Note that \leq is reflexive; it is antisymmetric if and only if the Picard group of A is trivial, and it is suspected to be transitive.

2. Preliminaries: ladders and ladder determinantal rings

This section concerns general properties of ladder determinantal rings, and in particular, details some isomorphisms between such rings. These results are applied in the main body of the paper when considering the rings' divisor class groups and semidualizing modules.

We will begin with a more detailed description of ladders. Consider an $m \times n$ matrix $X = (X_{ij})$ of indeterminates. Let $M = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Then we can consider X as a function $X : M \rightarrow \{X_{ij} \mid (i, j) \in M\}$, so that $X(i, j) = X_{ij}$. A **ladder** is a restriction Y of X to a subset $L \subseteq M$, such that if $(i, j), (h, k) \in L$ and $i \leq h, j \leq k$, then $(i, k), (h, j) \in L$, i.e., whenever the main diagonal of a minor of X is in Y then the minor is in Y . To avoid trivialities, we assume that for all i_0 with $1 \leq i_0 \leq m$ there exists a j with $1 \leq j \leq n$ such that $(i_0, j) \in L$, and that for all j_0 with $1 \leq j_0 \leq n$ there exists an i with $1 \leq i \leq m$ such that $(i, j_0) \in L$, i.e., that every row and column of Y (or L) is non-empty. This implies that X is the smallest matrix that contains Y , or M is the smallest rectangular grid in \mathbb{N}^2 that contains L .

Throughout this section, A will be a commutative ring with identity and Y is a ladder, not necessarily path-connected or t -connected. See [12, pp. 169–170] and [5] for more details regarding the definitions of, and notations for, properties about ladders.

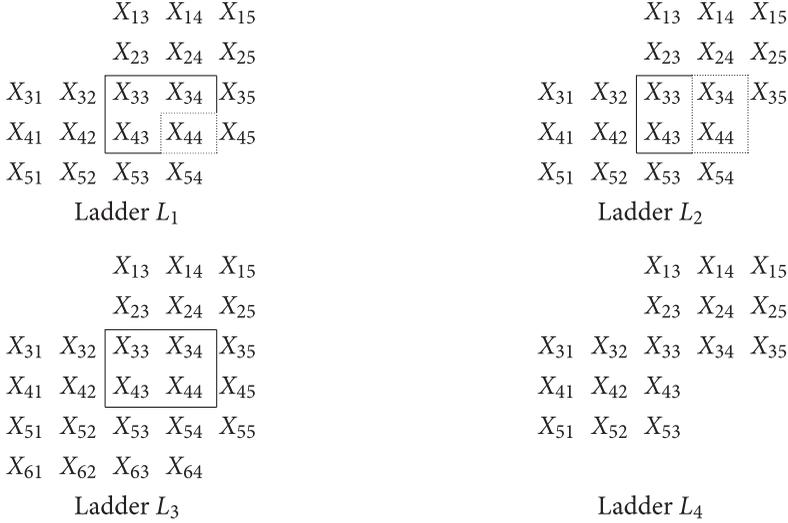
Definition 2.1. A ladder Y is **t -disconnected** [5, page 457] if there exist two subladders $\emptyset \neq Z_1, Z_2 \subseteq Y$ such that $Z_1 \cap Z_2 = \emptyset, Z_1 \cup Z_2 = Y$, and every t -minor of Y is contained in Z_1 or Z_2 . In this case, we say that Z_1, Z_2 form a **t -disconnection** of Y . A ladder Y is **t -connected** if it is not t -disconnected. A ladder Y is **path-connected** if there is a path between any two variables in Y , or equivalently, Y has only one **path-component**. A ladder is **path-disconnected** if it is not path-connected.

The indeterminates of the $m \times n$ matrix X , and of $Y \subset X$, will be identified with points of the set $\{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq m, 1 \leq j \leq n\}$.

Definition 2.2 ([5, p. 467]). A lower inside corner $S'_i = (a_i, b_i)$ of Y is said to be of **type 1** if the $(t-1)$ -minor based on (a_i, b_i) is contained in the ladder Y and contains at most one point of the upper border C of thickness 1; if so, then that point is $(a_i + t - 2, b_i + t - 2)$. If S'_i is not of type 1, then it is of **type 2**. The definitions of the upper inside corner $T'_j = (c_j, d_j)$ types are analogous. Let h^* be the number of S'_i of type 1 and k^* the number of T'_j of type 1. The ladder Y satisfies **Assumption (d)** if for each $S'_i = (a_i, b_i)$ the $(t-1)$ -minor based on (a_i, b_i) is contained in the ladder Y and contains no point of the upper border C .

Example 2.3 (Running Example). For a field k , consider $k_3(-)$ for each of the ladders below, and in particular, the 2-minor based at each inside corner $S'_1 = (3, 3)$. In the ladders in the left

column, (3,3) is a type 1 corner: the 2-minor based on X_{33} is contained in L_i and contains either one point of the border C (X_{44} in L_1 , indicated with a dotted box), or no points of C (L_3); for these examples, $h^* = 1 = k^*$. The corner (3,3) in each of the ladders in the right column is of type 2. In L_2 , the 2-minor based on X_{33} is contained in the ladder, but contains more than one point of C (X_{34} and X_{44}), while in L_4 , the 2-minor based on X_{33} is not contained in the ladder; for these examples, $h^* = 0 = k^*$.



The next result allows one to work with ladders that are not t -connected. Essentially, the sub-ladders Y_i are t -connected components of Y .

Lemma 2.4. *There are sub-ladders Y_1, \dots, Y_s of Y such that*

- (1) *each ladder Y_i is non-empty and t -connected;*
- (2) *Y is the disjoint union $Y_1 \cup \dots \cup Y_s$; and*
- (3) *every t -minor of Y is contained in one of the Y_i .*

Proof. Induct on $|Y|$. If Y is t -connected, then $s = 1$. Thus, assume that Y is t -disconnected. This implies that there are non-empty sub-ladders Y' and Y'' of Y such that Y is the disjoint union $Y' \cup Y''$ and every t -minor of Y is contained in Y' or Y'' . Now apply the induction hypothesis to Y' and Y'' . □

Remark 2.5. The subladders Y_i in Lemma 2.4 are unique, as one can show, so they will be referred to as the **t -components** of the ladder Y .

Because of the next result, Fact 1.2 may be used to find the semidualizing modules for $A_t(Y)$ when A is a normal domain.

Lemma 2.6. *If A is a normal domain, then so is $A_t(Y)$.*

Proof. Let Y_1, \dots, Y_s be the t -components of Y . Induct on s . When $s = 1$, Y is t -connected. By [4, Proposition 3.3], $R = k_t(Y)$ is a normal domain. Apply [3, Proposition 3.12]: since $S = \mathbb{Z}_t(Y)$ is faithfully flat over \mathbb{Z} and $R \cong S \otimes_{\mathbb{Z}} k$ is a normal domain, $A_t(Y)$ is a normal domain since A is a normal domain.

For the inductive step, note that if $Y' = Y \setminus Y_1$, then $A_t(Y) \cong A_t(Y_1) \otimes_A A_t(Y') \cong [A_t(Y_1)]_t(Y')$. By the base case, the ring $A_t(Y_1)$ is a normal domain, hence it follows, by induction, that the same is true for $A_t(Y) \cong [A_t(Y_1)]_t(Y')$. \square

The result below is a basic tool for the proof of [Lemma 2.9](#). It can be stated in much more generality than stated here, but this is the version that is needed.

Lemma 2.7. *Let $\phi : L \rightarrow M$ be a homomorphism between torsion-free abelian groups, i.e., between flat \mathbb{Z} -modules. The following conditions are equivalent:*

- (i) ϕ is an isomorphism;
- (ii) for each ring A , the map $\phi \otimes_{\mathbb{Z}} A$ is an isomorphism;
- (iii) for each field k , the map $\phi \otimes_{\mathbb{Z}} k$ is an isomorphism; and
- (iv) the map $\phi \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism and, for each prime number p , the map $\phi \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})$ is an isomorphism.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are routine.

(iv) \Rightarrow (i) Assume that the map $\phi \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism and, for each prime number p , the map $\phi \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})$ is an isomorphism. In other words, for each prime ideal $\mathfrak{p} \subseteq \mathbb{Z}$, the map $\phi \otimes_{\mathbb{Z}} \kappa(\mathfrak{p})$ is an isomorphism where $\kappa(\mathfrak{p})$ is the field of fractions of \mathbb{Z}/\mathfrak{p} . Consider the bounded chain complex of flat \mathbb{Z} -modules

$$X = \text{Cone}(\phi) = (0 \rightarrow L \xrightarrow{\phi} M \rightarrow 0)$$

which satisfies $X \otimes_{\mathbb{Z}} \kappa(\mathfrak{p}) \cong \text{Cone}(\phi \otimes_{\mathbb{Z}} \kappa(\mathfrak{p}))$. The fact that $\phi \otimes_{\mathbb{Z}} \kappa(\mathfrak{p})$ is an isomorphism for all \mathfrak{p} implies that $X \otimes_{\mathbb{Z}} \kappa(\mathfrak{p})$ is exact for all \mathfrak{p} . Since X is a bounded complex of flat \mathbb{Z} -modules, it follows from [7, Lemma 2.6] that X is exact, that is, that ϕ is an isomorphism. \square

The following notation, included for convenience, was introduced in [5, p. 462].

Notation 2.8. For $i = 1, \dots, h+1$ let $F_i \in A[Y]$ denote the $(t-1)$ -minor based on the lower outside corner S_i of Y , and let $f_i \in A_t(Y)$ be the residue of F_i . Set $F = F_1 \cdots F_{h+1}$ and $f = f_1 \cdots f_{h+1}$. For $j = 1, \dots, k+1$ let $G_j \in A[Y]$ denote the $(t-1)$ -minor based on the upper outside corner T_j of Y , and let $g_j \in A_t(Y)$ be the residue of G_j . Set $G = G_1 \cdots G_{k+1}$ and $g = g_1 \cdots g_{k+1}$.

Much of our work is based on careful localization. We describe our tools for this next, beginning with a version of [5, Lemma 4.1] with fewer restrictions on A . Recall that the indeterminates X_{ij} of Y are identified with the points (i, j) . Following convention, we will use the two partial orders below:

$$(i, j) \leq (h, k) \iff i \leq h \text{ and } j \leq k, \quad (i, j) \leqslant (h, k) \iff i \geq h \text{ and } j \leq k.$$

Lemma 2.9. *Assume that Y is t -connected. Let B be the set of points of the lower border with thickness $(t-1)$ of Y , and let C be the set of points of the upper border with thickness $(t-1)$ of Y . Set $Y_1 = \{P \in Y \mid P \leqslant (a_1, b_1 + t - 2)\}$ and $Y_2 = \{P \in Y \mid P \leqslant (c_1 - t + 2, d_1)\}$, and set $B_1 = B \setminus Y_1$ and $C_1 = C \setminus Y_2$. Then one has isomorphisms of localizations*

$$A_t(Y)_{f_1} \cong A_t(Y_1)[B_1]_{f_{11}} \quad A_t(Y)_{g_1} \cong A_t(Y_2)[C_1]_{g_{11}} \quad A_t(Y)_f \cong A[B]_F$$

where f_{11} is the residue of F_1 in $A_t(Y_1)[B_1]$ and g_{11} is the residue of G_1 in $A_t(Y_2)[C_1]$.

Proof. The first isomorphism will be verified; the others are verified similarly.

The case where A is a field is covered in [5, Lemma 4.1]. Moreover, the proof of [5, Lemma 4.1] provides a natural map

$$A_t(Y_1)[B_1]_{f_{11}} \xrightarrow{\psi_A} A_t(Y)_{f_i}$$

such that ψ_k is an isomorphism for each field k . Consider the map

$$\mathbb{Z}_t(Y_1)[B_1]_{f_{11}} \xrightarrow{\psi_{\mathbb{Z}}} \mathbb{Z}_t(Y)_{f_i}$$

between flat \mathbb{Z} -modules. It is straightforward to show that $\psi_k = \psi_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$; since this map is an isomorphism for each k , [Lemma 2.7](#) implies that $\psi_A = \psi_{\mathbb{Z}} \otimes_{\mathbb{Z}} A$ is an isomorphism for all A . \square

Next, the t -connected hypothesis is removed for the isomorphism used below.

Lemma 2.10. *Let B be the set of points of the lower border with thickness $(t-1)$ of Y . Then one has $A_t(Y)_f \cong A[B]_F$.*

Proof. Let Y_1, \dots, Y_s be the t -components of Y . Induct on s , where the base case $s=1$ follows from [Lemma 2.9](#). Re-order the Y_i if necessary to assume without loss of generality that $X_{1,n} \in Y_1$, and set $\tilde{Y} = Y \setminus Y_1$. Set

$$f_{(1)} = \prod_{X_{s_j} \in Y_1} f_j \quad \text{and} \quad \tilde{f} = f/f_{(1)} = \prod_{X_{s_j} \in Y'} f_j$$

Also, set $B_1 = B \cap Y_1$ and $\tilde{B} = B \setminus B_1$. Since Y_1 is part of a t -disconnection of Y , it follows that B_1 is the set of points of the lower border with thickness $(t-1)$ of Y_1 and \tilde{B} is the set of points of the lower border with thickness $(t-1)$ of \tilde{Y} . Thus, the base case and inductive assumption explain the fourth isomorphism in the following display.

$$\begin{aligned} A_t(Y)_f &\cong (A_t(Y_1))_t(\tilde{Y})_{f_{(1)}\tilde{f}} \cong [(A_t(Y_1))_t(\tilde{Y})_{f_{(1)}}]_{\tilde{f}} \cong (A_t(Y_1)_{f_{(1)}})_t(\tilde{Y})_{\tilde{f}} \\ &\cong (A[B_1]_{F_{(1)}})[\tilde{B}]_{\tilde{F}} \cong \left[(A[B_1])[\tilde{B}]_{F_{(1)}} \right]_{\tilde{F}} \cong A[B_1 \cup \tilde{B}]_{F_{(1)}\tilde{F}} \cong A[B]_F \end{aligned}$$

The first isomorphism is from the fact that Y_1 forms part of a t -disconnection of Y , along with the definitions above. The remaining steps are straightforward. \square

Assumption 2.11. Assume that Y is t -connected and $t > 2$. Let B_1 be the set of points of Y of the lower border with thickness 1, and set $Z = Y \setminus B_1$. Also, set $x = x_{s_1}x_{s_2} \cdots x_{s_{h+1}} \in \mathbf{k}_t(Y)$ and $\mathfrak{X} = X_{s_1}X_{s_2} \cdots X_{s_{h+1}} \in \mathbf{k}[Y]$. (We use \mathfrak{X} here instead of X since X is the $m \times n$ matrix of variables containing our ladder Y). See [Remark 2.21](#) for a discussion of the connectedness properties of Z and its corners.

The next goal is to prove [Proposition 2.23](#) which, for the t -connected ladder Y , gives an isomorphism

$$\mathbf{k}_t(Y)_x \cong \mathbf{k}_{t-1}(Z)[B_1]_{\mathfrak{X}}.$$

This result is proved by a series of lemmas. The proof outline is similar to that of [[4](#), Proposition 4.1 (2)], except that in the ladder Z , the variables are not relabeled after B_1 is deleted from Y . In pursuit of the isomorphism above, define endomorphisms ψ, χ on $\mathbf{k}[Y]_{\mathfrak{X}}$ as follows.

Definition 2.12. Continue with [Assumption 2.11](#). Let $X_{ij} \in Y$, for a fixed pair (i, j) , and let $U = U(i, j)$ be the interval $\{w \in \mathbb{N} \mid i > a_{w-1} \text{ and } j > b_w\}$. Define $\psi : \mathbf{k}[Y]_{\mathfrak{X}} \rightarrow \mathbf{k}[Y]_{\mathfrak{X}}$ by

$$\begin{aligned}\psi(X_{ij}) &= X_{ij} + \sum_{\substack{r \geq 1 \\ \{u_1 < u_2 < \dots < u_r\} \subseteq U}} \frac{X_{a_{u_1-1},j} X_{a_{u_2-1},b_{u_1}} \cdots X_{a_{u_r-1},b_{u_{r-1}}} X_{i,b_{u_r}}}{X_{a_{u_1-1},b_{u_1}} X_{a_{u_2-1},b_{u_2}} \cdots X_{a_{u_r-1},b_{u_r}}} \\ &= X_{ij} + \sum_{\substack{r \geq 1 \\ \{u_1 < u_2 < \dots < u_r\} \subseteq U}} \frac{X_{a_{u_1-1},j} X_{a_{u_2-1},b_{u_1}} \cdots X_{a_{u_r-1},b_{u_{r-1}}} X_{i,b_{u_r}}}{X_{S_{u_1}} X_{S_{u_2}} \cdots X_{S_{u_r}}}.\end{aligned}$$

In particular, $\psi(X_{ij}) = X_{ij}$ for all $X_{ij} \in B_1$ since $U(i,j) = \emptyset$ for these variables. Similarly, define $\chi : k[Y]_{\mathfrak{X}} \rightarrow k[Y]_{\mathfrak{X}}$ by

$$\chi(X_{ij}) = X_{ij} + \sum_{\substack{r \geq 1 \\ \{u_1 < u_2 < \dots < u_r\} \subseteq U}} (-1)^r \frac{X_{a_{u_1-1},j} X_{a_{u_2-1},b_{u_1}} \cdots X_{a_{u_r-1},b_{u_{r-1}}} X_{i,b_{u_r}}}{X_{S_{u_1}} X_{S_{u_2}} \cdots X_{S_{u_r}}}.$$

The example below will provide some clarity.

Example 2.13. In ladder L_3 of [Example 3.3](#), if $(i,j) = (2,4)$, then $U(2,4) = \{1\}$ since $a_0 = 1$ and $b_1 = 3$; hence $\psi(X_{24}) = X_{24} + \frac{X_{14}X_{23}}{X_{13}}$. For $(i,j) = (4,5)$, note that $U(4,5) = \{1,2\}$, hence, there are distinct chains $\{1\}, \{2\}, \{1 < 2\}$ in U . Thus,

$$\psi(X_{45}) = X_{45} + \frac{X_{15}X_{43}}{X_{13}} + \frac{X_{35}X_{41}}{X_{31}} + \frac{X_{15}X_{33}X_{41}}{X_{13}X_{31}}.$$

Lemma 2.14. *The maps ψ, χ from [Definition 2.12](#) are inverses of each other.*

Proof. It will be shown that $\chi(\psi(X_{ij})) = X_{ij}$. Similar arguments show that $\psi(\chi(X_{ij})) = X_{ij}$. Note that $\chi(\psi(X_{ij})) = X_{ij}$ for all $X_{ij} \in B_1$ since for these variables, $U(i,j) = \emptyset$. For $X_{ij} \notin B_1$, consider the expression

$$\chi\left(\frac{X_{a_{u_1-1},j} X_{a_{u_2-1},b_{u_1}} \cdots X_{a_{u_r-1},b_{u_{r-1}}} X_{i,b_{u_r}}}{X_{S_{u_1}} X_{S_{u_2}} \cdots X_{S_{u_r}}}\right). \quad (2.14.1)$$

Note that

$$\begin{aligned}\chi(X_{a_{u_1-1},j}) &= X_{a_{u_1-1},j} \\ &+ \sum_{u(i,j) \leq z_1 < z_2 < \dots < z_\ell \leq u_1-1} (-1)^\ell \frac{X_{a_{z_1-1},j} X_{a_{z_2-1},b_{z_1}} \cdots X_{a_{z_\ell-1},b_{z_{\ell-1}}} X_{a_{u_1-1},b_{z_\ell}}}{X_{S_{z_1}} X_{S_{z_2}} \cdots X_{S_{z_\ell}}} \\ \chi(X_{a_{u_2-1},b_{u_1}}) &= X_{a_{u_2-1},b_{u_1}} \\ &+ \sum_{u_1+1 \leq z_1 < z_2 < \dots < z_\ell \leq u_2-1} (-1)^\ell \frac{X_{a_{z_1-1},b_{u_1}} X_{a_{z_2-1},b_{z_1}} \cdots X_{a_{z_\ell-1},b_{z_{\ell-1}}} X_{a_{u_2-1},b_{z_\ell}}}{X_{S_{z_1}} X_{S_{z_2}} \cdots X_{S_{z_\ell}}} \\ &\vdots \\ \chi(X_{i,b_{u_r}}) &= X_{i,b_{u_r}} \\ &+ \sum_{u_r+1 \leq z_1 < z_2 < \dots < z_\ell \leq v(i,j)} (-1)^\ell \frac{X_{a_{z_1-1},b_{u_r}} X_{a_{z_2-1},b_{z_1}} \cdots X_{a_{z_\ell-1},b_{z_{\ell-1}}} X_{i,b_{z_\ell}}}{X_{S_{z_1}} X_{S_{z_2}} \cdots X_{S_{z_\ell}}}.\end{aligned}$$

Therefore, when expression (2.14.1) is expanded, all the terms are of the form

$$(-1)^{r-r} \frac{X_{a_{u_1-1},j} X_{a_{u_2-1},b_{u_1}} \cdots X_{a_{u_r-1},b_{u_{r-1}}} X_{i,b_{u_r}}}{X_{S_{u_1}} X_{S_{u_2}} \cdots X_{S_{u_r}}}, \quad (2.14.2)$$

where $u(i, j) \leq u_1 < u_2 < \dots < u_{r'} \leq v(i, j)$ and $r' \geq r \geq 1$. Note that all variables that appear in (2.14.2) have index $\not\leq(i, j)$, hence the coefficient of X_{ij} in $\chi(\psi(X_{ij}))$ is 1. Now in (2.14.2), we fix r' and vary r . Then the coefficient of

$$\frac{X_{a_{u_1-1}, j} X_{a_{u_2-1}, b_{u_1}} \cdots X_{a_{u_{r'}-1}, b_{u_{r'-1}}} X_{i, b_{u_{r'}}}}{X_{S_{u_1}} X_{S_{u_2}} \cdots X_{S_{u_{r'}}}}$$

is $\sum_{r=0}^{r'} \binom{r'}{r} (-1)^{r'-r} = 0$, where the coefficient $(-1)^{r'}$ appears in the definition of $\chi(X_{ij})$. Therefore, $\chi(\psi(X_{ij})) = X_{ij}$. \square

Remark 2.15. With the notation and assumptions of *Definition 2.12*, it holds that

$$\psi(X_{ij}) = X_{ij} + \sum_{w \in U} \frac{X_{i, b_w}}{X_{S_w}} \psi(X_{a_{w-1}, j}).$$

The next few lemmas are used to show that the maps ψ and χ respect certain ideals of minors. In them, the notation $|M| = [p_1, \dots, p_t | q_1, \dots, q_t]$ is used for the t -minor of the matrix M involving the variables located at the points (p_i, q_j) .

Lemma 2.16. *With assumptions as in 2.11, let (a_{i-1}, b_i) be a lower outside corner of Y with $1 \leq i \leq h + 1$. If there are integers ν_j for $j = 1, \dots, t - 1$ such that $b_i < \nu_1 < \nu_2 < \dots < \nu_{t-1}$ and the t -minor*

$$|M| = [a_{i-1}, a_{i-1} + 1, a_{i-1} + 2, \dots, a_{i-1} + t - 1 | b_i, \nu_1, \nu_2, \dots, \nu_{t-1}] \in I_t(Y)$$

then

$$\psi(|M|) = X_{S_i}([a_{i-1} + 1, a_{i-1} + 2, \dots, a_{i-1} + t - 1 | \nu_1, \nu_2, \dots, \nu_{t-1}] + E) \tag{2.16.1}$$

where E is a linear combination of $(t - 1)$ -minors of the form

$$[a_{i-1} + 1, a_{i-1} + 2, \dots, a_{i-1} + t - 1 | \sigma_1, \sigma_2, \dots, \sigma_{t-1}]$$

and $\sigma_j = \nu_j$ or $\sigma_j = b_w$ with $w \in U(a_{i-1}, \nu_j)$, and not all $\sigma_j = \nu_j$. In particular, $b_i < \sigma_j \leq \nu_j$.

Proof. Apply ψ to all the entries in the minor $|M|$. Prior to calculating the determinant of M perform elementary row operations, using *Remark 2.15* and the fact that the X_{S_w} are units, so that the first column is reduced to $[X_{S_i} \ 0 \ \cdots \ 0]^T$. The $(j + 1)$ st column becomes

$$\begin{bmatrix} X_{a_{i-1}, \nu_j} \\ X_{a_{i-1}+1, \nu_j} \\ \vdots \\ X_{a_{i-1}+t-1, \nu_j} \end{bmatrix} + \sum_{w \in U(a_{i-1}, \nu_j)} \frac{\psi(X_{a_{w-1}, \nu_j})}{X_{S_w}} \begin{bmatrix} X_{a_{i-1}, b_w} \\ X_{a_{i-1}+1, b_w} \\ \vdots \\ X_{a_{i-1}+t-1, b_w} \end{bmatrix}.$$

The Lemma then follows by expanding the determinant along the first column. \square

Lemma 2.17. *Continue with Assumption 2.11. Let (a_{i_0-1}, b_{i_0}) be a lower outside corner, $1 \leq i_0 \leq h + 1$. Let $\mu_j, \nu_j \in \mathbb{N}$ be such that $a_{i_0-1} < \mu_1 < \mu_2 < \dots < \mu_{t-1}$ and $b_{i_0} < \nu_1 < \nu_2 < \dots < \nu_{t-1}$, and set*

$$|M| := [a_{i_0-1}, \mu_1, \mu_2, \dots, \mu_{t-1} | b_{i_0}, \nu_1, \nu_2, \dots, \nu_{t-1}] \in I_t(Y).$$

Then one has

$$\psi(|M|) = X_{S_{i_0}}([\mu_1, \mu_2, \dots, \mu_{t-1} | \nu_1, \nu_2, \dots, \nu_{t-1}] + E) \tag{2.17.1}$$

where E is a linear combination of $(t - 1)$ -minors of the form

$$[\rho_1, \rho_2, \dots, \rho_{t-1} | \sigma_1, \sigma_2, \dots, \sigma_{t-1}]$$

where $\rho_i \in \{a_{w-1} | w \in U(\mu_i, b_{i_0})\} \cup \{\mu_i\}$, $\sigma_j \in \{b_w | w \in U(a_{i_0-1}, \nu_j)\} \cup \{\nu_j\}$, and it does not happen that all $\rho_i = \mu_i$ and $\sigma_j = \nu_j$ at the same time. In particular, one has $a_{i_0-1} < \rho_i \leq \mu_i$ and $b_{i_0} < \sigma_j \leq \nu_j$.

Proof. Let $W = \{a_{w-1} | w \in U(\mu_{t-1}, b_{i_0})\} \setminus \{\mu_1, \dots, \mu_{t-1}\}$. If $W = \emptyset$, then the proof is similar to that of Lemma 2.16. Otherwise, let $u = \min W$ and $v = \max W$. Form the minor

$$|M'| = \begin{array}{c|cccccc} & X_{a_u, b_{i_0}} & X_{a_u, \nu_1} & X_{a_u, \nu_2} & \cdots & X_{a_u, \nu_{t-1}} \\ I_{|W|} & \vdots & & & & \vdots \\ & X_{a_v, b_{i_0}} & X_{a_v, \nu_1} & X_{a_v, \nu_2} & \cdots & X_{a_v, \nu_{t-1}} \\ \hline \mathbf{0} & & & & & M \end{array},$$

so that $|M'| = |M|$. Apply ψ to all the entries in M' , and row reduce as in Lemma 2.16, so that $\psi(|M'|) = \psi(|M|)$ becomes

$$\begin{array}{c|cccccc} 1 & & 0 & X_{a_u, \nu_1} + \cdots & X_{a_u, \nu_2} + \cdots & \cdots & X_{a_u, \nu_{t-1}} + \cdots \\ & \ddots & & \vdots & & & \vdots \\ * & & 1 & X_{a_v, \nu_1} + \cdots & X_{a_v, \nu_2} + \cdots & \cdots & X_{a_v, \nu_{t-1}} + \cdots \\ \hline 0 & \cdots & 0 & X_{S_{i_0}} & X_{a_{i_0-1}, \nu_1} + \cdots & X_{a_{i_0-1}, \nu_2} + \cdots & \cdots & X_{a_{i_0-1}, \nu_{t-1}} + \cdots \\ & & & 0 & X_{\mu_1, \nu_1} + \cdots & X_{\mu_1, \nu_2} + \cdots & \cdots & X_{\mu_1, \nu_{t-1}} + \cdots \\ & & & \vdots & & & & \vdots \\ & & * & 0 & X_{\mu_{t-1}, \nu_1} + \cdots & X_{\mu_{t-1}, \nu_2} + \cdots & \cdots & X_{\mu_{t-1}, \nu_{t-1}} + \cdots \end{array}.$$

Expand the above determinant along the $(|W| + 1)$ st column, and then along the first $|W|$ columns. The conditions for σ_j then follow. Now note that when $\psi(|M'|)$ is row reduced, the row that contains $\psi(X_{a_w, b_{i_0}})$ is used to row reduce the row that contains $\psi(X_{\mu_i, b_{i_0}})$ if and only if $a_w < \mu_i$. So when the row-reduced determinant is expanded along the first $|W| + 1$ columns, the remaining rows either have index μ_1, \dots, μ_{t-1} , or some of the μ_i can be replaced with smaller indices among $\{a_{w-1} | w \in U(\mu_{t-1}, b_{i_0})\}$. Therefore $a_{i_0-1} < \rho_i \leq \mu_i$. \square

Corollary 2.18. *Continue with Assumption 2.11. Let $a_{i_0-1} < \mu_1 < \mu_2 < \cdots < \mu_{t-1}$ and $b_{i_0} < \nu_1 < \nu_2 < \cdots < \nu_{t-1}$ be such that $1 \leq i_0 \leq h + 1$ and*

$$[a_{i_0-1}, \mu_1, \mu_2, \dots, \mu_{t-1} | b_{i_0}, \nu_1, \nu_2, \dots, \nu_{t-1}] \in I_t(Y).$$

Let $|N| = [\mu_1, \mu_2, \dots, \mu_{t-1} | \nu_1, \nu_2, \dots, \nu_{t-1}]$. Then

$$\chi(|N|) = X_{S_{i_0}}^{-1}([a_{i_0-1}, \mu_1, \mu_2, \dots, \mu_{t-1} | b_{i_0}, \nu_1, \nu_2, \dots, \nu_{t-1}] + E) \quad (2.18.1)$$

where E is a linear combination of t -minors of the form

$$[a_{i_0-1}, \rho_1, \rho_2, \dots, \rho_{t-1} | b_{i_0}, \sigma_1, \sigma_2, \dots, \sigma_{t-1}]$$

with $a_{i_0-1} < \rho_i \leq \mu_i$ and $b_{i_0} < \sigma_j \leq \nu_j$. In particular, $\psi(\langle I_t(Y) \rangle) \supseteq \langle I_{t-1}(Z) \rangle$ in $\mathbf{k}[Y]_{\mathfrak{X}}$.

Proof. Let $\zeta = (\zeta_1, \dots, \zeta_{t-1})$ and $\xi = (\xi_1, \dots, \xi_{t-1})$, where $\zeta_i = \mu_i - (a_{i_0-1} + i)$ and $\xi_j = \nu_j - (b_{i_0} + j)$ for $1 \leq i, j \leq t - 1$. Note that these sequences are non-decreasing. The proof is by induction on (ξ, ζ) , where the values of ζ and ξ each form a finite subset of \mathbb{N}^{t-1} . Apply the reverse lexicographic order to

$(\xi, \zeta) \in \mathbb{N}^{2t-2}$. Consider the base case when $\xi = \zeta = 0$. Then (2.16.1) gives

$$\begin{aligned} \psi([a_{i_0-1}, \dots, a_{i_0-1} + t - 1 | b_{i_0}, \dots, b_{i_0} + t - 1]) &= X_{S_{i_0}} |N|, \text{ so} \\ \chi(|N|) &= X_{S_{i_0}}^{-1} [a_{i_0-1}, \dots, a_{i_0-1} + t - 1 | b_{i_0}, \dots, b_{i_0} + t - 1]. \end{aligned}$$

In the induction step, consider $(\xi, \zeta) \neq (0, 0)$. The term E in equation (2.17.1) is a linear combination of $(t - 1)$ -minors of the form $|N'| = [\rho_1, \rho_2, \dots, \rho_{t-1} | \sigma_1, \sigma_2, \dots, \sigma_{t-1}]$ where $a_{i_0-1} < \rho_i \leq \mu_i, b_{i_0} < \sigma_j \leq \nu_j$, and $\rho_i < \mu_i$ for at least one i or $\sigma_j < \nu_j$ for at least one j . Then apply χ to both sides of (2.17.1), rearrange, and apply the induction hypothesis to the terms $|N'|$ in E .

Finally, let $|N| = [\mu_1, \mu_2, \dots, \mu_{t-1} | \nu_1, \nu_2, \dots, \nu_{t-1}]$ be a generator of $I_{t-1}(Z)$. Then $X_{\mu_i, \nu_i} \notin B_1$. Let $i_0 \in U(\mu_1, \nu_1) \neq \emptyset$. Then (2.18.1) shows that $|N| \in \psi(I_t(Y))$. Therefore $\psi(\langle I_t(Y) \rangle) \supseteq \langle I_{t-1}(Z) \rangle$. \square

Lemma 2.19. *In addition to Assumption 2.11, suppose that Y does not have an outside corner at (μ_0, ν_0) . Let $\mu_0 < \mu_1 < \mu_2 < \dots < \mu_{t-1}$ and $\nu_0 < \nu_1 < \nu_2 < \dots < \nu_{t-1}$ be such that $|M| := [\mu_0, \mu_1, \mu_2, \dots, \mu_{t-1} | \nu_0, \nu_1, \nu_2, \dots, \nu_{t-1}] \in I_t(Y)$. Then*

$$\psi(|M|) = [\mu_0, \mu_1, \mu_2, \dots, \mu_{t-1} | \nu_0, \nu_1, \nu_2, \dots, \nu_{t-1}] + E$$

where E is a linear combination of t -minors of the form

$$[\rho_0, \rho_1, \rho_2, \dots, \rho_{t-1} | \sigma_0, \sigma_1, \sigma_2, \dots, \sigma_{t-1}]$$

where

$$\begin{aligned} \rho_i &\in \{a_{w-1} | w \in U(\mu_i, \nu_0) \setminus U(\mu_0, \nu_0)\} \cup \{\mu_i\} \\ \sigma_j &\in \{b_w | w \in U(\mu_0, \nu_j)\} \cup \{\nu_j\} \end{aligned}$$

and it does not happen that all $\rho_i = \mu_i$ and $\sigma_j = \nu_j$ at the same time. In particular, $\mu_0 \leq \rho_i \leq \mu_i$ and $\sigma_j \leq \nu_j$.

Proof. The proof is similar to that of Lemma 2.17 with some modifications. Let $W = \{a_{w-1} | w \in U(\mu_{t-1}, \nu_0) \setminus U(\mu_0, \nu_0)\} \setminus \{\mu_0, \mu_1, \dots, \mu_{t-1}\}$. When row reducing $\psi(|M'|)$, if $\mu_0 \in \{a_{w-1} | w \in U(\mu_{t-1}, \nu_0)\}$, use only the row that contains the entry $\psi(X_{\mu_0, \nu_0})$. Finally, after row reduction, expand along the first $|W|$ columns. \square

Notation 2.20. Continue with Assumption 2.11, and let $Z = Y \setminus B_1$. We extend the definitions of q_i, q'_i, p_j in [5, pp. 459, 463]. For each lower outside corner S_i as in [5, p. 457], set

$$\mathfrak{Q}_i = \begin{cases} \{X_{pq} \in Y | a_{i-1} \leq p \leq a_{i-1} + t - 2\} & \text{if } i = 1 \text{ or } S'_{i-1} \text{ has type 1} \\ \{X_{pq} \in Y | a_{i-1} \leq p \leq a_{i-1} + t - 2 \text{ and } q \leq d_j\} & \text{if } i > 1 \text{ and } S'_{i-1} \text{ has type 2} \end{cases}$$

where, in the second case, T'_j is the companion corner for S'_{i-1} . A slogan for this is

$$\mathfrak{Q}_i = \begin{cases} \{\text{rows of } Y \text{ involved in } F_i\} & \text{if } i = 1 \text{ or } S'_{i-1} \text{ has type 1} \\ \{\text{partial rows of } Y \text{ involved in } F_i\} & \text{if } i \neq 1 \text{ and } S'_{i-1} \text{ has type 2.} \end{cases}$$

Set

$$\begin{aligned} Q_i(Y) &= I_{t-1}(X_{pq} \in A[Y] | X_{pq} \in \mathfrak{Q}_i) + I_t(Y) \subseteq A[Y] \\ q_i(Y) &= Q_i(Y) / I_t(Y) \subseteq A_t(Y) \end{aligned}$$

and note that q_i is a height-1 prime ideal of $A_t(Y)$ containing f_i ; this is verified as in [5]. The ideal q'_i is defined similarly using columns associated to S_i :

$$\mathfrak{Q}'_i = \begin{cases} \{X_{pq} \in Y | b_i \leq q \leq b_i + t - 2\} & \text{if } i = h + 1 \text{ or } S'_i \text{ has type 1} \\ \{X_{pq} \in Y | b_i \leq q \leq b_i + t - 2 \text{ and } p \leq c_j\} & \text{if } i \leq h \text{ and } S'_i \text{ has type 2} \end{cases}$$

where, in the second case, T'_j is the companion corner for S'_i . A slogan for this is

$$\mathfrak{Q}'_i = \begin{cases} \{\text{columns of } Y \text{ involved in } F_i\} & \text{if } i = h + 1 \text{ or } S'_i \text{ has type 1} \\ \{\text{partial columns of } Y \text{ involved in } F_i\} & \text{if } i \leq h \text{ and } S'_i \text{ has type 2.} \end{cases}$$

Set $Q'_i(Y) = I_{t-1}(X_{pq} \in A[Y]|X_{pq} \in \mathfrak{Q}'_i) + I_t(Y)$ and $q'_i(Y) = Q'_i(Y)/I_t(Y)$ noting that q'_i is another height-1 prime ideal of $A_t(Y)$ containing f_i . For each inside corner T'_j , set

$$\mathfrak{P}_j = \begin{cases} \{X_{pq} \in Y|(p, q) \leq T'_j\} & \text{if } T'_j \text{ has type 1} \\ \emptyset & \text{if } T'_j \text{ has type 2} \end{cases}$$

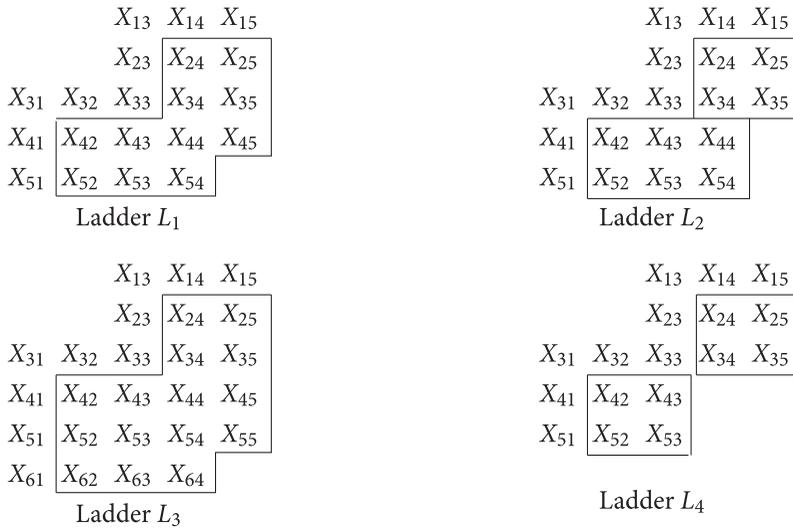
$$P_j(Y) = I_{t-1}(X_{pq} \in A[Y]|X_{pq} \in \mathfrak{P}_j) + I_t(Y) \subseteq A[Y]$$

$$p_j(Y) = P_j(Y)/I_t(Y) \subseteq A_t(Y)$$

and note that p_j is a height-1 prime in $A_t(Y)$ whenever $p_j \neq (0)$.

The ideals $q_i(Z), q'_i(Z), p_j(Z)$ are defined similarly, but we number these ideals as we do in Y , even if Z is not $(t - 1)$ -connected.

Remark 2.21. It is straightforward to build examples where $Z = Y \setminus B_1$ is $(t - 1)$ -disconnected, even though Y is t -connected; in particular, this is illustrated below with $t = 3$ for the ladders L_2 and L_4 from [Example 3.3](#), and the Z highlighted by the boxes. The ladders L_1 and L_3 are examples where Z is $(t - 1)$ -connected. In what follows, we wish to consider the ideals q_i, p_j associated to the rings $k_3(L_n)$.



Continuing with the running [Example 3.3](#), the q_i 's are displayed in [Table 1](#) below. The q'_i 's are similar, using columns.

The upper inside corners of Z are all upper inside corners of Y , but not vice versa in general. Thus, we analyze the cases here.

If T'_j is an upper inside corner of Y with type 1, then T'_j is also an upper inside corner of Z with type 1, and T'_j does not cause a $(t - 1)$ -disconnection of Z . (See ladders L_1 and L_3 above.) In this case, the definitions of q_i, q'_i, p_j coincide with those in [\[5\]](#). In particular, if Y satisfies Assumption (d), then Z is $(t - 1)$ -connected and satisfies Assumption (d) with respect to $t - 1$.

If T'_j is an upper inside corner of Y with type 2, then T'_j may or may not be an upper inside corner of Z , and T'_j may or may not cause a $(t - 1)$ -disconnection of Z . (See ladders L_2 and L_4 above.) In this case, let S'_j be the companion corner for T'_j . If $\max\{c_j - a_i, d_j - b_i\} < t - 2$, then $I_{t-1}(X_{pq} \in Y|(p, q) \leq T'_j) = (0)$, and similarly for Z if $T'_j \in Z$. If $T'_j \notin Z$, then $p_j(Z)$ is, in a sense, meaningless. (This is the case for ladder L_4 above.) If $d_j - b_i = t - 2$, then $I_{t-1}(X_{pq} \in$

Table 1. Prime ideals \mathfrak{q}_i of $k_3(Y)$ and $k_2(Z)$.

$Y = L_1 : \mathfrak{q}_1 = I_2 \begin{pmatrix} X_{13} & X_{14} & X_{15} \\ X_{23} & X_{24} & X_{25} \end{pmatrix}$	$Z : \mathfrak{q}_1 = (X_{24}, X_{25})$
$\mathfrak{q}_2 = I_2 \begin{pmatrix} X_{31} & X_{32} & X_{33} & X_{34} & X_{35} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} \end{pmatrix}$	$\mathfrak{q}_2 = (X_{42}, X_{43}, X_{44}, X_{45})$
$Y = L_2 : \mathfrak{q}_1 = I_2 \begin{pmatrix} X_{13} & X_{14} & X_{15} \\ X_{23} & X_{24} & X_{25} \end{pmatrix}$	$Z : \mathfrak{q}_1 = (X_{24}, X_{25})$
$\mathfrak{q}_2 = I_2 \begin{pmatrix} X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix}$	$\mathfrak{q}_2 = (X_{42}, X_{43}, X_{44})$
$Y = L_3 : \mathfrak{q}_1 = I_2 \begin{pmatrix} X_{13} & X_{14} & X_{15} \\ X_{23} & X_{24} & X_{25} \end{pmatrix}$	$Z : \mathfrak{q}_1 = (X_{24}, X_{25})$
$\mathfrak{q}_2 = I_2 \begin{pmatrix} X_{31} & X_{32} & X_{33} & X_{34} & X_{35} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} \end{pmatrix}$	$\mathfrak{q}_2 = (X_{42}, X_{43}, X_{44}, X_{45})$
$Y = L_4 : \mathfrak{q}_1 = I_2 \begin{pmatrix} X_{13} & X_{14} & X_{15} \\ X_{23} & X_{24} & X_{25} \end{pmatrix}$	$Z : \mathfrak{q}_1 = (X_{24}, X_{25})$
$\mathfrak{q}_2 = I_2 \begin{pmatrix} X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{pmatrix}$	$\mathfrak{q}_2 = (X_{42}, X_{43})$

Table 2. Prime ideals \mathfrak{p}_1 of $k_3(Y)$ and $k_2(Z)$.

$Y = L_1 : \mathfrak{p}_1 = I_2 \begin{pmatrix} & X_{13} & X_{14} \\ & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix}$	$Z : \mathfrak{p}_1 = I_1 \begin{pmatrix} & X_{24} \\ & X_{34} \\ X_{42} & X_{43} & X_{44} \end{pmatrix}$
$Y = L_2 : \mathfrak{p}_1 = (0)$	$Z : \mathfrak{p}_1 = (0)$
$Y = L_3 : \mathfrak{p}_1 = I_2 \begin{pmatrix} & X_{13} & X_{14} \\ & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \\ X_{51} & X_{52} & X_{53} & X_{54} \end{pmatrix}$	$Z : \mathfrak{p}_1 = I_1 \begin{pmatrix} & X_{24} \\ & X_{34} \\ X_{42} & X_{43} & X_{44} \\ X_{52} & X_{53} & X_{54} \end{pmatrix}$
$Y = L_4 : \mathfrak{p}_1 = (0)$	$Z : \mathfrak{p}_1 = (0)$

$Y|(p, q) \leq T'_j) + I_t(Y) = \mathfrak{q}'_i$ (as in ladder L_2 above). If $c_j - a_i = t - 2$, then $I_{t-1}(X_{pq} \in Y|(p, q) \leq T'_j) + I_t(Y) = \mathfrak{q}_{i+1}$. In all cases, T'_j does not contribute new ideals to $\text{Cl}(A_t(Y))$ nor $\text{Cl}(A_{t-1}(Z))$, hence, we may as well define $\mathfrak{p}_j(Y) = (0)$ and $\mathfrak{p}_j(Z) = (0)$.

Table 2 shows the ideals \mathfrak{p}_j , which is only \mathfrak{p}_1 , for the running example(s).

Corollary 2.22. *Continue with Assumption 2.11. Then $\psi(I_t(Y)) = I_{t-1}(Z), \psi(Q_i(Y)) = Q_i(Z)$ and $\psi(P_j(Y)) = P_j(Z)$ in $k[Y]_{\mathfrak{X}}$.*

Proof. First, the equality $\psi(I_t(Y)) = I_{t-1}(Z)$ will be established. The inclusion \supseteq is given by Corollary 2.18, hence it is only necessary to establish \subseteq .

Let $\mu_0 < \mu_1 < \mu_2 < \dots < \mu_{t-1}$ and $\nu_0 < \nu_1 < \nu_2 < \dots < \nu_{t-1}$ be such that $|M| := [\mu_0, \mu_1, \mu_2, \dots, \mu_{t-1} | \nu_0, \nu_1, \nu_2, \dots, \nu_{t-1}]$ is a generator of $I_t(Y)$. It is necessary to show that the terms of $\psi(|M|)$ from Lemma 2.19 are in $I_{t-1}(Z)$. By Lemma 2.17, it is sufficient to only consider the case when (μ_0, ν_0) is not an outside corner of Y . Consequently, $X_{\mu_0, \nu_0+1} \notin B_1$ or $X_{\mu_0+1, \nu_0} \notin B_1$. If $X_{\mu_0, \nu_0+1} \notin B_1$, expand all determinants along the first column to get $\psi(|M|) \in I_{t-1}(Z)$. Next, if $X_{\mu_0, \nu_0} \notin B_1$, then $X_{\mu_0, \nu_0+1} \notin B_1$, so one may assume that $X_{\mu_0+1, \nu_0} \notin B_1$, but $X_{\mu_0, \nu_0} \in B_1$. By Lemma 2.19, $\sigma_j \in \{b_w | w \in U(\mu_0, \nu_j)\} \cup \{\nu_j\}$, so $\nu_0 \leq \sigma_j \leq \nu_j$. Then expand all determinants in $\psi(|M|)$ along the first row to get $\psi(|M|) \in I_{t-1}(Z)$.

The proofs of $\psi(Q_i(Y)) = Q_i(Z)$ and $\psi(P_j(Y)) = P_j(Z)$ are similar (by considering $t - 1$ instead of t). □

Proposition 2.23. *The maps ψ and χ from Definition 2.12 induce isomorphisms*

$$\mathbf{k}_t(Y)_x \cong \mathbf{k}_{t-1}(Z)[B_1]_{\mathbf{x}}.$$

Proof. From Corollary 2.22 it follows that $\psi(I_t(Y)) = I_{t-1}(Z)$ in $\mathbf{k}[Y]_{\mathbf{x}}$, so the isomorphisms ψ and χ from Lemma 2.14 induce the given isomorphism. □

3. Divisor class groups

Assumption 3.1. Throughout this section, let A be a normal domain with field of fractions K , and let Y be a ladder of variables (not necessarily path-connected or t -connected). Let D be a subring of A that is a principal ideal domain, and let L denote the field of fractions of D . Let f be as in Notation 2.8. Recall that A is flat over D by Remark 1.1.

The results below use the following explicit description of $\text{Cl}(\mathbf{k}_t(Y))$, where Y is a t -connected ladder of variables. Conca [5, pp. 467–468] states this partially, but we require slightly more detail for our applications.

Fact 3.2. Assume that Y is t -connected. Conca [5, pp. 467–468] states that $\text{Cl}(\mathbf{k}_t(Y))$ is a free abelian group of rank $h + k^* + 1 = h^* + k + 1$. Moreover, using the sketch provided in *loc. cit.* with Notation 2.20, one finds that a basis for $\text{Cl}(\mathbf{k}_t(Y))$ is given by the classes of the ideals q_i where $i = 1, \dots, h + 1$, and the classes of the ideals p_j where j ranges through the T'_j of type 1. In particular, the basis for $\text{Cl}(\mathbf{k}_t(Y))$ is independent of the field \mathbf{k} , depending only on the shape of Y . From this, it follows that if $\mathbf{k} \rightarrow K$ is a field extension, then the induced map $\text{Cl}(\mathbf{k}_t(Y)) \rightarrow \text{Cl}(K_t(Y))$ is an isomorphism. A key ingredient of the proof is to show that the minimal primes of the element $f \in \mathbf{k}_t(Y)$ from Notation 2.8 are exactly the q_i , the q'_i , and the p_j (for $p_j \neq 0$). See Remark 3.8 for the t -disconnected case.

Example 3.3. (Running Example) The computations of $\text{Cl}(\mathbf{k}_3(L_i))$ for the ladders L_i are shown below. The details are provided in the Preliminaries section; in particular, all ideals listed appear in the corresponding rows in Tables 1 and 2 in Remark 2.21.

$X_{13} \ X_{14} \ X_{15}$	$X_{13} \ X_{14} \ X_{15}$
$X_{23} \ X_{24} \ X_{25}$	$X_{23} \ X_{24} \ X_{25}$
$X_{31} \ X_{32} \ X_{33} \ X_{34} \ X_{35}$	$X_{31} \ X_{32} \ X_{33} \ X_{34} \ X_{35}$
$X_{41} \ X_{42} \ X_{43} \ X_{44} \ X_{45}$	$X_{41} \ X_{42} \ X_{43} \ X_{44}$
$X_{51} \ X_{52} \ X_{53} \ X_{54}$	$X_{51} \ X_{52} \ X_{53} \ X_{54}$
Ladder L_1	Ladder L_2
$X_{13} \ X_{14} \ X_{15}$	$X_{13} \ X_{14} \ X_{15}$
$X_{23} \ X_{24} \ X_{25}$	$X_{23} \ X_{24} \ X_{25}$
$X_{31} \ X_{32} \ X_{33} \ X_{34} \ X_{35}$	$X_{31} \ X_{32} \ X_{33} \ X_{34} \ X_{35}$
$X_{41} \ X_{42} \ X_{43} \ X_{44} \ X_{45}$	$X_{41} \ X_{42} \ X_{43}$
$X_{51} \ X_{52} \ X_{53} \ X_{54} \ X_{55}$	$X_{51} \ X_{52} \ X_{53}$
$X_{61} \ X_{62} \ X_{63} \ X_{64}$	
Ladder L_3	Ladder L_4

- (1) In L_1 , the upper inside corner T'_1 has type 1, so $k^* = 1$. Since $h = 1$, $\text{Cl}(k_3(L_1)) \cong \mathbb{Z}^3$ with basis represented by the ideals q_1, q_2, p_1 .
- (2) In L_2 , the upper inside corner T'_1 has type 2, so $k^* = 0$. Since $h = 1$, $\text{Cl}(k_3(L_2)) \cong \mathbb{Z}^2$ with basis represented by the ideals q_1, q_2 .
- (3) Similarly, $\text{Cl}(k_3(L_3)) \cong \mathbb{Z}^3$ with basis represented by the ideals q_1, q_2, p_1 .
- (4) $\text{Cl}(k_3(L_4)) \cong \mathbb{Z}^2$ with basis represented by the ideals q_1, q_2 .

Our next result, which does not assume Conca’s Assumption (d), confirms a statement in [5, p. 457] about splitting divisor class groups and provides proof.

Proposition 3.4. *Assume that Y is t -connected, and let $A \xrightarrow{g_1} A_t(Y) \xrightarrow{h_2} K_t(Y)$ be the natural flat maps.*

- (a) The following sequence is split-exact:

$$0 \rightarrow \text{Cl}(A) \xrightarrow{\text{Cl}(g_1)} \text{Cl}(A_t(Y)) \xrightarrow{\text{Cl}(h_2)} \text{Cl}(K_t(Y)) \rightarrow 0. \tag{3.4.1}$$

In particular, $\text{Cl}(A_t(Y)) \cong \text{Cl}(A) \times \text{Cl}(K_t(Y))$.

- (b) *The ring A is a unique factorization domain if and only if the natural map $\text{Cl}(A_t(Y)) \xrightarrow{\text{Cl}(h_2)} \text{Cl}(K_t(Y))$ is an isomorphism.*

Proof. (a) Since $K_t(Y)$ is obtained from $A_t(Y)$ by inverting the non-zero elements of A , Nagata’s Theorem [6, Corollary 7.2] tells us that $\text{Cl}(h_2)$ is surjective with kernel generated by all the height-1 primes of $A_t(Y)$ containing non-zero elements of A . It is straightforward to show that these primes are exactly the ideals of $A_t(Y)$ extended from height-1 primes of A . Thus, $\text{Ker}(\text{Cl}(h_2)) = \text{Im}(\text{Cl}(g_1))$. Furthermore, $\text{Cl}(g_1)$ is injective since $A_t(Y)$ is faithfully flat over A . This establishes the exactness of the sequence (3.4.1). The sequence splits because $\text{Cl}(K_t(Y))$ is free by Fact 3.2.

- (b) This follows directly from part (a) as A is a UFD if and only if $\text{Cl}(A) = 0$. □

Next is a version of Proposition 3.4 with no t -connected assumption. Note that part (a) is an improvement of part of Fact 3.2; see also Proposition 3.6.

Corollary 3.5. *Let Y_1, \dots, Y_s be the t -components of Y .*

- (a) *There is an isomorphism $\text{Cl}(K_t(Y)) \cong \text{Cl}(K_t(Y_1)) \times \dots \times \text{Cl}(K_t(Y_s))$. In particular, $\text{Cl}(K_t(Y))$ is free of finite rank, and the rank is independent of the field.*
- (b) *The following sequence is split-exact:*

$$0 \rightarrow \text{Cl}(A) \xrightarrow{\text{Cl}(g_1)} \text{Cl}(A_t(Y)) \xrightarrow{\text{Cl}(h_2)} \text{Cl}(K_t(Y)) \rightarrow 0. \tag{3.5.1}$$

In particular, $\text{Cl}(A_t(Y)) \cong \text{Cl}(A) \times \text{Cl}(K_t(Y))$.

- (c) *The ring A is a unique factorization domain if and only if the natural map $\text{Cl}(A_t(Y)) \xrightarrow{\text{Cl}(h_2)} \text{Cl}(K_t(Y))$ is an isomorphism.*

Proof. (a) For the direct product decomposition, induct on s . The base case is trivial. In the inductive case, set $Y' = Y \setminus Y_1$, which is the disjoint union $Y_2 \cup \dots \cup Y_s$. Then $K_t(Y) \cong K_t(Y_1) \otimes_K K_t(Y') \cong (K_t(Y'))_t(Y_1)$. Apply Proposition 3.4(a) with the normal domain $A = K_t(Y')$

and the t -connected ladder Y_1 to conclude that $\text{Cl}(K_t(Y)) \cong \text{Cl}(K_t(Y_1)) \times \text{Cl}(K_t(Y'))$. Now apply the induction hypothesis to Y' to obtain the desired decomposition.

Since each abelian group $\text{Cl}(K_t(Y_i))$ is free of finite rank by [Fact 3.2](#), it follows that $\text{Cl}(K_t(Y)) \cong \text{Cl}(K_t(Y_1)) \times \cdots \times \text{Cl}(K_t(Y_s))$ is free of finite rank as well. [Fact 3.2](#) also establishes that the rank of each group $\text{Cl}(K_t(Y_i))$ is independent of the field K , hence so is the rank of $\text{Cl}(K_t(Y))$.

(b)–(c) These are proved like [Proposition 3.4](#) using part (a). \square

We improve part of [Fact 3.2](#) in the next Proposition.

Proposition 3.6. *Consider the field extension $L \rightarrow K$ from [3.1](#). The natural map $\text{Cl}(L_t(Y)) \rightarrow \text{Cl}(K_t(Y))$ is an isomorphism.*

Proof. Let Y_1, \dots, Y_s be the t -components of Y . [Fact 3.2](#) implies that each natural map $\text{Cl}(L_t(Y_i)) \rightarrow \text{Cl}(K_t(Y_i))$ is an isomorphism, hence, the product map $\prod_{i=1}^s \text{Cl}(L_t(Y_i)) \rightarrow \prod_{i=1}^s \text{Cl}(K_t(Y_i))$ is as well. One checks that the isomorphisms from [Corollary 3.5\(a\)](#) make the following diagram commute

$$\begin{array}{ccc} \text{Cl}(L_t(Y)) & \xrightarrow{\cong} & \prod_{i=1}^s \text{Cl}(L_t(Y_i)) \\ \downarrow & & \downarrow \cong \\ \text{Cl}(K_t(Y)) & \xrightarrow{\cong} & \prod_{i=1}^s \text{Cl}(K_t(Y_i)), \end{array}$$

and as a result, the unlabeled map is an isomorphism as well. \square

The next result is an improved version of [Corollary 3.5\(b\)](#). It contains a bit more information for use in finding semidualizing modules.

Theorem 3.7. *The following natural maps*

$$\text{Cl}(A) \times \text{Cl}(D_t(Y)) \rightarrow \text{Cl}(A_t(Y)) \rightarrow \text{Cl}(A_t(Y)_f) \times \text{Cl}(K_t(Y))$$

are bijections, hence

$$\text{Cl}(A_t(Y)) \cong \text{Cl}(A_t(Y)_f) \times \text{Cl}(K_t(Y)) \cong \text{Cl}(A) \times \text{Cl}(K_t(Y)).$$

Proof. Consider the following natural maps.

$$\begin{array}{ccc} A & \xrightarrow{g_1} & A_t(Y) \xrightarrow{h_1} & A_t(Y)_f \\ D_t(Y) & \xrightarrow{g_2} & A_t(Y) \xrightarrow{h_2} & K_t(Y) \end{array}$$

The map g_1 is flat (in fact, free and faithfully flat), and g_2 is flat because A is flat over D . Also, each h_i is a localization map, hence flat.

Next, there are commutative diagrams of flat maps of normal domains

$$\begin{array}{ccc} A & \xrightarrow{g_1} & A_t(Y) \\ \downarrow & & \downarrow h_2 \\ K & \longrightarrow & K_t(Y) \end{array} \quad \begin{array}{ccc} D_t(Y) & \xrightarrow{g_2} & A_t(Y) \\ \downarrow & & \downarrow h_1 \\ D_t(Y)_f & \longrightarrow & A_t(Y)_f \end{array}$$

which induce commutative diagrams on divisor class groups

$$\begin{array}{ccc} \text{Cl}(A) & \xrightarrow{\text{Cl}(g_1)} & \text{Cl}(A_t(Y)) \\ \downarrow & & \downarrow \text{Cl}(h_2) \\ 0 = \text{Cl}(K) & \longrightarrow & \text{Cl}(K_t(Y)) \end{array} \quad \begin{array}{ccc} \text{Cl}(D_t(Y)) & \xrightarrow{\text{Cl}(g_2)} & \text{Cl}(A_t(Y)) \\ \downarrow & & \downarrow \text{Cl}(h_1) \\ 0 = \text{Cl}(D_t(Y)_f) & \longrightarrow & \text{Cl}(A_t(Y)_f). \end{array} \quad (3.7.1)$$

The first vanishing in (3.7.1) comes from the fact that K is a field. For the second vanishing, recall that D is a principal ideal domain. Since it is a unique factorization domain, so is $D[B]_F \cong D_t(Y)_f$, where B and the isomorphism are from Lemma 2.10. These vanishings imply that, whenever $i \neq j$,

$$0 = \text{Cl}(h_i) \circ \text{Cl}(g_j) = \text{Cl}(h_i \circ g_j). \tag{3.7.2}$$

Next, consider the following diagram of group homomorphisms.

$$\begin{array}{ccc} \text{Cl}(A) \times \text{Cl}(D_t(Y)) & \xrightarrow{\text{Cl}(h_1 \circ g_1) \times \text{Cl}(h_2 \circ g_2)} & \text{Cl}(A_t(Y)_f) \times \text{Cl}(K_t(Y)) \\ & \searrow \text{Cl}(g_1) + \text{Cl}(g_2) & \nearrow (\text{Cl}(h_1), \text{Cl}(h_2)) \\ & & \text{Cl}(A_t(Y)) \end{array} \tag{3.7.3}$$

A straightforward diagram chase using (3.7.2) shows that this diagram commutes.

We claim that the horizontal map in diagram (3.7.3) is an isomorphism. To show this, it suffices to show that the component maps $\text{Cl}(h_1 \circ g_1)$ and $\text{Cl}(h_2 \circ g_2)$ are isomorphisms. The first of these comes from the following natural commutative diagrams, wherein the vertical isomorphism comes from the fact that the element F from Notation 2.8 is a product of prime elements of the polynomial ring $A[B]$ from Lemma 2.10.

$$\begin{array}{ccc} A & \xrightarrow{h_1 \circ g_1} & A_t(Y)_f \\ \downarrow & \nearrow \cong & \\ A[B]_F & & \end{array} \quad \begin{array}{ccc} \text{Cl}(A) & \xrightarrow{\text{Cl}(h_1 \circ g_1)} & \text{Cl}(A_t(Y)_f) \\ \cong \downarrow & \nearrow \cong & \\ \text{Cl}(A[B]_F) & & \end{array} \tag{3.7.4}$$

For the map $\text{Cl}(h_2 \circ g_2) : \text{Cl}(D_t(Y)) \rightarrow \text{Cl}(K_t(Y))$, recall that L is a subfield of K . Factor the map $h_2 \circ g_2 : D_t(Y) \rightarrow K_t(Y)$ as the composition of the natural flat maps $D_t(Y) \rightarrow L_t(Y) \rightarrow K_t(Y)$. It suffices to show that each map $\text{Cl}(D_t(Y)) \rightarrow \text{Cl}(L_t(Y)) \rightarrow \text{Cl}(K_t(Y))$ is an isomorphism, but this follows from Corollary 3.5(c) and Proposition 3.6, respectively. This establishes the claim.

Since the horizontal map in diagram (3.7.3) is an isomorphism, commutativity of the diagram implies in particular that the map $(\text{Cl}(h_1), \text{Cl}(h_2))$ is surjective. Thus, to complete the proof, it suffices to show that this map is also injective. So, let $[a] \in \text{Cl}(A_t(Y))$ be in $\text{Ker}(\text{Cl}(h_1), \text{Cl}(h_2)) = \text{Ker}(\text{Cl}(h_1)) \cap \text{Ker}(\text{Cl}(h_2))$. Corollary 3.5(b) implies that $\text{Ker}(\text{Cl}(h_2)) = \text{Im}(\text{Cl}(g_1))$. As a result, $[a] = \text{Cl}(g_1)([b])$ for some $[b] \in \text{Cl}(A)$. Then the condition $\text{Cl}(g_1)([b]) = [a] \in \text{Ker}(\text{Cl}(h_1))$ implies that

$$0 = \text{Cl}(h_1)(\text{Cl}(g_1)([b])) = \text{Cl}(h_1 \circ g_1)([b]).$$

Since it was established that $\text{Cl}(h_1 \circ g_1)$ is an isomorphism, $[b] = 0$, and thus, $[a] = \text{Cl}(g_1)([b]) = 0$, as desired. \square

Remark 3.8. Let Y_1, \dots, Y_s be the t -components of Y . For $\ell = 1, \dots, s$, let k_ℓ^* be the number of upper inside corners of Y_ℓ of type 1 (see Definition 2.2), and let h_ℓ be the number of lower inside corners of Y_ℓ . The above results show that $\text{Cl}(k_t(Y))$ has a basis given by $[(q_i(Y_\ell))]$, with $i = 1, \dots, h_\ell + 1$, and those $[(p_j(Y_\ell))]$ such that $T'_{\ell,j}$ is an upper inside corner of Y_ℓ of type 1.

The above paragraph shows how to calculate $\text{Cl}(k_t(Y)) \cong \text{Cl}(k_{t-1}(Z))$, which is used in Section 4, e.g., Corollary 4.6, and more. In particular, if Y is t -connected and Z is obtained from Y as in Assumption 2.11, then a basis for $\text{Cl}(k_{t-1}(Z))$ is given as follows: let Z_1, \dots, Z_s be the $(t-1)$ -components of Z i.e., each Z_ℓ is $(t-1)$ -connected. Then a basis for $\text{Cl}(k_{t-1}(Z))$ is given by $[(q_i(Z_\ell))]$, with i ranging through the lower outside corners of Z_ℓ , or equivalently, ranging through the lower outside corners of Y , and $[(p_j(Z_\ell))]$ such that $T'_{\ell,j}$ is an upper inside corner of

Z_ℓ of type 1, or equivalently, ranging through the upper inside corners of Y of type 1. See Remark 2.21 in regards to the running example.

4. Semidualizing modules

Assumption 4.1. Throughout this section, let A be a normal domain with field of fractions K , and let Y be a ladder of variables (not necessarily path-connected or t -connected). Let D be a subring of A that is a principal ideal domain, and let L denote the field of fractions of D . Let $f = f_1 \cdots f_{h+1}$ as in Notation 2.8. Let B_1 be the set of points of Y of the lower border with thickness 1. Set $Z = Y \setminus B_1$ and $x = x_{S_1} x_{S_2} \cdots x_{S_{h+1}} \in \mathfrak{k}_t(Y)$ and $\mathfrak{X} = X_{S_1} X_{S_2} \cdots X_{S_{h+1}} \in \mathfrak{k}[Y]$.

Lemma 4.2. *The natural map $\phi : \mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(L_t(Y))$ is injective.*

Proof. In the following commutative diagram

$$\begin{array}{ccc} \text{Cl}(D_t(Y)) & \xrightarrow{\cong} & \text{Cl}(L_t(Y)) \\ \uparrow & & \uparrow \\ \mathfrak{S}_0(D_t(Y)) & \xrightarrow{\phi} & \mathfrak{S}_0(L_t(Y)) \end{array}$$

the top isomorphism is from Corollary 3.5(c). A diagram chase shows that ϕ is injective. \square

The theorem below describes the semidualizing modules over $A_t(Y)$ where Y is arbitrary and A is a normal domain. Through a series of results, we are able to improve upon this by not only providing a more detailed description of $\mathfrak{S}_0(A_t(Y))$, but also by removing the conditions (1)–(2); see Theorem 4.16 below. Here and elsewhere, if U and V are ordered sets, then the notation $U \approx V$ means that there is a perfectly relation-respecting bijection from U to V .

Theorem 4.3. *Assume that at least one of the following conditions is satisfied:*

- (1) A contains a field, or
- (2) the natural map $\mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(L_t(Y))$ is surjective.

Then the natural maps

$$\mathfrak{S}_0(A) \times \mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(A_t(Y)) \rightarrow \mathfrak{S}_0(A_t(Y)_f) \times \mathfrak{S}_0(K_t(Y))$$

are bijections, hence

$$\mathfrak{S}_0(A_t(Y)) \approx \mathfrak{S}_0(A_t(Y)_f) \times \mathfrak{S}_0(K_t(Y)) \approx \mathfrak{S}_0(A) \times \mathfrak{S}_0(K_t(Y)).$$

Proof. Consider the following diagram where the vertical maps are the natural inclusions and the upper triagram commutes as per the proof of Theorem 3.7.

$$\begin{array}{ccc} \text{Cl}(A) \times \text{Cl}(D_t(Y)) & \xrightarrow{\text{Cl}(h_1 \circ g_1) \times \text{Cl}(h_2 \circ g_2)} & \text{Cl}(A_t(Y)_f) \times \text{Cl}(K_t(Y)) \\ & \searrow \cong & \nearrow \cong \\ & \text{Cl}(g_1) + \text{Cl}(g_2) & \text{Cl}(A_t(Y)) & \xrightarrow{(\text{Cl}(h_1), \text{Cl}(h_2))} & \text{Cl}(A_t(Y)_f) \times \text{Cl}(K_t(Y)) \\ & \uparrow & \uparrow & & \uparrow \\ \mathfrak{S}_0(A) \times \mathfrak{S}_0(D_t(Y)) & \xrightarrow{\mathfrak{S}_0(h_1 \circ g_1) \times \mathfrak{S}_0(h_2 \circ g_2)} & \mathfrak{S}_0(A_t(Y)_f) \times \mathfrak{S}_0(K_t(Y)) \\ & \searrow \mathfrak{S}_0(g_1) \otimes \mathfrak{S}_0(g_2) & \uparrow & \nearrow (\mathfrak{S}_0(h_1), \mathfrak{S}_0(h_2)) \\ & & \mathfrak{S}_0(A_t(Y)) & & \end{array} \tag{4.3.1}$$

Furthermore, it was established that the three maps in the upper triagram are isomorphisms. The map $\mathfrak{S}_0(g_1) \otimes \mathfrak{S}_0(g_2)$ is defined as $([C_1], [C_2]) \mapsto [C_1 \otimes_D C_2]$; the well definedness of this map is given in [Example 1.3](#). The three quadrilateral faces of this diagram commute by [[11](#), Theorem 4.4]. Since the vertical maps are one-to-one, a routine diagram chase shows that the lower triagram also commutes and that each map in the lower triagram is one-to-one, hence so is the map $\mathfrak{S}_0(h_2 \circ g_2)$. Also, the map $\mathfrak{S}_0(h_1 \circ g_1) : \mathfrak{S}_0(A) \rightarrow \mathfrak{S}_0(A[B]_F) \approx \mathfrak{S}_0(A_t(Y)_f)$ is a bijection by [[10](#), Corollary 3.11(b)], via a diagram as in ([3.7.4](#)).

We claim that each of the conditions (1) and (2) implies that $\mathfrak{S}_0(h_2 \circ g_2)$ is bijective.

- (1) Assume that A contains a field, so D contains a field k . Then the map $k_t(Y) \rightarrow K_t(Y)$ is faithfully flat and induces an isomorphism on divisor class groups, hence, the induced map $\mathfrak{S}_0(k_t(Y)) \rightarrow \mathfrak{S}_0(K_t(Y))$ is a bijection by [[10](#), Corollary 3.11(b)]. This map factors through the map $\mathfrak{S}_0(h_2 \circ g_2) : \mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(K_t(Y))$, therefore $\mathfrak{S}_0(h_2 \circ g_2)$ is surjective; since injectivity was already established, the argument in this case is complete.
- (2) Assume now that the natural map $\mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(L_t(Y))$ is surjective. Since this map is also injective, it is bijective. As already observed, the map $\mathfrak{S}_0(L_t(Y)) \rightarrow \mathfrak{S}_0(K_t(Y))$ is bijective, hence so is the composition $\mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(K_t(Y))$.

To complete the proof, use the bottom triagram of ([4.3.1](#)). Each map in this triagram is one-to-one. The claim and the paragraph preceding it show that the horizontal map in the triagram is a bijection. It follows that the diagonal map $(\mathfrak{S}_0(h_1), \mathfrak{S}_0(h_2))$ is also onto, hence a bijection. Thus, all the maps in this triagram are bijections, as desired. □

Corollary 4.4. *Let Y_1, \dots, Y_s be the t -components of Y . The following natural map is bijective*

$$\mathfrak{S}_0(k_t(Y_1)) \times \cdots \times \mathfrak{S}_0(k_t(Y_s)) \xrightarrow{\iota} \mathfrak{S}_0(k_t(Y_1) \otimes_k \cdots \otimes_k k_t(Y_s)) \approx \mathfrak{S}_0(k_t(Y)).$$

Proof. The fact that ι is well-defined and injective follows from [Example 1.3](#) and a standard induction argument. The argument for surjectivity proceeds by induction on $s \geq 1$, with the base case $s = 1$ being trivial. Inductively, assume $s > 1$ and set $\tilde{Y} = Y \setminus Y_s$. Then with $A = k_t(\tilde{Y})$ and $D = k = L$, one has

$$A_t(Y_s) \cong A \otimes_k k_t(Y_s) \cong k_t(\tilde{Y}) \otimes_k k_t(Y_s) \cong k_t(Y).$$

Thus, [Theorem 4.3\(1\)](#) implies that the map ι' in the next commutative diagram is a bijection, while $\tilde{\iota}$ is a bijection by the induction assumption.

$$\begin{array}{ccc} \mathfrak{S}_0(k_t(Y_1)) \times \cdots \times \mathfrak{S}_0(k_t(Y_{s-1})) \times \mathfrak{S}_0(k_t(Y_s)) & \xrightarrow[\approx]{\tilde{\iota} \times \mathfrak{S}_0(k_t(Y_s))} & \mathfrak{S}_0(k_t(\tilde{Y})) \times \mathfrak{S}_0(k_t(Y_s)) \\ & \searrow \tilde{\iota} & \downarrow \approx \iota' \\ & & \mathfrak{S}_0(k_t(Y)) \end{array}$$

It follows that ι is a bijection, as desired. □

Lemma 4.5. *Let B be an integral domain. If $t > 2$, then the element x_{s_i} in $B_t(Y)$ is prime for all $i = 1, \dots, h + 1$.*

Proof. Consider the following cases.

Case 1: $B = k$ and the ladder Y is t -connected. In this case, argue as in the proof in [[4](#), Proposition 4.8]. By [Fact 3.2](#), the minimal prime ideals of f are $q_1, \dots, q_{h+1}, q'_1, \dots, q'_{h+1}$, and p_j when T'_j has type 1. The ring $k_t(Y)$ is a Cohen-Macaulay domain and $x_{s_i} \notin q_i, q'_i, p_j$ for all $1 \leq$

$\ell \leq h + 1$, $1 \leq i \leq h + 1$ and $1 \leq j \leq k$. (Recall that x_{S_i} is a single variable and the prime ideals are defined by minors of size $t - 1 > 1$.) Thus, f, x is a regular sequence in $k_t(Y)$. By Lemma 2.9, the element x_{S_i} is prime in $k_t(Y)_f$. Hence x_{S_i} is prime in $k_t(Y)$.

Case 2: Y is t -connected. Let G be the field of fractions of B . By flatness, the quotient $B_t(Y)/(x_{S_i})$ is a subring of $G_t(Y)/(x_{S_i})$ which is an integral domain by Case 1. Hence, the subring $B_t(Y)/(x_{S_i})$ is an integral domain as well. Hence x_{S_i} is prime in $B_t(Y)$ in this case.

Case 3: Y is t -disconnected. In this case, let Y_1, \dots, Y_s be the t -components of Y . The assumption implies that $s > 1$. Re-order the Y_j 's if necessary to assume that $x_{S_i} \in Y_s$. Set $\tilde{Y} = Y \setminus Y_s$ and $\tilde{B} = B_t(\tilde{Y})$. Then $B_t(Y) \cong \tilde{B}_t(Y_s)$, and hence $B_t(Y)/(x_{S_i}) \cong \tilde{B}_t(Y_s)/(x_{S_i})$, which is a domain by Case 2. \square

We can now use Proposition 2.23 to give a new proof of [5, Theorem 5.1], where we call attention to the corrected values of λ_1 and λ_{h+1} in [5, Theorem 5.1]. (See [5, Example 7.3], specifically, $k_3(Y_1)$, which is Gorenstein).

Corollary 4.6. *Assume that Y is t -connected and satisfies Assumption (d). Let $[\omega]$ be the canonical class of $k_t(Y)$, and let*

$$[\omega] = \sum_{i=1}^{h+1} \lambda_i [q_i] + \sum_{j=1}^k \delta_j [p_j]$$

be the unique representation of $[\omega]$ with respect to the basis of $\text{Cl}(k_t(Y))$. Let $i_j = \min\{i : 1 \leq i \leq h + 1, a_i + t - 2 > c_j\}$. If $h > 0$, then:

$$\begin{aligned} \lambda_1 &= (a_1 + b_1) - (a_0 + b_0) + (t - 2), \\ \lambda_i &= (a_i + b_i) - (a_{i-1} + b_{i-1}) && \text{for } 1 < i \leq h, \\ \lambda_{h+1} &= (a_{h+1} + b_{h+1}) - (a_h + b_h) - (t - 2) && \text{and} \\ \delta_j &= (a_{i_j} + b_{i_j} + 2(t - 2)) - (c_j + d_j) && \text{for } 1 \leq j \leq k. \end{aligned}$$

If $h = 0$, then $\lambda_1 = (a_1 + b_1) - (a_0 + b_0)$.

Proof. Argue as in the proof of [4, Theorem 4.9 (b)] by induction on t . The case $t = 2$ is given by [4, Corollary 2.3 and Proposition 2.4]. For $t > 2$, the ideal (x_{S_i}) is prime for all $i = 1, \dots, h + 1$ by Lemma 4.5; recall that $x = x_{S_1} x_{S_2} \cdots x_{S_{h+1}}$. In the following commutative diagram, the first and third isomorphisms are from Nagata's Theorem [6, Corollary 7.2], and the second isomorphism is from Proposition 2.23.

$$\text{Cl}(k_t(Y)) \xrightarrow{\cong} \text{Cl}(k_t(Y)_x) \xrightarrow[\psi]{\cong} \text{Cl}(k_{t-1}(Z)[B_1]_{\bar{x}}) \xleftarrow{\cong} \text{Cl}(k_{t-1}(Z))$$

The third isomorphism is induced by the natural flat homomorphism $k_{t-1}(Z) \rightarrow k_{t-1}(Z)[B_1]_{\bar{x}}$, thus, it respects canonical classes; and similarly for the other two isomorphisms. Corollary 2.22 implies that we have $\bar{\psi}([(q_i(Y))]) = [(q_i(Z))]$ and $\bar{\psi}([(p_j(Y))]) = [(p_j(Z))]$, hence, one can read the coefficients for $[\omega]$ from the corresponding coefficients for $[\omega_{k_{t-1}(Z)}]$; as we note in Remark 2.21, since Y satisfies Assumption (d), the ladder Z is $(t - 1)$ -connected and satisfies Assumption (d) with respect to $t - 1$, therefore, the natural basis of $\text{Cl}(k_{t-1}(Z))$ corresponds exactly to the natural basis of $\text{Cl}(k_t(Y))$.

Note that if Y has corners (a_i, b_i) with $0 \leq i \leq h + 1$, then Z has corners $(a_0 + 1, b_0)$, $(a_{h+1}, b_{h+1} + 1)$, and $(a_i + 1, b_i + 1)$ for $1 \leq i \leq h$ (without relabeling the variables in Z). From this, it is straightforward to check that the λ 's and δ 's coming from Z are the same as the ones coming from Y . Thus, the desired result follows by induction. \square

All semidualizing modules of the rings from Corollary 4.6 are described below.

Theorem 4.7. *Assume that Y is t -connected and satisfies Assumption (d). Then $\mathbf{k}_t(Y)$ has only trivial semidualizing modules.*

Proof. The proof is similar to that of Corollary 4.6 by induction on t . The case $t=2$ is given by [12, Theorem 3.10]. For the induction step augment the diagram from the proof of Corollary 4.6.

$$\begin{array}{ccccccc} \mathrm{Cl}(\mathbf{k}_t(Y)) & \xrightarrow{\cong} & \mathrm{Cl}(\mathbf{k}_t(Y)_x) & \xrightarrow{\cong} & \mathrm{Cl}(\mathbf{k}_{t-1}(Z)[B_1]_x) & \xleftarrow{\cong} & \mathrm{Cl}(\mathbf{k}_{t-1}(Z)) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathfrak{S}_0(\mathbf{k}_t(Y)) & \xrightarrow{\rho} & \mathfrak{S}_0(\mathbf{k}_t(Y)_x) & \xrightarrow{\approx} & \mathfrak{S}_0(\mathbf{k}_{t-1}(Z)[B_1]_x) & \xleftarrow{\approx} & \mathfrak{S}_0(\mathbf{k}_{t-1}(Z)) \end{array}$$

The vertical maps are the natural inclusions. The second horizontal map on the bottom row is a bijection because it is induced by the ring isomorphism $\mathbf{k}_t(Y)_x \xrightarrow{\cong} \mathbf{k}_{t-1}(Z)[B_1]_x$. The third horizontal map on the bottom row is a bijection by [10, Corollary 3.11(b)]. A diagram chase shows that the map ρ is injective. To show that it is surjective, note that the induction hypothesis implies that $\mathbf{k}_{t-1}(Z)$ has only trivial semidualizing modules, hence so does $\mathbf{k}_t(Y)_x$. It is straightforward to show that the two trivial semidualizing modules (free and dualizing) are in the image of ρ , that is, that ρ is also surjective, as desired. \square

Definition 4.8. Let Y be t -connected (noting that Assumption 4.1 is still active). Recalling that an upper inside corner $T'_j = (c_j, d_j)$ is of type 1 if the $(t-1)$ -minor based on T'_j contains at most one point in B_1 , we say that $T'_j = (c_j, d_j)$ is of **type 1.1** if the $(t-1)$ -minor based on T'_j contains at least one point in B_1 .

Remark 4.9. With Y as in the definition, let k^\bullet be the number of upper inside corners of Y with type 1.1. Conca [5, p. 458] shows that there are sub-ladders $Y_0, \dots, Y_{k^\bullet}$ of Y such that $\mathbf{k}_t(Y) = \Lambda^k / (\mathcal{L})$ where $\Lambda^k = \mathbf{k}_t(Y_0) \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} \mathbf{k}_t(Y_{k^\bullet})$ and $\mathcal{L} = \ell_1, \dots, \ell_s$ is a sequence of linear forms that is Λ^k -regular. Moreover, each linear form ℓ_i is a difference of variables; the point is that the construction takes disjoint ladders and glues them together by identifying certain entries. In particular, the ℓ_i are independent of the coefficient ring. Furthermore, each sub-ladder Y_i has no upper inside corners of type 1.1, i.e., Y_i satisfies Assumption (d). In the case $t=2$, we have $Y = Y_0 \# \dots \# Y_{k^\bullet}$ as in [13, Notation 3.1].

Corollary 4.4 implies that the map

$$\iota : \mathfrak{S}_0(\mathbf{k}_t(Y_0)) \times \dots \times \mathfrak{S}_0(\mathbf{k}_t(Y_{k^\bullet})) \rightarrow \mathfrak{S}_0(\Lambda^k)$$

given by $([C_0], \dots, [C_{k^\bullet}]) \mapsto [C_0 \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} C_{k^\bullet}]$ is well-defined and bijective; and [13, Proposition 2.1(6)] implies that the base-change map $\mathfrak{S}_0(\Lambda^k) \rightarrow \mathfrak{S}_0(\Lambda^k / (\mathcal{L}))$ is well-defined and injective. One point of the next result is that the base-change map is bijective as well. Another point is that one can explicitly describe the image of this map, as follows.

Theorem 4.7 implies that each ring $\mathbf{k}_t(Y_i)$ has only the trivial semidualizing modules $\mathbf{k}_t(Y_i)$ and $\omega_{\mathbf{k}_t(Y_i)}$. The natural map $\mathbf{k}_t(Y_i) \rightarrow \Lambda^k$ is flat; in fact, free. We abuse notation and let $[\omega_{\mathbf{k}_t(Y_i)}] \in \mathrm{Cl}(\Lambda^k)$ denote the image of $[\omega_{\mathbf{k}_t(Y_i)}] \in \mathrm{Cl}(\mathbf{k}_t(Y_i))$ under the induced map $\mathrm{Cl}(\mathbf{k}_t(Y_i)) \rightarrow \mathrm{Cl}(\Lambda^k)$, and we let $[\omega_{\mathbf{k}_t(Y_i)}] \in \mathrm{Cl}(\Lambda^k / (\mathcal{L}))$ denote the image of $[\omega_{\mathbf{k}_t(Y_i)}] \in \mathrm{Cl}(\Lambda^k)$ under the induced map $\mathrm{Cl}(\Lambda^k) \rightarrow \mathrm{Cl}(\Lambda^k / (\mathcal{L}))$. The preceding paragraph with the triviality of $\mathfrak{S}_0(\mathbf{k}_t(Y_i))$ implies that every semidualizing Λ^k -module is of the form $C \cong C_0 \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} C_{k^\bullet}$ where for each i we have $C_i \cong \mathbf{k}_t(Y_i)$ or $C_i \cong \omega_{\mathbf{k}_t(Y_i)}$. Set $\theta_i = 0$ if $C_i \cong \mathbf{k}_t(Y_i)$ and $\theta_i = 1$ otherwise. Then in $\mathrm{Cl}(\Lambda^k)$, we have $[C] = \sum_{i=1}^{k^\bullet} \theta_i [\omega_{\mathbf{k}_t(Y_i)}]$, and the image of $[C]$ in $\mathrm{Cl}(\Lambda^k / (\mathcal{L})) = \mathrm{Cl}(\mathbf{k}_t(Y))$ is $\sum_{i=1}^{k^\bullet} \theta_i [\omega_{\mathbf{k}_t(Y_i)}]$ as well.

A criterion for the Gorensteinness of $\mathbf{k}_t(Y_j)$ in the next result is in [5, Theorem 5.2].

Theorem 4.10. *Assume that Y is t -connected, and use the notation from Remark 4.9. Then*

$$\mathfrak{S}_0(\mathbf{k}_t(Y)) = \left\{ \sum_{j=0}^{k^*} \theta_j [\omega_{\mathbf{k}_t(Y_j)}] : \theta_j = 0 \text{ or } 1 \right\} \approx \prod_{j=0}^{k^*} \mathfrak{S}_0(\mathbf{k}_t(Y_j)).$$

In particular, $|\mathfrak{S}_0(\mathbf{k}_t(Y))| = |\mathfrak{S}_0(\mathbf{k}_t(Y_0))| \cdots |\mathfrak{S}_0(\mathbf{k}_t(Y_{k^}))| = 2^{\varepsilon_0 + \cdots + \varepsilon_{k^*}}$, where $\varepsilon_j = 0$ if $\mathbf{k}_t(Y_j)$ is Gorenstein and $\varepsilon_j = 1$ otherwise.*

Proof. The proof is similar to that of Corollary 4.6 by induction on t . In light of Remark 4.9, it suffices to show that $|\mathfrak{S}_0(\mathbf{k}_t(Y))| \leq |\mathfrak{S}_0(\mathbf{k}_t(Y_0))| \cdots |\mathfrak{S}_0(\mathbf{k}_t(Y_{k^*}))|$.

The case $t=2$ follows from [13, Theorem 3.12]. If $t > 2$, then we consider the diagram from the proof of Theorem 4.7.

$$\begin{array}{ccccccc} \mathrm{Cl}(\mathbf{k}_t(Y)) & \xrightarrow{\cong} & \mathrm{Cl}(\mathbf{k}_t(Y)_x) & \xrightarrow{\cong} & \mathrm{Cl}(\mathbf{k}_{t-1}(Z)[B_1]_x) & \xleftarrow{\cong} & \mathrm{Cl}(\mathbf{k}_{t-1}(Z)) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathfrak{S}_0(\mathbf{k}_t(Y)) & \xrightarrow{\rho} & \mathfrak{S}_0(\mathbf{k}_t(Y)_x) & \xrightarrow{\cong} & \mathfrak{S}_0(\mathbf{k}_{t-1}(Z)[B_1]_x) & \xleftarrow{\cong} & \mathfrak{S}_0(\mathbf{k}_{t-1}(Z)) \end{array}$$

Because of the injectivity/bijjectivity of the maps in the bottom row, it suffices to show that $|\mathfrak{S}_0(\mathbf{k}_{t-1}(Z))| = |\mathfrak{S}_0(\mathbf{k}_t(Y_0))| \cdots |\mathfrak{S}_0(\mathbf{k}_t(Y_{k^*}))|$.

To this end, the analysis of Remark 2.21 shows that there are $(t-1)$ -connected subladders Z_0, \dots, Z_{k^*} of Z corresponding exactly to the subladders Y_0, \dots, Y_{k^*} of Y such that

- (1) each Z_ℓ satisfies Assumption (d),
- (2) $\mathbf{k}_{t-1}(Z_\ell)$ is Gorenstein if and only if $\mathbf{k}_t(Y_\ell)$ is Gorenstein, and
- (3) there are integers u and $0 = k_0 < \cdots < k_u < k_{u+1} = k^* + 1$ such that the ladders $Z_{k_i} \cup \cdots \cup Z_{k_{i+1}-1}$ for $i = 0, \dots, u$ are the $(t-1)$ -components of Z .

Thus, our argument will be complete once we verify the next sequence of equalities.

$$\begin{aligned} |\mathfrak{S}_0(\mathbf{k}_{t-1}(Z))| &= |\mathfrak{S}_0(\mathbf{k}_{t-1}(Z_0 \cup \cdots \cup Z_{k_1-1}))| \cdots |\mathfrak{S}_0(\mathbf{k}_{t-1}(Z_{k_u} \cup \cdots \cup Z_{k^*}))| \\ &= |\mathfrak{S}_0(\mathbf{k}_{t-1}(Z_0))| \cdots |\mathfrak{S}_0(\mathbf{k}_{t-1}(Z_{k^*}))| \\ &= |\mathfrak{S}_0(\mathbf{k}_t(Y_0))| \cdots |\mathfrak{S}_0(\mathbf{k}_t(Y_{k^*}))| \end{aligned}$$

The first equality is from Corollary 4.4, and the second equality is from our induction hypothesis applied to the $(t-1)$ -connected ladders $Z_{k_i} \cup \cdots \cup Z_{k_{i+1}-1}$. The third equality is from Theorem 4.7, with help from conditions (1)–(2) above, since each Y_ℓ satisfies Assumption (d). \square

Example 4.11. We compute $\mathfrak{S}_0(\mathbf{k}_3(L_i))$ for the ladders L_i from Section 3. The sub-ladders Y_ℓ of each L_i , if they exist, are shown below; see also Example 2.3.

$\begin{array}{cccc} & X_{13} & X_{14} & X_{15} \\ & X_{23} & X_{24} & X_{25} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \\ X_{51} & X_{52} & X_{53} & X_{54} \end{array}$ <p style="text-align: center;">Ladders Y_1, Y_2 for L_1</p>	$\begin{array}{ccc} X_{13} & X_{14} & X_{15} \\ X_{23} & X_{24} & X_{25} \\ X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \\ X_{51} & X_{52} & X_{53} \end{array}$ <p style="text-align: center;">Ladders Y_1, Y_2 for L_2</p>
$\begin{array}{cccc} X_{13} & X_{14} & X_{15} & \\ X_{23} & X_{24} & X_{25} & \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \\ X_{51} & X_{52} & X_{53} & X_{54} \\ X_{61} & X_{62} & X_{63} & X_{64} \end{array}$ <p style="text-align: center;">Ladder L_3: no Y_ℓ</p>	$\begin{array}{ccc} X_{13} & X_{14} & X_{15} \\ X_{23} & X_{24} & X_{25} \\ X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \\ X_{51} & X_{52} & X_{53} \end{array}$ <p style="text-align: center;">Ladders Y_1, Y_2 for L_4</p>

In $Y = L_1$, the upper inside corner T'_1 has type 1.1, as per [Definition 4.8](#), so $k^\bullet = 1$. For $\ell = 1, 2$, since Y_ℓ is rectangular and not square, the ring $k_3(Y_\ell)$ is not Gorenstein by [[5](#), [Theorem 5.2](#)]. Thus, [Theorem 4.10](#) implies that $|\mathfrak{S}_0(k_3(L_1))| = 4$.

For $Y = L_2$, decompose Y similarly into Y_1 and Y_2 as displayed above. In this case, each ladder Y_ℓ is rectangular, but only one is square. Thus, one of the rings $k_3(Y_\ell)$ is Gorenstein, while the other is not, again by [[5](#), [Theorem 5.2](#)], thus, it follows that $|\mathfrak{S}_0(k_3(L_2))| = 2$.

In $Y = L_3$, since $k^\bullet = 0$, there is no decomposition of L_3 . In this case, since the underlying matrix of variables X is not square, $k_3(Y)$ is not Gorenstein by [[5](#), [Theorem 5.2](#)], hence, $|\mathfrak{S}_0(k_3(L_3))| = 2$.

Lastly, $Y = L_4$ decomposes into two square subladders Y_i , thus each ring $k_3(Y_\ell)$ is Gorenstein by [[5](#), [Theorem 5.2](#)], and hence, $|\mathfrak{S}_0(k_3(L_4))| = 1$. In particular, $k_3(L_4)$ is Gorenstein.

The next result is a version of [Theorem 4.10](#) for t -disconnected ladders. □

Corollary 4.12. *Let Y_1, \dots, Y_s be the t -components of Y . For $i = 1, \dots, s$, let k_i^\bullet be the number of upper inside corners of Y_i of type 1.1, and let $Y_{i,0}, \dots, Y_{i,k_i^\bullet}$ be subladders of Y_i as in [Remark 4.9](#). Then*

$$\mathfrak{S}_0(k_t(Y)) \approx \prod_{i=1}^s \prod_{j=0}^{k_i^\bullet} \mathfrak{S}_0(k_t(Y_{i,j})).$$

In particular, $|\mathfrak{S}_0(k_t(Y))| = \prod_{i=1}^s \prod_{j=0}^{k_i^\bullet} |\mathfrak{S}_0(k_t(Y_{i,j}))| = 2^{(\sum_{i=1}^s \sum_{j=0}^{k_i^\bullet} \varepsilon_{i,j})}$, where $\varepsilon_{i,j} = 0$ if $k_t(Y_{i,j})$ is Gorenstein and $\varepsilon_{i,j} = 1$ otherwise.

Proof. Combine [Corollary 4.4](#) and [Theorem 4.10](#). □

Now we return to our goal of removing the assumptions (1) and (2) from [Theorem 4.3](#).

Remark 4.13. Assume that Y is t -connected and use the notation of [Remark 4.9](#). Set $\Lambda^D = D_t(Y_0) \otimes_D \dots \otimes_D D_t(Y_{k^\bullet})$. Since the ℓ_i 's from [Remark 4.9](#) are differences of variables, it is straightforward to show that $\Lambda^D/(\underline{\ell}) \cong D_t(Y)$. Note that Λ^D is a normal domain by [Lemma 2.6](#) applied to the disjoint union of the Y_i .

We claim that the sequence ℓ is Λ^D -regular. Let $K_D = K^{\Lambda^D}(\ell)$ be the Koszul complex over Λ^D , and let K_D^+ denote the augmented Koszul complex

$$K_D^+ : \quad 0 \rightarrow \Lambda^D \rightarrow \cdots \rightarrow \Lambda^D \rightarrow \Lambda^D/(\ell) \rightarrow 0.$$

As noted in [Remark 4.9](#), the sequence ℓ is Λ^k -regular for every field k , hence $K_k^+ \cong K_{\mathbb{Z}}^+ \otimes_{\mathbb{Z}} k$ is exact. Now, use [[7](#), Lemma 2.6] as in the proof of [Lemma 2.7](#) to conclude that K_D^+ is exact. Since the sequence ℓ is homogeneous, the exactness of K_D^+ implies that the sequence is regular, as desired.

The following lemmas allow [Theorem 4.3](#) to be invoked in the proof of our main result, namely [Theorem 4.16](#), which is a culmination of our efforts to describe the semidualizing modules of $A_t(Y)$, where Y is not necessarily t -connected.

Lemma 4.14. *When Y is t -connected, the natural map $\mathfrak{S}_0(D_t(Y)) \xrightarrow{\phi} \mathfrak{S}_0(L_t(Y))$ is bijective.*

Proof. [Lemma 4.2](#) shows that ϕ is injective; it remains to show that ϕ is surjective. Use the notation from [Remarks 4.9](#) and [4.13](#). Then [Theorem 4.10](#) shows that an arbitrary $[C] \in \mathfrak{S}_0(L_t(Y))$ is the image of $[C_0 \otimes_L \cdots \otimes_L C_{k^*}]$ under the natural map $\mathfrak{S}_0(\Lambda^L) \rightarrow \mathfrak{S}_0(\Lambda^L/(\ell))$. Moreover, *loc. cit.* shows that for $i = 0, \dots, k^*$ we have $C_i \cong L_t(Y_i) \cong L_t(Y_i) \otimes_{D_t(Y_i)} D_t(Y_i)$ or $C_i \cong \omega_{L_t(Y_i)} \cong L_t(Y_i) \otimes_{D_t(Y_i)} \omega_{D_t(Y_i)}$ since the map $D_t(Y) \rightarrow L_t(Y)$ is a localization map. It follows that there are $[B_i] \in \mathfrak{S}_0(D_t(Y_i))$ equal to $[D_t(Y_i)]$ or $[\omega_{D_t(Y_i)}]$ such that $[C_0 \otimes_L \cdots \otimes_L C_{k^*}]$ is the image of $[B_0 \otimes_L \cdots \otimes_L B_{k^*}] \in \mathfrak{S}_0(\Lambda^D)$ under the natural map $\mathfrak{S}_0(\Lambda^D) \rightarrow \mathfrak{S}_0(\Lambda^L)$. Let $[B] \in \mathfrak{S}_0(\Lambda^D/(\ell))$ be the image of $[B_0 \otimes_L \cdots \otimes_L B_{k^*}]$ under the natural map $\mathfrak{S}_0(\Lambda^D) \rightarrow \mathfrak{S}_0(\Lambda^D/(\ell))$, as in the next diagram.

$$\begin{array}{ccccc} [B_0 \otimes_L \cdots \otimes_L B_{k^*}] & \xrightarrow{\hspace{10em}} & & & [C_0 \otimes_L \cdots \otimes_L C_{k^*}] \\ \downarrow & & & & \downarrow \\ & & \mathfrak{S}_0(\Lambda^D) \xrightarrow{\hspace{2em}} \mathfrak{S}_0(\Lambda^L) & & \\ & & \downarrow \qquad \qquad \downarrow & & \\ & & \mathfrak{S}_0(\Lambda^D/(\ell)) \xrightarrow{\phi} \mathfrak{S}_0(\Lambda^D/(\ell)) & & [C] \\ \downarrow & & & & \downarrow \\ [B] & & & & \end{array}$$

Commutativity of the diagram shows that $[C] = \phi([B])$, so ϕ is surjective. □

The version of [Lemma 4.14](#) given next considers ladders that are t -disconnected.

Lemma 4.15. *The natural map $\phi : \mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(L_t(Y))$ is bijective.*

Proof. Let Y_1, \dots, Y_s be the t -components of Y . [Lemma 4.14](#) implies that each induced map $\phi_i : \mathfrak{S}_0(D_t(Y_i)) \rightarrow \mathfrak{S}_0(L_t(Y_i))$ is bijective. Using the isomorphisms $D_t(Y) \cong D_t(Y_1) \otimes_D \cdots \otimes_D D_t(Y_s)$ and $L_t(Y) \cong L_t(Y_1) \otimes_L \cdots \otimes_L L_t(Y_s)$, the horizontal maps in the following commutative diagram and the bijectivity of one of them are from [Example 1.3](#) and [Corollary 4.4](#).

$$\begin{array}{ccc} \prod_{i=1}^s \mathfrak{S}_0(D_t(Y_i)) & \xrightarrow{\hspace{2em}} & \mathfrak{S}_0(D_t(Y)) \\ \approx \downarrow \prod_i \phi_i & & \downarrow \phi \\ \prod_{i=1}^s \mathfrak{S}_0(L_t(Y_i)) & \xrightarrow{\approx} & \mathfrak{S}_0(L_t(Y)) \end{array}$$

The map ϕ is injective by [Lemma 2.14](#). A straightforward diagram chase shows that ϕ is also surjective, as desired. □

The main theorem of this paper is below. It is a version of [Theorem 4.3](#) without the conditions (1)–(2). Note that the ladder Y may be t -disconnected. See [Example 1.3](#) for a discussion of transitivity for \trianglelefteq .

Theorem 4.16. *Let A be a normal domain and Y a ladder as in [Assumption 4.1](#). The natural maps below are bijections:*

$$\mathfrak{S}_0(A) \times \mathfrak{S}_0(D_t(Y)) \rightarrow \mathfrak{S}_0(A_t(Y)) \rightarrow \mathfrak{S}_0(A_t(Y)_f) \times \mathfrak{S}_0(K_t(Y)).$$

In particular, with notation as in [Corollary 4.12](#),

$$\begin{aligned} \mathfrak{S}_0(A_t(Y)) &\approx \mathfrak{S}_0(A_t(Y)_f) \times \mathfrak{S}_0(K_t(Y)) \approx \mathfrak{S}_0(A) \times \mathfrak{S}_0(K_t(Y)) \\ &\approx \mathfrak{S}_0(A) \times \prod_{i=1}^s \prod_{j=0}^{k_i^*} \mathfrak{S}_0(\mathfrak{k}_t(Y_{i,j})) \approx \mathfrak{S}_0(A) \times \{0, 1\}^e, \end{aligned}$$

where e is the number of ladders $Y_{i,j}$ such that $K_t(Y_{i,j})$ is not Gorenstein. Thus, $\mathfrak{S}_0(A_t(Y))$ is finite if and only if $\mathfrak{S}_0(A)$ is finite, and the relation \trianglelefteq on $\mathfrak{S}_0(A_t(Y))$ is transitive if and only if the relation \trianglelefteq on $\mathfrak{S}_0(A)$ is so.

Proof. The bijections follow from [Theorem 4.3\(2\)](#), [Corollary 4.12](#), and [Lemma 4.15](#). Consequently, since $\{0, 1\}^e$ is finite, it is clear that $\mathfrak{S}_0(A_t(Y))$ and $\mathfrak{S}_0(A)$ are either both finite or neither is. Next, the set $\{0, 1\}$ is totally ordered by \trianglelefteq (that is, \leq). Then, in particular, \trianglelefteq is a partial order on any finite product of copies of $\{0, 1\}$; i.e., \trianglelefteq is transitive. Therefore, if \trianglelefteq is a partial order on $\mathfrak{S}_0(A)$, then it is a partial order on $\mathfrak{S}_0(A_t(Y))$, and vice versa. \square

Corollary 4.17. *If Y is t -connected and satisfies [Assumption \(d\)](#), then*

$$\mathfrak{S}_0(A_t(Y)) \approx \mathfrak{S}_0(A) \times \mathfrak{S}_0(K_t(Y)) \approx \begin{cases} \mathfrak{S}_0(A) & \text{if } K_t(Y) \text{ is Gorenstein} \\ \mathfrak{S}_0(A) \times \{0, 1\} & \text{if } K_t(Y) \text{ is not Gorenstein.} \end{cases}$$

Proof. This is the case of [Theorem 4.16](#) with $s = 1$ and $k_0^* = 0$. \square

Remark 4.18. It is natural to ask what $\mathfrak{S}_0(\mathfrak{k}[Y]/I_t(Y))$ looks like, as is done for $\mathfrak{k}[X]/I_t(X)$ in [10]. In that work, the fact that $\mathfrak{k}_t(X)$ satisfies Serre’s condition (R_2) is crucial, since it allows one to conclude that $\mathfrak{S}_0(\mathfrak{k}[X]/I_t(X)) \approx \mathfrak{S}_0(\mathfrak{k}_t(X))$; see [10, Corollary 3.13]. If $\mathfrak{k}_t(Y)$ were to satisfy (R_2) , then it would similarly hold that $\mathfrak{S}_0(\mathfrak{k}_t[Y]/I_t(Y)) \approx \mathfrak{S}_0(\mathfrak{k}_t(Y))$, and furthermore, it would simplify some of our regular sequence arguments above. Thus, we pose the following question.

Question 4.19. Under [Assumption 4.1](#), must $\mathfrak{k}_t(Y)$ satisfy (R_2) ?

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