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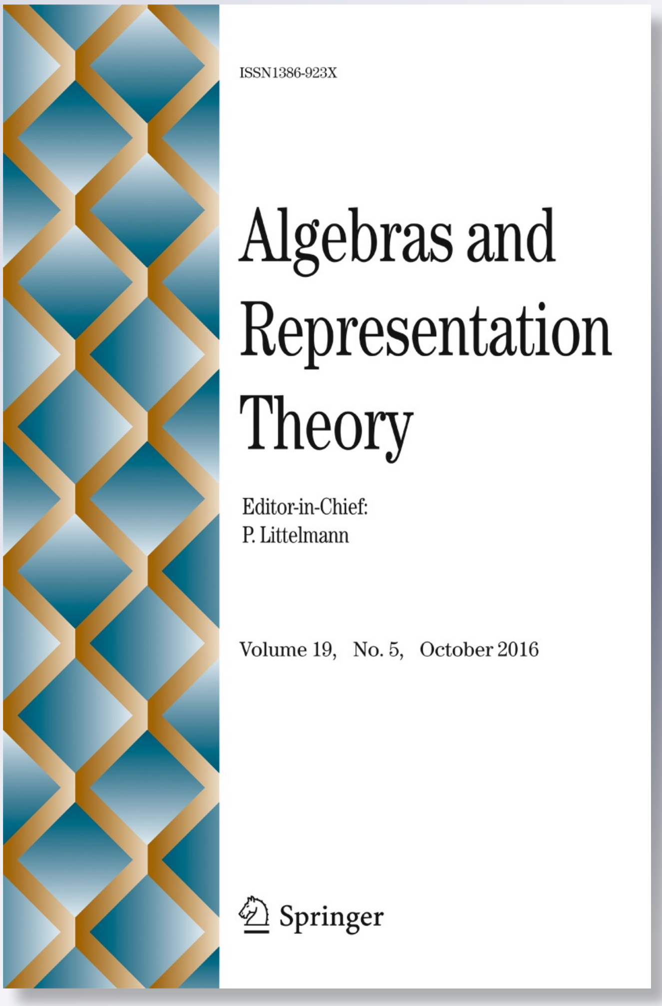

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Extended Local Cohomology and Local Homology

Sean Sather-Wagstaff[1] **· Richard Wicklein**[2]

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Abstract We present an in-depth exploration of the module structures of local (co)homology modules (moreover, for complexes) over the completion $\widehat{R}^{\mathfrak{a}}$ of a commutative noetherian ring R with respect to a proper ideal $\mathfrak{a}$. In particular, we extend Greenlees-May Duality and MGM Equivalence to track behavior over $\widehat{R}^{\mathfrak{a}}$, not just over R. We apply this to the study of two recent versions of homological finiteness for complexes, and to certain isomorphisms, with a view toward further applications. We also discuss subtleties and simplifications in the computations of these functors.

Keywords Adic finiteness · Cohomologically cofinite complexes · Derived local cohomology · Derived local homology · Greenlees-May duality · MGM equivalence · Support

Mathematics Subject Classification (2010) 13B35 · 13C12 · 13D09 · 13D45

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1 Introduction

Throughout this paper let R be a commutative noetherian ring, let $\mathfrak{a} \subsetneq R$ be a proper ideal of R, and let $\widehat{R}^{\mathfrak{a}}$ be the $\mathfrak{a}$-adic completion of R. Let K denote the Koszul complex over R on a finite generating sequence for $\mathfrak{a}$. We work in the derived category $\mathcal{D}(R)$ with objects the R-complexes indexed homologically $X = \cdots \to X_i \to X_{i-1} \to \cdots$ and the full subcategory $\mathcal{D}_b(R)$ of complexes with bounded homology. Isomorphisms in $\mathcal{D}(R)$ are marked by the symbol $\simeq$. The right derived functor of Hom is $\mathbf{R}\mathrm{Hom}_R(-,-)$, and the left derived functor of $- \otimes_R -$ is $- \otimes^{\mathbf{L}}_R -$. See, e.g., [13, 37, 38] for foundations and Section 2 for background.

This work is part 4 in a series of papers on derived local cohomology and derived local homology. It builds on our previous papers [26, 29, 30], and it is applied in the papers [27, 28].

The starting point for this paper is the following fact. Given an R-module M, each local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion, so it has a natural $\widehat{R}^{\mathfrak{a}}$-module structure. The completion $\widehat{M}^{\mathfrak{a}}$ also has a natural $\widehat{R}^{\mathfrak{a}}$-module structure. More generally, given an R-complex X, the derived local cohomology complex $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$ and the derived local homology complex $\mathbf{L}\Lambda^{\mathfrak{a}}(X)$ are naturally complexes over $\widehat{R}^{\mathfrak{a}}$. These complexes are constructed by applying the torsion and completion functors, respectively, to appropriate resolutions of X. For clarity, we write $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)$ and $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)$ when we are working over $\widehat{R}^{\mathfrak{a}}$. See Section 2 for definitions and notation. Note that Section 3 documents some subtleties and a simplification involved in these constructions.

In this paper, we investigate how standard facts for the R-complexes $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$ and $\mathbf{L}\Lambda^{\mathfrak{a}}(X)$ extend to the $\widehat{R}^{\mathfrak{a}}$-complexes $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)$ and $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)$. Our primary motivation comes from work of Alonso Tarrío, Jeremías López, and Lipman [1]; Greenlees and May [12]; Matlis [19, 20]; and Porta, Shaul, and Yekutieli [23]. For instance, the main results of Section 4, summarized next, extend Greenlees-May Duality and MGM Equivalence (named for Matlis, Greenlees, and May) to this setting. See Theorems 4.8, 4.12, and 4.13 in the body of the paper.

Theorem 1.1 *Let $X, Y \in \mathcal{D}(R)$ be given.*

(a) *There are natural isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$:*

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y)) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(Y)) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y)).\end{aligned}$$

(b) *The functor $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \colon \mathcal{D}(R)_{\mathfrak{a}\text{-tor}} \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}}$ is a quasi-equivalence with quasi-inverse given by the forgetful functor $Q \colon \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}} \to \mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$.*

(c) *The functor $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \colon \mathcal{D}(R)_{\mathfrak{a}\text{-comp}} \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}}$ is a quasi-equivalence with quasi-inverse given by the forgetful functor $Q \colon \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}} \to \mathcal{D}(R)_{\mathfrak{a}\text{-comp}}$.*

Here $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ and $\mathcal{D}(R)_{\mathfrak{a}\text{-comp}}$ are the full subcategories of $\mathcal{D}(R)$ consisting of the complexes X and Y, respectively, such that the natural morphisms $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \to X$ and $Y \to \mathbf{L}\Lambda^{\mathfrak{a}}(Y)$ are isomorphisms.

Section 5 investigates the flat and injective dimensions of the complexes $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)$ and $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)$ over $\widehat{R}^{\mathfrak{a}}$. In most cases, we bound these above by flat and injective dimensions of X over R.

In Section 6, we use these constructions to explain the connection between the "cohomologically $\mathfrak{a}$-adically cofinite" complexes of Porta, Shaul, and Yekutieli [24] and our "$\mathfrak{a}$-adically finite" complexes from [30]. The first of these notions is only defined when R

is $\mathfrak{a}$-adically complete; in this setting, we show that our notion is equivalent; see Proposition 6.2. In general, Theorem 6.3 shows that the category of $\mathfrak{a}$-adically finite complexes over R is quasi-equivalent to the category of homologically finite complexes over $\widehat{R}^{\mathfrak{a}}$, hence to the category of cohomologically $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-adically cofinite complexes over $\widehat{R}^{\mathfrak{a}}$.

The concluding Section 7 exhibits some isomorphisms for use in [27, 28]. For instance, the following result is Theorem 7.3 from the body of the paper.

Theorem 1.2 *Let $R \to S$ be a homomorphism of commutative noetherian rings, and let $X \in \mathcal{D}_b(R)$ be $\mathfrak{a}$-adically finite over R. If $S \otimes^{\mathbf{L}}_R X \in \mathcal{D}_b(S)$, e.g., if $\mathrm{fd}_R(S) < \infty$, then there is an isomorphism in $\mathcal{D}(\widehat{S}^{\mathfrak{a}S})$:*

$$\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}\left(S \otimes^{\mathbf{L}}_R X\right).$$

When X is homologically finite, this is a straightfoward consequence of the isomorphisms $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X$ and $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_R X) \simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R X)$. In general, though, this is more subtle. And while it may seem esoteric, it is key for understanding some base-change properties in [28].

2 Background

2.1 Derived Categories

In addition to the categories mentioned in Section 1, we also consider the following full triangulated subcategories of $\mathcal{D}(R)$:

$\mathcal{D}_+(R)$: objects are the complexes X with $\mathrm{H}_i(X) = 0$ for $i \ll 0$.
$\mathcal{D}_-(R)$: objects are the complexes X with $\mathrm{H}_i(X) = 0$ for $i \gg 0$.
$\mathcal{D}^{\mathrm{f}}(R)$: objects are the complexes X with $\mathrm{H}_i(X)$ finitely generated for all i.

Doubly ornamented subcategories are defined as intersections, for instance, $\mathcal{D}^{\mathrm{f}}_+(R) := \mathcal{D}^{\mathrm{f}}(R) \bigcap \mathcal{D}_+(R)$.

2.2 Resolutions

An R-complex F is *semi-flat*[1] if the functor $F \otimes_R -$ respects quasiisomorphisms and each module F_i is flat over R, that is, if $F \otimes_R -$ respects injective quasiisomorphisms (see [2, 1.2.F]). A *semi-flat resolution* of an R-complex X is a quasiisomorphism $F \xrightarrow{\simeq} X$ such that F is semi-flat. The *flat dimension* of X

$$\mathrm{fd}_R(X) := \inf\{\sup\{i \in \mathbb{Z} \mid F_i \neq 0\} \mid F \xrightarrow{\simeq} X \text{ is a semi-flat resolution}\}$$

is the length of the shortest bounded semi-flat resolution of X, if one exists. The projective and injective versions (semi-projective, etc.) are defined similarly.

For the following items, consult [2, Section 1] or [3, Chapters 3 and 5]. Bounded below complexes of projective R-modules are semi-projective, bounded below complexes of flat R-modules are semi-flat, and bounded above complexes of injective R-modules are semi-injective. Semi-projective R-complexes are semi-flat, and every R-complex admits a

[1] In the literature, semi-flat complexes are sometimes called "K-flat" or "DG-flat".

semi-projective resolution (hence, a semi-flat one) and a semi-injective resolution, by work of Spaltenstein [35].

2.3 Support and Co-support

The following notions are due to Foxby [9] and Benson, Iyengar, and Krause [5].

Definition 2.1 Let $X \in \mathcal{D}(R)$. The *small support* and *small co-support* of X are

$$\begin{aligned} \operatorname{supp}_R(X) &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \not\simeq 0\} \\ \text{co-supp}_R(X) &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), X) \not\simeq 0\} \end{aligned}$$

where $\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

Much of the following is from [9] when X and Y are appropriately bounded and from [4, 5] in general. We refer to [30] as a matter of convenience.

Fact 2.2 Let $X, Y \in \mathcal{D}(R)$. Then we have $\operatorname{supp}_R(X) = \emptyset$ if and only if $X \simeq 0$ if and only if $\text{co-supp}_R(X) = \emptyset$, because of [30, Fact 3.4 and Proposition 4.7(a)]. Also, by [30, Propositions 3.12 and 4.10] we have

$$\begin{aligned} \operatorname{supp}_R(X \otimes^{\mathbf{L}}_R Y) &= \operatorname{supp}_R(X) \bigcap \operatorname{supp}_R(Y) \\ \text{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(X, Y)) &= \operatorname{supp}_R(X) \bigcap \text{co-supp}_R(Y). \end{aligned}$$

2.4 Derived Local (Co)homology

The next notions go back to Grothendieck [14], and Matlis [19, 20], respectively; see also [1, 18]. Let $\Lambda^{\mathfrak{a}}$ denote the $\mathfrak{a}$-adic completion functor, and $\Gamma_{\mathfrak{a}}$ is the $\mathfrak{a}$-torsion functor, i.e., for an R-module M we have

$$\Lambda^{\mathfrak{a}}(M) = \widehat{M}^{\mathfrak{a}} \qquad\qquad \Gamma_{\mathfrak{a}}(M) = \{x \in M \mid \mathfrak{a}^n x = 0 \text{ for } n \gg 0\}.$$

A module M is $\mathfrak{a}$-torsion if $\Gamma_{\mathfrak{a}}(M) = M$.

The associated left and right derived functors (i.e., *derived local homology and cohomology* functors) are $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. Specifically, given an R-complex $X \in \mathcal{D}(R)$ and a semi-flat resolution $F \xrightarrow{\simeq} X$ and a semi-injective resolution $X \xrightarrow{\simeq} I$, then we have $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(F)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(I)$. Note that these definitions yield natural transformations $\mathbf{R}\Gamma_{\mathfrak{a}} \xrightarrow{\varepsilon_{\mathfrak{a}}} \mathrm{id} \xrightarrow{\vartheta^{\mathfrak{a}}} \mathbf{L}\Lambda^{\mathfrak{a}}$, induced by the natural morphisms $\Gamma_{\mathfrak{a}}(I) \xrightarrow{\iota^I_{\mathfrak{a}}} I$ and $F \xrightarrow{\nu^{\mathfrak{a}}_F} \Lambda^{\mathfrak{a}}(F)$. Let $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ denote the full subcategory of $\mathcal{D}(R)$ of all complexes X such that the morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \xrightarrow{\varepsilon^X_{\mathfrak{a}}} X$ is an isomorphism, and let $\mathcal{D}(R)_{\mathfrak{a}\text{-comp}}$ denote the full subcategory of $\mathcal{D}(R)$ of all complexes Y such that the morphism $Y \xrightarrow{\vartheta^{\mathfrak{a}}_Y} \mathbf{L}\Lambda^{\mathfrak{a}}(Y)$ is an isomorphism.

The definitions of $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$ and $\mathbf{L}\Lambda^{\mathfrak{a}}(X)$ yield complexes over the completion $\widehat{R}^{\mathfrak{a}}$, and we denote by $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ and $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ the associated functors $\mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$. If $Q\colon \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$ is the forgetful functor, then it follows readily that $Q \circ \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \simeq \mathbf{L}\Lambda^{\mathfrak{a}}$ and $Q \circ \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \simeq \mathbf{R}\Gamma_{\mathfrak{a}}$.

Fact 2.3 If $X \in \mathcal{D}^{\mathrm{f}}_{+}(R)$, then there is a natural isomorphism $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} X$ in $\mathcal{D}(R)$ by [10, Proposition 2.7]. Moreover, the proof of this result shows that there is a natural isomorphism $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} X$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$.[2] By [1, Theorem (0.3) and Corollary (3.2.5.i)], there are natural isomorphisms of functors

$$\mathbf{R}\Gamma_{\mathfrak{a}}(-) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_{R} - \qquad \mathbf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathbf{R}\mathrm{Hom}_{R}(\mathbf{R}\Gamma_{\mathfrak{a}}(R), -).$$

More generally, from [32, Theorems 3.2 and 3.6] there are natural isomorphisms of functors $\mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$

$$\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(-) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} \mathbf{R}\Gamma_{\mathfrak{a}}(-) \simeq \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_{R} -$$
$$\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(-) \simeq \mathbf{R}\mathrm{Hom}_{R}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(R), -) \simeq \mathbf{R}\mathrm{Hom}_{R}(\widehat{R}^{\mathfrak{a}}, \mathbf{L}\Lambda^{\mathfrak{a}}(-)).$$

Here are Greenlees-May duality and MGM equivalence.

Fact 2.4 Given $X, Y \in \mathcal{D}(R)$, we have natural isomorphisms in $\mathcal{D}(R)$

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{R}(\mathbf{R}\Gamma_{\mathfrak{a}}(X), \mathbf{R}\Gamma_{\mathfrak{a}}(Y)) &\xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_{R}(\mathbf{R}\Gamma_{\mathfrak{a}}(X), Y)\\ &\xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_{R}(\mathbf{R}\Gamma_{\mathfrak{a}}(X), \mathbf{L}\Lambda^{\mathfrak{a}}(Y))\\ &\xleftarrow{\simeq} \mathbf{R}\mathrm{Hom}_{R}(X, \mathbf{L}\Lambda^{\mathfrak{a}}(Y))\\ &\xleftarrow{\simeq} \mathbf{R}\mathrm{Hom}_{R}(\mathbf{L}\Lambda^{\mathfrak{a}}(X), \mathbf{L}\Lambda^{\mathfrak{a}}(Y))\end{aligned}$$

induced by $\varepsilon_{\mathfrak{a}}$ and $\vartheta^{\mathfrak{a}}$; see [1, Theorem (0.3)*].[3], From [1, Corollary to Theorem (0.3)*] and [23, Theorem 1.2] the next natural morphisms are isomorphisms:

$$\mathbf{R}\Gamma_{\mathfrak{a}} \circ \mathrm{id} \xrightarrow[\simeq]{\mathbf{R}\Gamma_{\mathfrak{a}} \circ \vartheta^{\mathfrak{a}}} \mathbf{R}\Gamma_{\mathfrak{a}} \circ \mathbf{L}\Lambda^{\mathfrak{a}} \qquad \mathbf{L}\Lambda^{\mathfrak{a}} \circ \mathbf{R}\Gamma_{\mathfrak{a}} \xrightarrow[\simeq]{\mathbf{L}\Lambda^{\mathfrak{a}} \circ \varepsilon_{\mathfrak{a}}} \mathbf{L}\Lambda^{\mathfrak{a}} \circ \mathrm{id}$$
$$\mathbf{R}\Gamma_{\mathfrak{a}} \circ \mathbf{R}\Gamma_{\mathfrak{a}} \xrightarrow[\simeq]{\varepsilon_{\mathfrak{a}} \circ \mathbf{R}\Gamma_{\mathfrak{a}}} \mathrm{id} \circ \mathbf{R}\Gamma_{\mathfrak{a}} \qquad \mathrm{id} \circ \mathbf{L}\Lambda^{\mathfrak{a}} \xrightarrow[\simeq]{\vartheta^{\mathfrak{a}} \circ \mathbf{L}\Lambda^{\mathfrak{a}}} \mathbf{L}\Lambda^{\mathfrak{a}} \circ \mathbf{L}\Lambda^{\mathfrak{a}}.$$

The second row of isomorphisms here shows that the essential image of $\mathbf{R}\Gamma_{\mathfrak{a}}$ in $\mathcal{D}(R)$ is $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$, and the essential image of $\mathbf{L}\Lambda^{\mathfrak{a}}$ in $\mathcal{D}(R)$ is $\mathcal{D}(R)_{\mathfrak{a}\text{-comp}}$.

Fact 2.5 Let $X \in \mathcal{D}(R)$. Then we know that $\mathrm{supp}_{R}(X) \subseteq \mathrm{V}(\mathfrak{a})$ if and only if $X \in \mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ if and only if each homology module $\mathrm{H}_{i}(X)$ is $\mathfrak{a}$-torsion, by [30, Proposition 5.4] and [23, Corollary 4.32].[4]

Fact 2.6 Let $Y \in \mathcal{D}(R)$, and consider the following exact triangles in $\mathcal{D}(R)$.

$$\mathbf{R}\Gamma_{\mathfrak{a}}(Y) \xrightarrow{\varepsilon^{Y}_{\mathfrak{a}}} Y \to B \to \qquad\qquad Y \xrightarrow{\vartheta^{\mathfrak{a}}_{Y}} \mathbf{L}\Lambda^{\mathfrak{a}}(Y) \to C \to$$

[2] This is based on the fact that, for a finitely generated free R-module L, induction on the rank of L shows that the natural isomorphism $\widehat{R}^{\mathfrak{a}} \otimes_{R} L \cong \widehat{L}^{\mathfrak{a}}$ is $\widehat{R}^{\mathfrak{a}}$-linear.

[3] See also [23, Thoerem 6.12]. In addition, we have [23, Remark 6.14] for a discussion of some aspects of this result, and [25] for a correction.

[4] The affiliated characterization of $\mathcal{D}(R)_{\mathfrak{a}\text{-comp}}$ in terms of co-support is not needed here.

By [5, Corollary 4.9] one has

$$\operatorname{supp}_R(B) \bigcap \mathrm{V}(\mathfrak{a}) = \emptyset = \operatorname{co-supp}_R(B) \bigcap \mathrm{V}(\mathfrak{a})$$
$$\operatorname{supp}_R(C) \bigcap \mathrm{V}(\mathfrak{a}) = \emptyset = \operatorname{co-supp}_R(C) \bigcap \mathrm{V}(\mathfrak{a}).$$

Fact 2.7 The following natural transformations are isomorphisms

$$\mathbf{R}\Gamma_{\mathfrak{a}} \circ \mathbf{R}\Gamma_{\mathfrak{a}} \xrightarrow[\simeq]{\mathbf{R}\Gamma_{\mathfrak{a}} \circ \varepsilon_{\mathfrak{a}}} \mathbf{R}\Gamma_{\mathfrak{a}} \circ \mathrm{id} \qquad \mathbf{L}\Lambda^{\mathfrak{a}} \circ \mathrm{id} \xrightarrow[\simeq]{\mathbf{L}\Lambda^{\mathfrak{a}} \circ \vartheta^{\mathfrak{a}}} \mathbf{L}\Lambda^{\mathfrak{a}} \circ \mathbf{L}\Lambda^{\mathfrak{a}}$$

by [4, Lemma 3.4(a)], [5, (4.2)], and [18, Proposition 3.5.3]. Note the slight difference between these and the last two isomorphisms in Fact 2.4; while 2.4 establishes the idempotence of $\mathbf{R}\Gamma_{\mathfrak{a}}$ and $\mathbf{L}\Lambda^{\mathfrak{a}}$ (which holds in more generality than we state here), we actually need these specific morphisms to be isomorphisms here. Note also that one can obtain these isomorphisms as the special case $X = \mathbf{R}\Gamma_{\mathfrak{a}}(R)$ of the next result.

Lemma 2.8 *Let* $X \in \mathcal{D}(R)$ *be such that* $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ *or* $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$*; e.g.,* $X \simeq K$*. Then the following natural transformations are isomorphisms.*

$$X \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(-) \xrightarrow[\simeq]{X \otimes^{\mathbf{L}}_R \varepsilon_{\mathfrak{a}}} X \otimes^{\mathbf{L}}_R - \xrightarrow[\simeq]{X \otimes^{\mathbf{L}}_R \vartheta^{\mathfrak{a}}} X \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(-)$$
$$\mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\Gamma_{\mathfrak{a}}(-)) \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}(X, \varepsilon_{\mathfrak{a}})} \mathbf{R}\mathrm{Hom}_R(X, -) \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}(X, \vartheta^{\mathfrak{a}})} \mathbf{R}\mathrm{Hom}_R(X, \mathbf{L}\Lambda^{\mathfrak{a}}(-))$$
$$\mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(-), X) \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}(\vartheta^{\mathfrak{a}}, X)} \mathbf{R}\mathrm{Hom}_R(-, X) \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}(\varepsilon_{\mathfrak{a}}, X)} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(-), X)$$

Proof Let $Y \in \mathcal{D}(R)$, and consider the exact triangle

$$\mathbf{R}\Gamma_{\mathfrak{a}}(Y) \xrightarrow{\varepsilon^Y_{\mathfrak{a}}} Y \to B \to$$

in $\mathcal{D}(R)$, and the following induced triangle.

$$X \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(Y) \xrightarrow{X \otimes^{\mathbf{L}}_R \varepsilon^Y_{\mathfrak{a}}} X \otimes^{\mathbf{L}}_R Y \to X \otimes^{\mathbf{L}}_R B \to \tag{2.8.1}$$

Facts 2.5 and 2.6 yield the next sequence

$$\operatorname{supp}_R(X \otimes^{\mathbf{L}}_R B) = \operatorname{supp}_R(X) \bigcap \operatorname{supp}_R(B) \subseteq \mathrm{V}(\mathfrak{a}) \bigcap \operatorname{supp}_R(B) = \emptyset.$$

We conclude that $X \otimes^{\mathbf{L}}_R B \simeq 0$, so the exact triangle (2.8.1) implies that $X \otimes^{\mathbf{L}}_R \varepsilon^Y_{\mathfrak{a}}$ is an isomorphism in $\mathcal{D}(R)$. The other isomorphisms from the statement of this result follow similarly. □

2.5 Adic Finiteness

The next two items take their cues from work of Hartshorne [15], Kawasaki [16, 17], and Melkersson [21].

Fact 2.9 ([30, Theorem 1.3]) For $X \in \mathcal{D}_{\mathrm{b}}(R)$, the next conditions are equivalent.

(i) One has $K^R(\underline{y}) \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ for some (equivalently for every) generating sequence $\underline{y}$ of $\mathfrak{a}$.

(ii) One has $X \otimes^{\mathbf{L}}_R R/\mathfrak{a} \in \mathcal{D}^{\mathrm{f}}(R)$.
(iii) One has $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X) \in \mathcal{D}^{\mathrm{f}}(R)$.
(iv) One has $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$.

Definition 2.10 An R-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ is $\mathfrak{a}$-adically finite if it satisfies the equivalent conditions of Fact 2.9 and $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

Example 2.11 Let $X \in \mathcal{D}_{\mathrm{b}}(R)$ be given.

(a) If $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then we have $\mathrm{supp}_R(X) = \mathrm{V}(\mathfrak{b})$ for some ideal $\mathfrak{b}$, and it follows that X is $\mathfrak{a}$-adically finite whenever $\mathfrak{a} \subseteq \mathfrak{b}$. (The case $\mathfrak{a} = 0$ is from [30, Proposition 7.8(a)], and the general case follows readily.)
(b) K and $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ are $\mathfrak{a}$-adically finite, by [30, Fact 3.4 and Theorem 7.10].
(c) The homology modules of X are artinian if and only if there is an ideal $\mathfrak{a}$ of finite colength (i.e., such that $R/\mathfrak{a}$ is artinian) such that X is $\mathfrak{a}$-adically finite, by [29, Proposition 5.11].

3 Computing Derived Functors

Lipman [18, Lemma 3.5.1] shows that, to compute $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$, one need not use a semi-injective resolution of X; it suffices to use an injective resolution of X, i.e., a quasiisomorphism $X \xrightarrow{\simeq} I$ where I is a complex of injective R-modules. This fact, along with the fact that $\Gamma_{\mathfrak{a}}$ transforms complexes of injective modules into complexes of injective modules, is the essence of Lipman's proof of the first isomorphism in Fact 2.7. Our next example shows that [18, Lemma 3.5.1] is crucial here, as it shows that $\Gamma_{\mathfrak{a}}$ need not respect the class of semi-injective complexes.

Example 3.1 We consider the following special case of a construction of Chen and Iyengar [6, Proposition 2.7]. Let k be a field, set $R = k[\![X, Y]\!]/(X^2)$, and let x and y denote the residues in R of X and Y, respectively. Set $\mathfrak{n} = (x, y)R$ and $\mathfrak{p} = xR$, and consider the injective hull $E = E_R(R/\mathfrak{n})$. We consider the complexes

$$F := (\cdots \xrightarrow{x} R \xrightarrow{x} R \to 0) \qquad G := \bigoplus_{n\in\mathbb{Z}} \Sigma^n F$$

$$I := \mathrm{Hom}_R(F, E) \cong (0 \to E \xrightarrow{x} E \xrightarrow{x} \cdots) \qquad J := \mathrm{Hom}_R(G, E) \cong \prod_{n\in\mathbb{Z}} \Sigma^n I$$

The complex F gives a semi-projective (hence, semi-flat) resolution of R/xR, and I yields a semi-injective resolution of $M := \mathrm{Hom}_R(R/xR, E)$. Also, G describes a semi-projective (hence semi-flat) resolution of $\bigoplus_{n\in\mathbb{Z}} \Sigma^n R/xR$, and J provides a semi-injective resolution of $\prod_{n\in\mathbb{Z}} \Sigma^n M$. Furthermore, $\mathfrak{p}$ is an associated prime of each module $J_i \cong \prod_{n\geqslant i} E$, but $J_{\mathfrak{p}}$ is not semi-injective over $R_{\mathfrak{p}}$: Chen and Iyengar prove this last claim by noting that $J_{\mathfrak{p}}$ is acyclic but not contractible over $R_{\mathfrak{p}}$; it follows that $J_{\mathfrak{p}}$ is acyclic but not contractible over R, so not semi-injective over R.

Claim: the complex $\Gamma_{\mathfrak{m}}(J)$ is not semi-injective over R. To see this, note that each module J_i is a direct sum of copies of E and copies of $E_R(R/\mathfrak{p})$. It follows that we have the following natural short exact sequence of complexes:

$$0 \to \Gamma_{\mathfrak{m}}(J) \to J \to J_{\mathfrak{p}} \to 0.$$

Since J is semi-injective over R, the fact that these are complexes of injective R-modules implies that $\Gamma_{\mathfrak{m}}(J)$ is semi-injective over R if and only if $J_{\mathfrak{p}}$ is so; hence, by the previous paragraph, the claim is established.

The following lemma is used in the subsequent example.

Lemma 3.2 *Assume that* $(R, \mathfrak{m}, k)$ *is local and that* $\operatorname{Spec}(R) = \{\mathfrak{m}, \mathfrak{p}\}$ *with* $\mathfrak{p} \subsetneq \mathfrak{m}$. *Let* L *be a free* R*-module, and set* $L' := \widehat{L}^{\mathfrak{m}}/L$. *Then* $\widehat{L}^{\mathfrak{m}}$ *is flat over* R, *and* L' *is a flat* $R_{\mathfrak{p}}$*-module (hence,* L' *is also flat over* R*). If* L *is not finitely generated over* R, *then* $L' \neq 0$.

Proof For the sake of brevity, set $\widehat{L} := \widehat{L}^{\mathfrak{m}}$.

Krull's Intersection Theorem implies that L is $\mathfrak{m}$-adically separated, so the natural map $L \to \widehat{L}$ is injective. Thus, the definition of L' makes sense. The module $\widehat{L}$ is flat over R by [8, Theorem 5.3.28]. Also, the proof of [8, Proposition 6.7.6] shows that the inclusion $L \to \widehat{L}$ is pure. It follows that L' is also flat over R.

Claim: L' is naturally an $R_{\mathfrak{p}}$-module. To see this, first note that [24, Corollary 1.9(1)] shows that the natural map $L/\mathfrak{m}L \to \widehat{L}/\mathfrak{m}\widehat{L}$ is an isomorphism. Right-exactness of $- \otimes_R k$ applied to the natural sequence

$$0 \to L \xrightarrow{\nu_L^{\mathfrak{m}}} \widehat{L} \to L' \to 0 \tag{3.2.1}$$

implies that $L' \otimes_R k = 0$. The fact that L' is flat implies that $\operatorname{Tor}_i^R(L', k) = 0$ for all $i \neq 0$, and we conclude that $\mathfrak{m} \notin \operatorname{supp}_R(L')$. It follows from [9, Remark 2.9] that the minimal injective resolution of L' over R contains no summand of the form $E_R(k)$. Thus, the minimal injective resolution of L' over R has the form

$$0 \to E_R(R/\mathfrak{p})^{(\mu^0)} \xrightarrow{\partial_0} E_R(R/\mathfrak{p})^{(\mu^1)} \to \cdots.$$

Since each module $E_R(R/\mathfrak{p})^{(\mu^i)}$ is naturally an $R_{\mathfrak{p}}$-module, the R-module homomorphism ∂_0 is $R_{\mathfrak{p}}$-linear. Since L' is isomorphic to $\operatorname{Ker}(\partial_0)$, the claim follows.

We include here a second proof of the claim, as it sheds a different light on the module L', which is somewhat mysterious to us. For this second proof, it suffices to show that $L' \cong \operatorname{Ext}_R^1(R_{\mathfrak{p}}, L)$, since this Ext-module inherits an $R_{\mathfrak{p}}$-structure from the first slot. (Note that this proof is intimately related to results of [23, 39].) Since L is flat over R, there is an isomorphism $\mathbf{L}\Lambda^{\mathfrak{m}}(L) \simeq \widehat{L}$ in $\mathcal{D}(R)$. Thus, the exact sequence (3.2.1) provides an exact triangle

$$L \xrightarrow{\vartheta_L^{\mathfrak{m}}} \mathbf{L}\Lambda^{\mathfrak{m}}(L) \to L' \to \tag{3.2.2}$$

in $\mathcal{D}(R)$. Using the structure of $\operatorname{Spec}(R)$ again, as in the proof of the claim in Example 3.1, we have the next exact triangle

$$\mathbf{R}\Gamma_{\mathfrak{m}}(R) \xrightarrow{\varepsilon_{\mathfrak{m}}^R} R \to R_{\mathfrak{p}} \to$$

in $\mathcal{D}(R)$. In the language of [23, Section 7], this says that we have $\mathbf{R}\Gamma_{0/\mathfrak{m}}(R) \simeq R_{\mathfrak{p}}$. The proof of [23, Lemma 7.2] exhibits an exact triangle of the following form.

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{0/\mathfrak{m}}(R), L) \to L \xrightarrow{\vartheta_L^{\mathfrak{m}}} \mathbf{L}\Lambda^{\mathfrak{m}}(L) \to$$

Rotating this triangle, we obtain the next one.

$$L \xrightarrow{\vartheta_L^{\mathfrak{m}}} \mathbf{L}\Lambda^{\mathfrak{m}}(L) \to \Sigma\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{0/\mathfrak{m}}(R), L) \to$$

Combining this with the triangle in (3.2.2), we conclude that

$$L' \simeq \Sigma\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{0/\mathfrak{m}}(R), L) \simeq \Sigma\mathbf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, L).$$

Applying H_0, we obtain the next isomorphism

$$L' \cong \mathrm{H}_0(L') \cong \mathrm{H}_0(\Sigma\mathbf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, L)) \cong \mathrm{Ext}^1_R(R_{\mathfrak{p}}, L).$$

This concludes the second proof of the claim.

We conclude the proof of the lemma. Since L' is a flat R-module, the localization $L'_{\mathfrak{p}} \cong L'$ is a flat $R_{\mathfrak{p}}$-module; the isomorphism follows from the above claim. Finally, assume that L is not finitely generated. To show that $L' \neq 0$, it suffices to show that L is not complete. Since L is free of infinite rank, consider a sequence $e_1, e_2, \ldots$ of distinct elements of a basis of L. From our assumption on $\mathrm{Spec}(R)$, the nilradical of R is $\mathfrak{p}$. Let $y \in \mathfrak{m} \setminus \mathfrak{p}$, which is not nilpotent. It follows that the Cauchy sequence $\left\{\sum_{i=1}^{n} y^i e_i\right\}_{n=1}^{\infty}$ in L does not converge in L, so L is not complete, as desired. □

Similar to the previous example, the next one shows that $\Lambda^{\mathfrak{a}}$ does not respect the class of semi-flat complexes.

Example 3.3 We continue with the set-up of Example 3.1.

Claim: the complex $\Lambda^{\mathfrak{m}}(G)$ is not semi-flat over R. Since each module G_i is free over R, Lemma 3.2 provides a short exact sequence

$$0 \to G \xrightarrow{\nu_G^{\mathfrak{m}}} \Lambda^{\mathfrak{m}}(G) \to G' \to 0$$

where each module G'_i is a non-zero flat $R_{\mathfrak{p}}$-module. Since $R_{\mathfrak{p}}$ is Gorenstein and artinian, it follows that G'_i is injective over $R_{\mathfrak{p}}$, hence

$$G_i \cong E_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}))^{(\mu_i)} \cong E_R(R/\mathfrak{p})^{(\mu_i)} \cong R_{\mathfrak{p}}^{(\mu_i)}$$

for some set μ_i.

Since G is semi-flat, and the modules $\Lambda^{\mathfrak{m}}(G)$ and G'_i are flat over R for all i, to establish the claim, it suffices to show that G' is not semi-flat over R. To accomplish this, we follow the lead of Chen and Iyengar [6] by showing that G' is exact and minimal. (Recall that a complex Z is *minimal* if every homotopy equivalence $Z \to Z$ is an isomorphism.) If G' were semi-flat, this would imply $G' = 0$, which we know to be false.

Note that each homology module $\mathrm{H}_i(G) \cong R/xR$ is $\mathfrak{m}$-adically complete, since R is so. Hence, from [39, Theorem 3] we conclude that the natural morphism $\vartheta_G^{\mathfrak{m}}: G \to \mathbf{L}\Lambda^{\mathfrak{m}}(G)$ is an isomorphism in $\mathcal{D}(R)$. In other words, the chain map $\nu_G^{\mathfrak{m}}: G \to \Lambda^{\mathfrak{m}}(G)$ is a quasiisomorphism. It follows that G' is exact.

Since G' is a complex of injective R-modules, to show that G' is minimal, it suffices to show that the inclusion $\oplus_{i\in\mathbb{Z}} \mathrm{Ker}(\partial_i^{G'}) \subseteq \oplus_{i\in\mathbb{Z}} G'_i$ is an injective envelope over R; see, e.g., [7, Lemma 5.4.16]. As every R-module has an injective envelope, we see from [8, Corollary 6.4.4] that it suffices to show that each inclusion $\mathrm{Ker}(\partial_i^{G'}) \subseteq G'_i$ is an injective envelope over R, i.e., over $R_{\mathfrak{p}}$ as G' is an $R_{\mathfrak{p}}$-complex.

Because $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p}, \kappa(\mathfrak{p}))$ is a local ring, the inclusion $\mathrm{Soc}_{R_\mathfrak{p}}(E_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))^{(\mu_i)}) \subseteq E_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))^{(\mu_i)}$ is an injective envelope. That is, the inclusion $\mathrm{Soc}_{R_\mathfrak{p}}(G'_i) \subseteq G'_i$ is an injective envelope. On the other hand, the isomorphism $G'_i \cong R_\mathfrak{p}^{(\mu_i)}$ works with the conditions $\mathfrak{p}R_\mathfrak{p} = xR_\mathfrak{p} \neq 0$ and $x^2 = 0$ to imply that $\mathrm{Soc}_{R_\mathfrak{p}}(G'_i) = xG'_i = (0 :_{G_i} x)$. Thus, we are reduced to showing that $\mathrm{Ker}(\partial_i^{G'}) = xG'_i$.

By construction, we have $\partial_j^G(G_j) \subseteq xG_{j-1}$ for all j, so $\partial_j^G(xG_j) \subseteq x^2G_{j-1} = 0$. In other words, the composition

$$G_j \xrightarrow{x} G_j \xrightarrow{\partial_j^G} G_{j-1}$$

is 0. Applying the functor $\Lambda^\mathfrak{m}$, we see that the composition

$$\Lambda^\mathfrak{m}(G_j) \xrightarrow{x} \Lambda^\mathfrak{m}(G_j) \xrightarrow{\Lambda^\mathfrak{m}(\partial_j^G)} \Lambda^\mathfrak{m}(G_{j-1})$$

is 0, that is, the composition

$$\Lambda^\mathfrak{m}(G)_j \xrightarrow{x} \Lambda^\mathfrak{m}(G)_j \xrightarrow{\partial_j^{\Lambda^\mathfrak{m}(G)}} \Lambda^\mathfrak{m}(G)_{j-1}$$

is 0. Since $\partial^{G'}$ is induced by $\partial_j^{\Lambda^\mathfrak{m}(G)}$, it follows that the composition

$$G'_j \xrightarrow{x} G'_j \xrightarrow{\partial_j^{G'}} G'_{j-1}$$

is 0. We conclude that $0 = \partial_j^{G'}(xG'_j) = x\partial_j^{G'}(G'_j)$. The first equality here implies that $\mathrm{Ker}(\partial_i^{G'}) \supseteq xG'_i$. Using the second equality here, we see that $\mathrm{Ker}(\partial_i^{G'}) = \partial_{i+1}^{G'}(G'_{i+1}) \subseteq (0 :_{G_i} x) = xG'_i$. We conclude that $\mathrm{Ker}(\partial_i^{G'}) = xG'_i$. This establishes the claim and concludes the example.

As with [18, Lemma 3.5.1], our next result shows that one can use flat resolutions to compute $\mathbf{L}\Lambda^\mathfrak{a}$.

Proposition 3.4 *Let $X \in \mathcal{D}(R)$, and let F be a complex of flat R-modules such that $F \simeq X$ in $\mathcal{D}(R)$. Then we have $\mathbf{L}\Lambda^\mathfrak{a}(X) \simeq \Lambda^\mathfrak{a}(F)$ in $\mathcal{D}(R)$.*

Proof Claim: given any exact complex G of flat R-modules, one has $\Lambda^\mathfrak{a}(G) \simeq 0$. To establish the claim, let $i \in \mathbb{Z}$ be given; we need to show that $\mathrm{H}_i(\Lambda^\mathfrak{a}(G)) = 0$. Assume that the ideal $\mathfrak{a}$ is generated by a sequence of length n. Then the "telescope complex" T is a projective resolution of $\mathbf{R}\Gamma_\mathfrak{a}(R)$ concentrated in degrees $0, \ldots, -n$; see [23]. From this, we conclude that

$$\mathrm{H}_{n+1}(\mathbf{L}\Lambda^\mathfrak{a}(M)) \cong \mathrm{H}_{n+1}(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_\mathfrak{a}(R), M)) = 0$$

for each R-module M.

Set $M := \mathrm{Im}(\partial_{i-n-1}^G)$, so we have a flat resolution

$$\cdots \xrightarrow{\partial_{i+1}^G} G_i \xrightarrow{\partial_i^G} \cdots \xrightarrow{\partial_{i-n}^G} G_{i-n-1} \to M \to 0.$$

From this and the previous paragraph, we conclude that

$$0 = \mathrm{H}_{n+1}(\mathbf{L}\Lambda^\mathfrak{a}(M)) \cong \mathrm{H}_i(\Lambda^\mathfrak{a}(G)).$$

Since i is arbitrary, this establishes the claim.

Now we prove our proposition. Let $P \xrightarrow{\simeq} X$ be a semi-projective resolution. The isomorphism $X \simeq F$ in $\mathcal{D}(R)$ provides a quasiisomorphism $\phi\colon P \xrightarrow{\simeq} F$. The mapping cone $G := \operatorname{Cone}(\phi)$ is an exact complex of flat R-modules, so the above claim implies that

$$0 \simeq \Lambda^{\mathfrak{a}}(G) = \Lambda^{\mathfrak{a}}(\operatorname{Cone}(\phi)) \simeq \operatorname{Cone}(\Lambda^{\mathfrak{a}}(\phi)).$$

We conclude that $\Lambda^{\mathfrak{a}}(\phi)\colon \Lambda^{\mathfrak{a}}(P) \to \Lambda^{\mathfrak{a}}(F)$ is a quasiisomorphism, so $\Lambda^{\mathfrak{a}}(F) \simeq \Lambda^{\mathfrak{a}}(P) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(X)$, as desired. □

Corollary 3.5 *Let $X \in \mathcal{D}(R)$, and let F be a complex of flat R-modules such that $F \simeq X$ in $\mathcal{D}(R)$. Then we have $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(F)$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$.*

4 Extended Greenlees-May Duality and MGM Equivalence

In this section, we extend previous isomorphisms to cover the functors $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ and $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$, beginning with extended versions of parts of Fact 2.4.

Lemma 4.1 *The natural transformations*

$$\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \circ \mathbf{R}\Gamma_{\mathfrak{a}} \xrightarrow[\simeq]{\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}\circ\varepsilon_{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \circ \mathrm{id} \xrightarrow[\simeq]{\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}\circ\vartheta^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \circ \mathbf{L}\Lambda^{\mathfrak{a}}$$

are isomorphisms of functors $\mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$.

Proof For the first isomorphism, let $X \in \mathcal{D}(R)$ be given, and choose semi-projective resolutions $P \xrightarrow{\simeq} X$ and $Q \xrightarrow{\simeq} \mathbf{R}\Gamma_{\mathfrak{a}}(X)$. Let $\phi\colon Q \to P$ be a chain map representing the natural morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \xrightarrow{\varepsilon_{\mathfrak{a}}^X} X$. Then the induced morphism $\mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(X)) \to \mathbf{L}\Lambda^{\mathfrak{a}}(X)$ is an isomorphism in $\mathcal{D}(R)$ by Fact 2.4, and it is represented by $\Lambda^{\mathfrak{a}}(\phi)\colon \Lambda^{\mathfrak{a}}(Q) \to \Lambda^{\mathfrak{a}}(P)$. It follows that $\Lambda^{\mathfrak{a}}(\phi)$ is a quasi-isomorphism. Since $\Lambda^{\mathfrak{a}}(\phi)$ also represents the natural morphism $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(X)) \to \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, this morphism is also an isomorphism, as desired.

For $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \circ \vartheta^{\mathfrak{a}}$, argue similarly with Fact 2.7 in place of Fact 2.4. □

Lemma 4.2 *There is a natural isomorphism $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \circ \mathbf{R}\Gamma_{\mathfrak{a}} \simeq \mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \circ \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ of functors $\mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$*

Proof Let $X \in \mathcal{D}(R)$ be given, and choose a semi-flat resolution $F \xrightarrow{\simeq} \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)$ over $\widehat{R}^{\mathfrak{a}}$. Since $\widehat{R}^{\mathfrak{a}}$ is flat over R, the complex F is also semi-flat over R, so it is a semi-flat resolution of $Q(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(X)$ over R. This explains the isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ in the next display

$$\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(X)) \simeq \Lambda^{\mathfrak{a}}(F) = \Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(F) \simeq \mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X))$$

The equality comes from the fact that F is an $\widehat{R}^{\mathfrak{a}}$-complex. □

Theorem 4.3 *The natural transformation*

$$\mathrm{id} \circ \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \xrightarrow[\simeq]{\vartheta^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \circ \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}} \mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \circ \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$$

is an isomorphism of functors $\mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$. In other words, the essential image of $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ is contained in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}-\mathrm{comp}}$.

Proof Let $X \in \mathcal{D}(R)$ be given. According to Fact 2.4, it suffices to show that $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)$ is of the form $\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(Y)$ for some $Y \in \mathcal{D}(\widehat{R}^{\mathfrak{a}})$. To this end, the next isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, from Lemmas 4.1 and 4.2

$$\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(X)) \simeq \mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X))$$

show that the complex $Y = \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)$ satisfies this condition. □

The next few results are proved like the preceding ones, using semi-injective resolutions for the first two.

Lemma 4.4 *The natural transformations*

$$\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \circ \mathbf{R}\Gamma_{\mathfrak{a}} \xrightarrow[\simeq]{\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \circ \varepsilon_{\mathfrak{a}}} \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \circ \mathrm{id} \xrightarrow[\simeq]{\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \circ \vartheta^{\mathfrak{a}}} \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \circ \mathbf{L}\Lambda^{\mathfrak{a}}$$

are isomorphisms of functors $\mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$.

Lemma 4.5 *There is a natural isomorphism $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \circ \mathbf{L}\Lambda^{\mathfrak{a}} \simeq \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \circ \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ of functors $\mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$.*

Theorem 4.6 *The natural transformation*

$$\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \circ \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \xrightarrow[\simeq]{\varepsilon_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \circ \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}} \mathrm{id} \circ \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$$

is an isomorphism of functors $\mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$. In other words, the essential image of $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ is contained in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}}$.

Remark 4.7 After we announced the results of this paper, we learned from Liran Shaul that he has obtained some of the results of this section independently and in more generality. For instance, the isomorphism $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \simeq \mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \circ \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ from Lemmas 4.1 and 4.2 is [31, Theorem 1.7] in a non-noetherian setting.

Here is a version of Greenlees-May Duality 2.4 for our extended functors. It is Theorem 1.1(a) from the introduction. Note that an important special case of this result and Proposition 4.11 below can be found in [33, Lemma 2.5].

Theorem 4.8 *Given $X, Y \in \mathcal{D}(R)$, there are natural isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$:*

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y)) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(Y)) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y)).\end{aligned}$$

Proof The first isomorphism follows from the next sequence

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X),\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y)) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)),\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y)))\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)),\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y)))\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(X)),\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(Y)))\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X),\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(Y))\end{aligned}$$

wherein the isomorphisms are from Theorem 4.3, Greenlees-May duality 2.4, and Lemma 4.5, and Lemma 4.4, respectively.

The second isomorphism follows from the next sequence

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X),\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(Y)) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(X)),\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(Y)))\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)),\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y)))\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)),\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y))\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(X)),\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y))\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X),\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Y))\end{aligned}$$

which is justified similarly. □

Remark 4.9 It is reasonable to ask whether versions of other isomorphisms from Greenlees-May duality 2.4 hold in our set-up. For instance, given $X, Y \in \mathcal{D}(R)$, we have the natural isomorphism

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X),\mathbf{R}\Gamma_{\mathfrak{a}}(Y)) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X),Y).$$

In our set-up, the naive question would ask whether we have

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X),\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(Y)) \overset{?}{\simeq} \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X),Y)$$

in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. However, this doesn't make sense as Y does not come equipped with an $\widehat{R}^{\mathfrak{a}}$-structure. On the other hand, see Propositions 4.10–4.11 and their proofs for some isomorphisms involving $\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X),Y)$ when $Y \in \mathcal{D}(\widehat{R}^{\mathfrak{a}})$.

The next two results are akin to [32, Corollary 3.9].

Proposition 4.10 *Let $X \in \mathcal{D}(R)$ and $Y \in \mathcal{D}(\widehat{R}^{\mathfrak{a}})$ be given. Then there are isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$*

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X),Y) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X),\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(Y))\\ \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y,\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(Y),\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X))\end{aligned}$$

which are natural in X and Y.

Proof We verify the second isomorphism; the verification of the first one is similar. In the following display, the first step is from Theorem 4.3:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y,\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y,\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)))\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(Y),\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X))\end{aligned}$$

The second step is Greenlees-May duality 2.4. □

Proposition 4.11 *Let $X \in \mathcal{D}(R)$ and $Y \in \mathcal{D}(\widehat{R}^{\mathfrak{a}})$ be given, and let $Q \colon \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$ denote the forgetful functor. Then there are isomorphisms in $\mathcal{D}(R)$*

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X), Y) \simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{L}\Lambda^{\mathfrak{a}}(Q(Y)))$$
$$\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(Q(Y)), X)$$

which are natural in X and Y.

Proof We verify the second isomorphism; the verification of the first one is similar. In the following display, the first step is from Fact 2.3:

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(Y, \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, \mathbf{L}\Lambda^{\mathfrak{a}}(X)))\\
&\simeq \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} Y, \mathbf{L}\Lambda^{\mathfrak{a}}(X))\\
&\simeq \mathbf{R}\mathrm{Hom}_R(Q(Y), \mathbf{L}\Lambda^{\mathfrak{a}}(X))\\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(Q(Y)), X).
\end{aligned}$$

The second step is Hom-tensor adjointness, and the third one is tensor-cancellation. The last step is an adjointness isomorphism that follows from Fact 2.3. □

The next two results form our extension of MGM equivalence, as described in parts (b) and (c) of Theorem 1.1 from the introduction; see Remark 4.15.

Theorem 4.12 *The functor $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \colon \mathcal{D}(R)_{\mathfrak{a}\text{-tor}} \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}}$ and the forgetful functor $Q \colon \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}} \to \mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ are quasi-inverse equivalences.*

Proof Note that $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ maps $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ into $\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}}$ by Theorem 4.6. The forgetful functor Q maps $\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}}$ into $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ by Fact 2.9 and [29, Lemma 5.3].

Next, we show that the composition $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \circ Q$ is equivalent to the identity. For this, consider a complex $X \in \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}}$; it suffices to show that $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(Q(X)) \simeq X$ over $\widehat{R}^{\mathfrak{a}}$. Fix a semi-injective resolution $X \xrightarrow{\simeq} I$ over $\widehat{R}^{\mathfrak{a}}$. Since $\widehat{R}^{\mathfrak{a}}$ is flat over R, this yields a semi-injective resolution $Q(X) \xrightarrow{\simeq} Q(I)$. Also, the torsion functors $\Gamma_{\mathfrak{a}}$ and $\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}$ are the same when restricted to $\widehat{R}^{\mathfrak{a}}$-complexes. By definition of $\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}}$, the inclusion morphism $\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(I) \to I$ is a quasiisomorphism. Thus, we have

$$\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(Q(X)) \simeq \Gamma_{\mathfrak{a}}(Q(I)) = \Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(I) \simeq I \simeq X$$

over $\widehat{R}^{\mathfrak{a}}$, hence the desired conclusion.

Lastly, the composition $Q \circ \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ is $\mathbf{R}\Gamma_{\mathfrak{a}}$. When restricted to $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$, this is isomorphic to the identity by definition of $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$. □

The next result is proved like the previous one, using a semi-flat resolution.

Theorem 4.13 *The functor $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \colon \mathcal{D}(R)_{\mathfrak{a}\text{-comp}} \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}}$ and the forgetful functor $Q \colon \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}} \to \mathcal{D}(R)_{\mathfrak{a}\text{-comp}}$ are quasi-inverse equivalences.*

The next result is a consequence of the proofs of Theorems 4.12 and 4.13; it is a special case of [23, Theorem 6.5].

Corollary 4.14 *There are natural isomorphisms $Q \circ \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \simeq \mathbf{R}\Gamma_{\mathfrak{a}} \circ Q$ and $Q \circ \mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \simeq \mathbf{L}\Lambda^{\mathfrak{a}} \circ Q$ of functors $\mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$.*

Remark 4.15 Theorems 4.12 and 4.13 have several consequences. First, they augment Theorems 4.3 and 4.6 by showing that the essential images of $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ and $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ are *equal to* $\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}}$ and $\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}}$, respectively.

Second, they show that MGM equivalences over R and $\widehat{R}^{\mathfrak{a}}$ are essentially the same. These equivalences appear in the rows of the following diagram

$$\begin{array}{ccc}
\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}} & \underset{\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}}{\overset{\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}}{\underset{\simeq}{\rightleftarrows}}} & \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}} \\
\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \uparrow \ \simeq \ \downarrow Q & & \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \uparrow \ \simeq \ \downarrow Q \\
\mathcal{D}(R)_{\mathfrak{a}\text{-tor}} & \underset{\mathbf{R}\Gamma_{\mathfrak{a}}}{\overset{\mathbf{L}\Lambda^{\mathfrak{a}}}{\underset{\simeq}{\rightleftarrows}}} & \mathcal{D}(R)_{\mathfrak{a}\text{-comp}}
\end{array}$$

while our results provide the equivalences in the columns. Various versions of this diagram commute, up to natural isomorphism. For instance, the first diagram in the next display commutes by Lemmas 4.1 and 4.2. The second one is from Lemmas 4.4 and 4.5.

$$\begin{array}{ccc}
\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}} & \xrightarrow[\simeq]{\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}} & \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}} \\
\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \uparrow \simeq & & \simeq \uparrow \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \\
\mathcal{D}(R)_{\mathfrak{a}\text{-tor}} & \xrightarrow[\simeq]{\mathbf{L}\Lambda^{\mathfrak{a}}} & \mathcal{D}(R)_{\mathfrak{a}\text{-comp}}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}} & \xleftarrow[\simeq]{\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}} & \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}} \\
\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \uparrow \simeq & & \simeq \uparrow \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \\
\mathcal{D}(R)_{\mathfrak{a}\text{-tor}} & \xleftarrow[\simeq]{\mathbf{R}\Gamma_{\mathfrak{a}}} & \mathcal{D}(R)_{\mathfrak{a}\text{-comp}}
\end{array}$$

Corollary 4.14 explains the commutativity of next two diagrams.

$$\begin{array}{ccc}
\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}} & \xrightarrow[\simeq]{\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}} & \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}} \\
Q \downarrow \simeq & & \simeq \downarrow Q \\
\mathcal{D}(R)_{\mathfrak{a}\text{-tor}} & \xrightarrow[\simeq]{\mathbf{L}\Lambda^{\mathfrak{a}}} & \mathcal{D}(R)_{\mathfrak{a}\text{-comp}}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}} & \xleftarrow[\simeq]{\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}} & \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}} \\
Q \downarrow \simeq & & \simeq \downarrow Q \\
\mathcal{D}(R)_{\mathfrak{a}\text{-tor}} & \xleftarrow[\simeq]{\mathbf{R}\Gamma_{\mathfrak{a}}} & \mathcal{D}(R)_{\mathfrak{a}\text{-comp}}
\end{array}$$

5 Flat and Injective Dimensions

In this section, we provide bounds on the flat and injective dimensions over $\widehat{R}^{\mathfrak{a}}$ of $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)$ and $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)$, for use in [28].

Proposition 5.1 *Let $X \in \mathcal{D}_b(R)$ be given. Then there are inequalities*

$$\begin{aligned}
\operatorname{id}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) &\leqslant \operatorname{id}_R(X) \\
\operatorname{fd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)) &\leqslant \operatorname{fd}_R(X).
\end{aligned}$$

Proof The Čech complex over $\widehat{R}^{\mathfrak{a}}$ on a generating sequence for $\mathfrak{a}$ shows that we have $\mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(R)) \leqslant 0$. Thus, by the isomorphism $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(R), X)$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ from Fact 2.3, we have

$$\mathrm{id}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) = \mathrm{id}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(R), X)) \leqslant \mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(R)) + \mathrm{id}_R(X) = \mathrm{id}_R(X)$$

by [2, Theorem 4.1(F)]. Similarly, from the isomorphism $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X) \simeq \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R X$, we have $\mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)) \leqslant \mathrm{fd}_R(X)$ by [2, Theorem 4.1(F)]. □

We end this section with similar bounds for $\mathrm{id}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X))$ and $\mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X))$, after the following lemma.

Lemma 5.2 *Let N be an $\widehat{R}^{\mathfrak{a}}$-module and is either $\mathfrak{a}$-adically complete or $\mathfrak{a}$-torsion. Then one has*

$$\mathrm{fd}_R(N) = \mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(N) \qquad \mathrm{id}_R(N) = \mathrm{id}_{\widehat{R}^{\mathfrak{a}}}(N).$$

In particular, N is flat over R if and only if it is flat over $\widehat{R}^{\mathfrak{a}}$, and N is injective over R if and only if it is injective over $\widehat{R}^{\mathfrak{a}}$.

Proof We verify the first displayed equality in the statement of the lemma; the second one is verified similarly, and the subsequent statements follow from these directly. Since $\widehat{R}^{\mathfrak{a}}$ is flat over R, the inequality $\mathrm{fd}_R(N) \leqslant \mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(N)$ is from [2, Corollary 4.1(b)(F)]. (Note that this does not use the assumption that N is $\mathfrak{a}$-adically complete or $\mathfrak{a}$-torsion.)

Claim. We have

$$\mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(N) = \sup\{\sup((\widehat{R}^{\mathfrak{a}}/P) \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} N) \mid P \in \mathrm{V}(\mathfrak{a}\widehat{R}^{\mathfrak{a}})\}. \tag{5.2.1}$$

If N is $\mathfrak{a}$-adically complete, then this is by [34, Proposition 2.1]. If N is $\mathfrak{a}$-torsion, it is straightforward to show that we have $\mathrm{Supp}_{\widehat{R}^{\mathfrak{a}}}(N) \subseteq \mathrm{V}(\mathfrak{a}\widehat{R}^{\mathfrak{a}})$, so the claim follows from [2, Proposition 5.3.F]. (The corresponding formula for injective dimension is from [34, Proposition 3.2] and [2, Proposition 5.3.I].)

Now we complete the proof by verifying the inequality $\mathrm{fd}_R(N) \geqslant \mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(N)$. The natural isomorphism $\widehat{R}^{\mathfrak{a}}/\mathfrak{a}\widehat{R}^{\mathfrak{a}} \cong R/\mathfrak{a}$ shows that every $P \in \mathrm{V}(\mathfrak{a}\widehat{R}^{\mathfrak{a}})$ is of the form $P = \mathfrak{p}\widehat{R}^{\mathfrak{a}}$ for a unique prime ideal $\mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \subseteq \mathrm{Spec}(R)$. This also implies that

$$\widehat{R}^{\mathfrak{a}}/P = \widehat{R}^{\mathfrak{a}}/\mathfrak{p}\widehat{R}^{\mathfrak{a}} \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R (R/\mathfrak{p})$$

so we find that

$$(\widehat{R}^{\mathfrak{a}}/P) \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} N \simeq (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R (R/\mathfrak{p})) \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} N \simeq (R/\mathfrak{p}) \otimes^{\mathbf{L}}_R N.$$

From this, we have

$$\sup((\widehat{R}^{\mathfrak{a}}/P) \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} N) = \sup((R/\mathfrak{p}) \otimes^{\mathbf{L}}_R N) \leqslant \mathrm{fd}_R(N).$$

Thus, the inequality $\mathrm{fd}_R(N) \geqslant \mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(N)$ follows from (5.2.1). □

Proposition 5.3 *Let $X \in \mathcal{D}_b(R)$ be given. Then one has $\mathrm{id}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)) \leqslant \mathrm{id}_R(X)$ and $\mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \leqslant \mathrm{fd}_R(X)$.*

Proof For the first inequality, assume that $\mathrm{id}_R(X) < \infty$, and let $X \xrightarrow{\simeq} J$ be a bounded semi-injective resolution over R such that $J_i = 0$ for all $i < -\mathrm{id}_R(X)$. It follows that the R-complex $\Gamma_{\mathfrak{a}}(J)$ is a bounded semi-injective resolution of $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$ over R such that $\Gamma_{\mathfrak{a}}(J)_i = \Gamma_{\mathfrak{a}}(J_i) = 0$ for all $i < -\mathrm{id}_R(X)$. Since each module in this complex is $\mathfrak{a}$-torsion, the complex $\Gamma_{\mathfrak{a}}(J)$ is an $\widehat{R}^{\mathfrak{a}}$-complex, and Lemma 5.2 implies that it consists of injective $\widehat{R}^{\mathfrak{a}}$-modules. Thus, the $\widehat{R}^{\mathfrak{a}}$-complex $\Gamma_{\mathfrak{a}}(J)$ is a bounded semi-injective resolution of $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)$ over $\widehat{R}^{\mathfrak{a}}$ such that $\Gamma_{\mathfrak{a}}(J)_i = \Gamma_{\mathfrak{a}}(J_i) = 0$ for all $i < -\mathrm{id}_R(X)$. The inequality $\mathrm{id}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X)) \leqslant \mathrm{id}_R(X)$ follows.

The proof of the inequality $\mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \leqslant \mathrm{fd}_R(X)$ is similar, using the fact that the $\mathfrak{a}$-adic completion of any flat R-module is flat; see [36, Tag 0AGW]. □

6 Cohomological Adic Cofiniteness

Next, we discuss the connection between $\mathfrak{a}$-adically finite complexes and the following similar notion from [24].

Definition 6.1 Assume that R is $\mathfrak{a}$-adically complete. An R-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ is *cohomologically* $\mathfrak{a}$*-adically cofinite* if there is a complex $N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ such that $X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(N)$.

Our first result in this direction, given next, shows that, when it makes sense to compare these two notions, they are the same. It is primarily from [24].

Proposition 6.2 *Assume that R is $\mathfrak{a}$-adically complete. An R-complex $X \in \mathcal{D}_b(R)$ is cohomologically $\mathfrak{a}$-adically cofinite if and only if it is $\mathfrak{a}$-adically finite.*

Proof Assume first that X is cohomologically $\mathfrak{a}$-adically cofinite, so by definition there is a complex $N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ such that $X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(N)$. Fact 2.4 implies that X is in $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$, so we have $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ by Fact 2.5, and we have $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X) \in \mathcal{D}^{\mathrm{f}}(R)$ by [24, Theorem 0.4]. Thus, X is $\mathfrak{a}$-adically finite.

Conversely, assume that X is $\mathfrak{a}$-adically finite. Then we have $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ by definition, so X is in $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ by Fact 2.5. Also by definition, we have $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X) \in \mathcal{D}^{\mathrm{f}}(R)$, so according to [24, Theorem 0.4], the complex X is cohomologically $\mathfrak{a}$-adically cofinite. □

The next result gives a similar characterization of $\mathfrak{a}$-adically finite complexes in the incomplete setting.

Theorem 6.3 *Let $Q\colon \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$ be the forgetful functor, and let $\mathcal{D}(R)_{\mathfrak{a}\text{-}fin}$ be the full subcategory of $\mathcal{D}(R)$ consisting of all $\mathfrak{a}$-adically finite R-complexes.*

(a) *An R-complex $X \in \mathcal{D}_b(R)$ is $\mathfrak{a}$-adically finite if and only if there is a complex $N \in \mathcal{D}^f_b(\widehat{R}^{\mathfrak{a}})$ such that $X \simeq Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N))$.*
(b) *The functor $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ induces an equivalence of categories $\mathcal{D}(R)_{\mathfrak{a}\text{-}fin} \to \mathcal{D}^f_b(\widehat{R}^{\mathfrak{a}})$ with quasi-inverse induced by $Q \circ \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}$.*
(c) *If there is a complex $N \in \mathcal{D}^f_b(R)$ such that $X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(N)$, then $X \in \mathcal{D}(R)_{\mathfrak{a}\text{-}fin}$.*

Proof Claim 1: if $N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$, then we have $N \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N)))$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. Indeed, the first isomorphism in the following sequence is from Corollary 4.14.

$$\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N))) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(Q(N))) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Q(N)) \simeq N$$

The second isomorphism is from Lemma 4.1. The third one is from Theorem 4.13; to apply this result, we use the conditions $N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}}) \subseteq \mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-comp}}$.

Claim 2: if $N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$, then we have $Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N)) \in \mathcal{D}(R)_{\mathfrak{a}\text{-fin}}$. Indeed, Corollary 4.14 implies that $Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Q(N))$, so we have

$$\operatorname{supp}_R(Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N))) = \operatorname{supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(Q(N))) \subseteq \mathrm{V}(\mathfrak{a})$$

by [30, Proposition 3.6]. Also, Claim 1 shows that $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N))) \simeq N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$, so $Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N))$ is $\mathfrak{a}$-adically finite by definition.

Claim 3: if $X \in \mathcal{D}(R)_{\mathfrak{a}\text{-fin}}$, then $X \simeq Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)))$ in $\mathcal{D}(R)$. The first three isomorphisms in the following sequence are from Lemma 4.5, Lemma 4.4, and Theorem 4.12, respectively.

$$Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X))) \simeq Q(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(X))) \simeq Q(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(X))) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq X$$

The fourth isomorphism is by Fact 2.5, as we have $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ by assumption.

Now we complete the proof of the result. By definition, if $X \in \mathcal{D}(R)_{\mathfrak{a}\text{-fin}}$, then $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$, that is, the functor $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ maps $\mathcal{D}(R)_{\mathfrak{a}\text{-fin}}$ to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$. Claim 2 shows that $Q \circ \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}$ maps $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$ to $\mathcal{D}(R)_{\mathfrak{a}\text{-fin}}$, and Claim 1 shows that the composition $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \circ Q \circ \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}$ is isomorphic to the identity on $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$. Claim 3 shows that the composition $Q \circ \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \circ \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ is isomorphic to the identity on $\mathcal{D}(R)_{\mathfrak{a}\text{-fin}}$. This establishes part (b) of the theorem, and part (a) follows.

For part (c), assume that there is a complex $N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ such that $X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(N)$. Then the complex $N' := \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(N) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$ satisfies

$$\begin{aligned} Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N')) &\simeq Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(N))) \\ &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Q(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(N))) \\ &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(N)) \\ &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}(N) \\ &\simeq X; \end{aligned}$$

see Fact 2.3. So, we have $X \simeq Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N')) \in \mathcal{D}(R)_{\mathfrak{a}\text{-fin}}$ by part (a). □

The next example shows that the converse of Theorem 6.3(c) fails in general. Thus, the characterization in Theorem 6.3(a) cannot be simplified (at least not in the naive manner suggested by Theorem 6.3(c)).

Example 6.4 Let $(R, \mathfrak{m}, k)$ be a local ring that does not admit a dualizing complex. Such a ring exists by work of Ogoma [22]. The injective hull $E := E_R(k)$ is $\mathfrak{m}$-adically finite by [30, Proposition 7.8(b)]. Suppose that there is an R-complex $N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ such that $E \simeq \mathbf{R}\Gamma_{\mathfrak{m}}(N)$. In [28, Example 6.7] we show that this implies that N is dualizing for R, contradicting our assumption on R.

7 Induced Isomorphisms

This section consists of useful isomorphisms derived from our preceding results. We begin with extended versions of Lemma 2.8.

Proposition 7.1 *Let $X \in \mathcal{D}(\widehat{R}^{\mathfrak{a}})$ be such that* $\operatorname{supp}_{\widehat{R}^{\mathfrak{a}}}(X) \subseteq \mathrm{V}(\mathfrak{a}\widehat{R}^{\mathfrak{a}})$. *Given an R-complex $M \in \mathcal{D}(R)$, there are isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$*

$$X \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M) \simeq X \otimes^{\mathbf{L}}_{R} M \simeq X \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} M)$$

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(X, \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) \simeq \mathbf{R}\mathrm{Hom}_{R}(X, M) \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(X, \mathbf{R}\mathrm{Hom}_{R}(\widehat{R}^{\mathfrak{a}}, M)).$$

Proof We verify the first two isomorphisms; the others are verified similarly. The first isomorphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ in the following sequence is from Fact 2.3.

$$\begin{aligned} X \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M) &\simeq X \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} \mathbf{R}\Gamma_{\mathfrak{a}}(M)) \\ &\simeq X \otimes^{\mathbf{L}}_{R} \mathbf{R}\Gamma_{\mathfrak{a}}(M) \\ &\simeq X \otimes^{\mathbf{L}}_{R} M \\ &\simeq X \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} M) \end{aligned}$$

The second and fourth ones are tensor-cancellation. The third one is the natural one $X \otimes^{\mathbf{L}}_{R} \mathbf{R}\Gamma_{\mathfrak{a}}(M) \xrightarrow{X \otimes^{\mathbf{L}}_{R} \varepsilon^{M}_{\mathfrak{a}}} X \otimes^{\mathbf{L}}_{R} M$; this is an isomorphism in $\mathcal{D}(R)$ by Lemma 2.8, and it respects the $\widehat{R}^{\mathfrak{a}}$-structure coming from the left. □

Corollary 7.2 *Let $\widehat{K}$ be the Koszul complex over $\widehat{R}^{\mathfrak{a}}$ on the generating sequence $\underline{x}$ for $\mathfrak{a}$. Given an R-complex $M \in \mathcal{D}(R)$, there are isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$*

$$\widehat{K} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M) \simeq \widehat{K} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} M) \simeq \widehat{K} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$$

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{K}, \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{K}, \mathbf{R}\mathrm{Hom}_{R}(\widehat{R}^{\mathfrak{a}}, M)) \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{K}, \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)).$$

The next result is Theorem 1.2 from the introduction. Note that it is straightforward when $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, by Fact 2.3. See [28, Theorems 5.6 and 5.7] for applications.

Theorem 7.3 *Let $R \to S$ be a homomorphism of commutative noetherian rings, and let $X \in \mathcal{D}_b(R)$ be $\mathfrak{a}$-adically finite over R. If $S \otimes^{\mathbf{L}}_{R} X \in \mathcal{D}_b(S)$, e.g., if* $\operatorname{fd}_R(S) < \infty$, *then there is an isomorphism in $\mathcal{D}(\widehat{S}^{\mathfrak{a}S})$:*

$$\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_{R} X).$$

Proof Our finiteness assumption on X implies that $X \in \mathcal{D}_{\mathrm{b}}(R)$ and $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$. Thus, we have $\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \in \mathcal{D}^{\mathrm{f}}_{+}(\widehat{S}^{\mathfrak{a}S})$. Also, from [29, Theorem 5.10], we have $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_{R} X) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{S}^{\mathfrak{a}S})$. For clarity, we set $K^R := K = K^R(\underline{x})$, where $\underline{x}$ is a finite generating sequence for $\mathfrak{a}$, and set $K^{\widehat{R}^{\mathfrak{a}}} := K^{\widehat{R}^{\mathfrak{a}}}(\underline{x})$, and similarly for K^S and $K^{\widehat{S}^{\mathfrak{a}S}}$.

Claim 1: there is an isomorphism $K^{\widehat{R}^{\mathfrak{a}}} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} (K^{R} \otimes^{\mathbf{L}}_{R} X)$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. This follows from the next sequence of isomorphisms:

$$\begin{aligned}
K^{\widehat{R}^{\mathfrak{a}}} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) &\simeq K^{\widehat{R}^{\mathfrak{a}}} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} X) \\
&\simeq (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} K^{R}) \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} X) \\
&\simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} (K^{R} \otimes^{\mathbf{L}}_{R} X).
\end{aligned}$$

The first isomorphism is from Corollary 7.2, and the others are routine.

Claim 2: we have $\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{S}^{\mathfrak{a}S})$. To this end, recall that the first paragraph of this proof shows that we have $\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \in \mathcal{D}^{\mathrm{f}}_{+}(\widehat{S}^{\mathfrak{a}S})$. Thus, we need only show that $\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \in \mathcal{D}_{\mathrm{b}}(\widehat{S}^{\mathfrak{a}S})$. For this, note first that every maximal ideal of $\widehat{S}^{\mathfrak{a}S}$ contains $\mathfrak{a}\widehat{S}^{\mathfrak{a}S}$. In other words, we have $\mathrm{supp}_{\widehat{S}^{\mathfrak{a}S}}(K^{\widehat{S}^{\mathfrak{a}S}}) = \mathrm{V}(\mathfrak{a}\widehat{S}^{\mathfrak{a}S}) \supseteq \text{m-Spec}(\widehat{S}^{\mathfrak{a}S})$. Thus, according to [11, Theorem 4.2(b)], to establish the claim, it suffices to show that we have $K^{\widehat{S}^{\mathfrak{a}S}} \otimes^{\mathbf{L}}_{\widehat{S}^{\mathfrak{a}S}} (\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{S}^{\mathfrak{a}S})$. To this end, we consider the following sequence of isomorphisms in $\mathcal{D}(\widehat{S}^{\mathfrak{a}S})$.

$$\begin{aligned}
K^{\widehat{S}^{\mathfrak{a}S}} \otimes^{\mathbf{L}}_{\widehat{S}^{\mathfrak{a}S}} (\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) &\simeq (\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} K^{\widehat{R}^{\mathfrak{a}}}) \otimes^{\mathbf{L}}_{\widehat{S}^{\mathfrak{a}S}} (\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (K^{\widehat{R}^{\mathfrak{a}}} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} (K^{R} \otimes^{\mathbf{L}}_{R} X)) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{S} (S \otimes^{\mathbf{L}}_{R} (K^{R} \otimes^{\mathbf{L}}_{R} X)) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{S} ((S \otimes^{\mathbf{L}}_{R} K^{R}) \otimes^{\mathbf{L}}_{S} (S \otimes^{\mathbf{L}}_{R} X)) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{S} (K^{S} \otimes^{\mathbf{L}}_{S} (S \otimes^{\mathbf{L}}_{R} X))
\end{aligned}$$

The third isomorphism is from Claim 1, and the others are standard. Since we have $S \otimes^{\mathbf{L}}_{R} X \in \mathcal{D}_{\mathrm{b}}(S)$, by assumption, the condition $\mathrm{pd}_{S}(K^{S}) < \infty$ implies that $K^{S} \otimes^{\mathbf{L}}_{S} (S \otimes^{\mathbf{L}}_{R} X) \in \mathcal{D}_{\mathrm{b}}(S)$. Thus the flatness of $\widehat{S}^{\mathfrak{a}S}$ over S implies that we have

$$K^{\widehat{S}^{\mathfrak{a}S}} \otimes^{\mathbf{L}}_{\widehat{S}^{\mathfrak{a}S}} (\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{S} (K^{S} \otimes^{\mathbf{L}}_{S} (S \otimes^{\mathbf{L}}_{R} X)) \in \mathcal{D}_{\mathrm{b}}(\widehat{S}^{\mathfrak{a}S}).$$

This establishes Claim 2.

Since the complexes $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_{R} X)$ and $\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)$ are in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{S}^{\mathfrak{a}S})$, to show that they are isomorphic, Theorem 6.3(b) says that it suffices to show that $\mathbf{R}\Gamma_{\mathfrak{a}\widehat{S}^{\mathfrak{a}S}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_{R} X)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}\widehat{S}^{\mathfrak{a}S}}(\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X))$ in $\mathcal{D}(\widehat{S}^{\mathfrak{a}S})$. To verify this isomorphism, we compute as follows.

$$\begin{aligned}
\mathbf{R}\Gamma_{\mathfrak{a}\widehat{S}^{\mathfrak{a}S}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_{R} X)) &\simeq \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_{R} X) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{S} \mathbf{R}\Gamma_{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_{R} X) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{S} (S \otimes^{\mathbf{L}}_{R} X) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{R} X
\end{aligned}$$

The first isomorphism is by Lemmas 4.4 and 4.5 The second isomorphism is from Fact 2.3. For the third isomorphism, note that [29, Lemma 5.7] shows that $\mathrm{supp}_{S}(S \otimes^{\mathbf{L}}_{R} X) \subseteq \mathrm{V}(\mathfrak{a}S)$, so Fact 2.5 implies that $\mathbf{R}\Gamma_{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_R X) \simeq S \otimes^{\mathbf{L}}_R X$ in $\mathcal{D}(S)$. The last isomorphism is tensor-cancellation. The next isomorphisms are justified similarly.

$$
\begin{aligned}
\mathbf{R}\Gamma_{\mathfrak{a}\widehat{S}^{\mathfrak{a}S}}(\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) &\simeq \mathbf{R}\Gamma_{\mathfrak{a}\widehat{S}^{\mathfrak{a}S}}(\widehat{S}^{\mathfrak{a}S}) \otimes^{\mathbf{L}}_{\widehat{S}^{\mathfrak{a}S}} (\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \\
&\simeq (\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}})) \otimes^{\mathbf{L}}_{\widehat{S}^{\mathfrak{a}S}} (\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}) \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X)) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(X) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(X)) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} (\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X) \\
&\simeq \widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_R X
\end{aligned}
$$

These two sequences give the desired isomorphism, completing the proof. □

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