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# Adic semidualizing complexes

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We introduce and study a class of objects that encompasses Christensen and Foxby's semidualizing modules and complexes and Kubik's quasi-dualizing modules: the class of $\mathfrak{a}$-adic semidualizing modules and complexes. We give examples and equivalent characterizations of these objects, including a characterization in terms of the more familiar semidualizing property. As an application, we give a proof of the existence of dualizing complexes over complete local rings that does not use the Cohen Structure Theorem.

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## 1. Introduction

Throughout this paper, let $R$ be a commutative noetherian ring, let $\mathfrak{a} \subsetneq R$ be a proper ideal of $R$ and let $\widehat{R}^{\mathfrak{a}}$ be the $\mathfrak{a}$-adic completion of $R$. Let $K$ denote the Koszul complex over $R$ on a finite generating sequence for $\mathfrak{a}$.

This work is part 5 in a series of papers on derived local cohomology and derived local homology. It builds on our previous papers [35–38], and it is applied in the paper [34].

Duality is a powerful tool in many areas of mathematics. For instance, over a complete Cohen–Macaulay local ring, Grothendieck's local duality [20] uses

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Matlis duality to relate local cohomology modules to Ext-modules (i.e. derived dual-modules) with respect to the ring's canonical module. When the ring is not Cohen–Macaulay, Grothendieck [19] uses a dualizing complex to obtain similar[a] conclusions. This allows one to study local cohomology by studying Ext, and vice versa, which is incredibly useful.

Because of this and many other applications, dualizing complexes have become important in commutative algebra and algebraic geometry. The standard proof of the existence of a dualizing complex for a complete local ring $R$ uses the powerful Cohen Structure Theorem [10]: one surjects onto $R$ with a complete regular local ring $Q$, takes an injective resolution $I$ of $Q$ over itself, and shows that the complex $\mathrm{Hom}_Q(R, I)$ is dualizing for $R$.

One consequence of our work in this paper is the following alternate construction of a dualizing complex which avoids the Cohen Structure Theorem; see Theorem 6.1(b) below.

**Theorem 1.1.** *Assume that $(R, \mathfrak{m}, k)$ is local with $E_R(k)$ the injective hull of $k$ over $R$. Let $F$ be a flat resolution of $E$ over $R$. Then the $\mathfrak{m}$-adic completion $\widehat{F}^{\mathfrak{m}}$ is a dualizing complex over $\widehat{R}^{\mathfrak{m}}$.*

With the power of dualizing complexes in mind, much work has been devoted to the identification of good objects for use in dualities. For instance, Foxby[11] introduced the "PG modules of rank 1" now known as *semidualizing modules*; these are the finitely generated (i.e. noetherian) $R$-modules $C$ such that the natural homothety map $\chi : R \to \mathrm{Hom}_R(C, C)$ given by $\chi(r)(c) := rc$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geq 1$. Canonical modules over Cohen–Macaulay local rings are special cases of these. Christensen[8] extended this to the "semidualizing complexes", a notion that is flexible enough to encompass both the semidualizing modules as well as the dualizing complexes. This theory is very useful, capturing not only the dualizing complexes, but also Avramov and Foxby's [3] "relative dualizing complexes", but it misses other important situations, e.g. Matlis duality.

The work of Kubik[24] begins to fill this gap by introducing the "quasi-dualizing modules" over a local ring $(R, \mathfrak{m}, k)$: an artinian $R$-module $T$ is $\mathfrak{m}$-torsion, so it is a module over $\widehat{R}^{\mathfrak{m}}$ and $T$ is *quasi-dualizing* if the natural homothety map $\widehat{R}^{\mathfrak{m}} \to \mathrm{Hom}_R(T, T)$ is an isomorphism, and $\mathrm{Ext}^i_R(T, T) = 0$ for all $i \geq 1$. This includes the injective hull $E_R(k)$, i.e. the base for Matlis duality, as a special case. However, this does not include the semidualizing objects as special cases, unless the ring is artinian and local, though it does come tantalizingly close, with the same Ext-vanishing condition, a similar endomorphism algebra isomorphism, and a related finiteness conditions.

The primary goal of this paper is to fill this gap completely by introducing a single notion that recovers all the aforementioned examples as special cases: that of an "$\mathfrak{a}$-adic semidualizing complex". The general definition is necessarily somewhat

[a]or, depending on your perspective, the same.

technical, building on our papers[35–38] as well as the established literature on semidualizing complexes; see Definition 4.1. For modules, though, the definition is more straightforward: an $\mathfrak{a}$-torsion $R$-module $M$ has the structure of a module over $\widehat{R}^{\mathfrak{a}}$ and $M$ is $\mathfrak{a}$-*adically semidualizing* if $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i$, the natural homothety map $\widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(M, M)$ is an isomorphism, and $\operatorname{Ext}^i_R(M, M) = 0$ for all $i \geq 1$. In particular, the special case $\mathfrak{a} = 0$ recovers the semidualizing modules, and the maximal ideal $\mathfrak{a} = \mathfrak{m}$ of a local ring yields the quasi-dualizing modules; see Propositions 4.4 and 4.5.

Section 4 of this paper is devoted to the foundational properties of $\mathfrak{a}$-adic semidualizing complexes, with the help of some preparatory lemmas from Sec. 3. The main result of Sec. 4 is Theorem 4.6, a characterization of the adic semidualizing property. It shows first that any isomorphism $\widehat{R}^{\mathfrak{a}} \cong \operatorname{Hom}_R(M, M)$ implies that the homothety map $\widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(M, M)$ is an isomorphism, which is somewhat surprising since $\widehat{R}^{\mathfrak{a}}$ and $M$ are not assumed to be finitely generated, a crucial feature of the definition. Second, it characterizes this property in terms of a semidualizing condition over $\widehat{R}^{\mathfrak{a}}$. We state a partial version for modules here for perspective.

**Theorem 1.2.** *Let $M$ be an $R$-module with flat resolution $F$. Then the following conditions are equivalent*:

(i) *$M$ is $\mathfrak{a}$-adically semidualizing over $R$.*
(ii) *$M$ is $\mathfrak{a}$-torsion, the module $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i$, and one has $\widehat{R}^{\mathfrak{a}} \cong \operatorname{Hom}_R(M, M)$ and $\operatorname{Ext}^i_R(M, M) = 0$ for all $i \geq 1$.*
(iii) *$M$ is $\mathfrak{a}$-torsion, and the completed complex $\widehat{F}^{\mathfrak{a}}$ is semidualizing over $\widehat{R}^{\mathfrak{a}}$.*

Section 5 is devoted to describing the connections between various flavors of semidualizing objects, though one can already see hints of this in Theorem 1.2. As a sample, the next result contains parts of Theorems 5.10 and 5.14; see Remark 5.16 for a diagrammatic representation of this and more.

**Theorem 1.3.** *The following sets are in natural bijection* (*in all possible pairs*):

(a) *the set of shift-isomorphism classes of $\mathfrak{a}$-adic semidualizing $R$-complexes,*
(b) *the set of shift-isomorphism classes of semidualizing $\widehat{R}^{\mathfrak{a}}$-complexes and*
(c) *the set of shift-isomorphism classes of $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-adic semidualizing $\widehat{R}^{\mathfrak{a}}$-complexes.*

Another result worth mentioning from this section is Theorem 5.7, which states that the adic semidualizing property is local.

Section 6 contains Theorem 1.1 and other results about dualizing complexes. It also includes characterizations of the adic semidualizing complexes in the case where they should all be trivial in some sense: when $R$ is Gorenstein.

While we have phrased much of this introduction in terms of modules, the bulk of this paper deals with the more general situation of chain complexes. Specifically, we work primarily in the derived category. Section 2 contains some background material on this topic.

This work is largely inspired by the papers mentioned above, especially those of Christensen and Foxby [8, 11]. We explore other $\mathfrak{a}$-adic aspects of these works in our subsequent paper [34].

## 2. Background

**Derived Categories.** We work in the derived category $\mathcal{D}(R)$ with objects the $R$-complexes indexed homologically $X = \cdots \to X_i \to X_{i-1} \to \cdots$; see [19, 39, 40]. Isomorphisms in $\mathcal{D}(R)$ are identified by the symbol $\simeq$. The $n$th shift (or suspension) of $X$ is denoted $\Sigma^n X$. We also consider the next full triangulated subcategories:

$$\mathcal{D}_+(R) : \text{objects are homologically bounded below } R\text{-complexes},$$

$$\mathcal{D}_-(R) : \text{objects are homologically bounded above } R\text{-complexes},$$

$$\mathcal{D}_\mathrm{b}(R) : \text{objects are homologically bounded } R\text{-complexes},$$

$$\mathcal{D}^\mathrm{f}(R) : \text{objects are homologically degree-wise finite } R\text{-complexes}.$$

Intersections of these categories are designated with two ornaments, e.g. $\mathcal{D}^\mathrm{f}_\mathrm{b}(R) = \mathcal{D}_\mathrm{b}(R) \cap \mathcal{D}^\mathrm{f}(R)$.

**Resolutions.** An $R$-complex $F$ is *semi-flat*[b] if the functor $- \otimes_R F$ respects injective quasiisomorphisms, that is, if each module $F_i$ is flat over $R$ and the functor $- \otimes_R F$ respects quasiisomorphisms. A *semi-flat resolution* of an $R$-complex $X$ is a quasiisomorphism $F \xrightarrow{\simeq} X$ such that $F$ is semi-flat; for $X \in \mathcal{D}_\mathrm{b}(R)$, the *flat dimension* $\mathrm{fd}_R(X)$ is the length of the shortest bounded semi-flat resolution of $X$, if one exists:

$$\mathrm{fd}_R(X) := \inf\{\sup\{i \in \mathbb{Z} \mid F_i \neq 0\} \mid F \simeq X \text{ is a semi-flat resolution}\}.$$

The *injective* and *projective* versions of these notions are defined similarly.

For the following items, consult [2, Sec. 1] or [5, Chaps. 3 and 5]. Bounded below complexes of flat $R$-modules are semi-flat and bounded above complexes of injective $R$-modules are semi-injective. Every $R$-complex admits a semi-flat resolution (hence, a semi-projective one) and a semi-injective resolution.

**Derived Functors.** The right derived functor of Hom is $\mathbf{R}\mathrm{Hom}_R(-,-)$, which is computed via a semi-projective resolution in the first slot or a semi-injective resolution in the second slot. The left derived functor of tensor product is $- \otimes^{\mathbf{L}}_R -$, which is computed via a semi-flat resolution in either slot.

Local cohomology and local homology, described next, play a major role in this work. These notions go back to Grothendieck [20] and Matlis [27, 28], respectively; see also [1, 26]. Let $\Lambda^\mathfrak{a}(-)$ denote the $\mathfrak{a}$-adic completion functor, and $\Gamma_\mathfrak{a}(-)$ is the $\mathfrak{a}$-torsion functor, i.e. for an $R$-module $M$ we have

$$\Lambda^\mathfrak{a}(M) = \widehat{M}^\mathfrak{a} \quad \Gamma_\mathfrak{a}(M) = \{x \in M \mid \mathfrak{a}^n x = 0 \text{ for } n \gg 0\}.$$

A module $M$ is $\mathfrak{a}$-*torsion* if $\Gamma_\mathfrak{a}(M) = M$.

[b]In the literature, semi-flat complexes are sometimes called "K-flat" or "DG-flat".

The associated left and right derived functors (i.e. *derived local homology and cohomology* functors) are $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. Specifically, given an $R$-complex $X \in \mathcal{D}(R)$ with a semi-flat resolution $F \xrightarrow{\simeq} X$ and a semi-injective resolution $X \xrightarrow{\simeq} I$, we have $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(F)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(I)$. Note that these definitions yield natural transformations $\mathbf{R}\Gamma_{\mathfrak{a}} \to \mathrm{id} \to \mathbf{L}\Lambda^{\mathfrak{a}}$, induced by the natural morphisms $\Gamma_{\mathfrak{a}}(I) \to I$ and $F \to \Lambda^{\mathfrak{a}}(F)$. Let $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ denote the full subcategory of $\mathcal{D}(R)$ of all complexes $X$ such that the morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \to X$ is an isomorphism.

The definitions of $\mathbf{R}\Gamma_{\mathfrak{a}}(X)$ and $\mathbf{L}\Lambda^{\mathfrak{a}}(X)$ yield complexes over the completion $\widehat{R}^{\mathfrak{a}}$, and we denote by $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ and $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ the associated functors $\mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$.

**Fact 2.1.** If $Q : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$ is the forgetful functor, then it follows readily that $Q \circ \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}} \simeq \mathbf{L}\Lambda^{\mathfrak{a}}$ and $Q \circ \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} \simeq \mathbf{R}\Gamma_{\mathfrak{a}}$. If $X \in \mathcal{D}^{\mathrm{f}}_{+}(R)$, then there is a natural isomorphism $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} X$ by [15, Proposition 2.7]. Moreover, the proof of this result shows that there is a natural isomorphism $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} X$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$.[c] More generally, by [35, Theorem 5.1] we have $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(-) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} \mathbf{R}\Gamma_{\mathfrak{a}}(-)$. From [1, Theorem (0.3) and Corollary (3.2.5.i)], there are natural isomorphisms

$$\mathbf{R}\Gamma_{\mathfrak{a}}(-) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_{R} - \quad \mathbf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), -).$$

Here is an important feature of these constructions, sometimes called MGM equivalence (after Matlis, Greenlees and May).

**Fact 2.2.** From [1, Corollary to Theorem (0.3)*; 32, Theorem 1.2] the following natural morphisms are isomorphisms:

$$\mathbf{R}\Gamma_{\mathfrak{a}} \circ \mathrm{id} \xrightarrow{\simeq} \mathbf{R}\Gamma_{\mathfrak{a}} \circ \mathbf{L}\Lambda^{\mathfrak{a}}, \quad \mathbf{L}\Lambda^{\mathfrak{a}} \circ \mathbf{R}\Gamma_{\mathfrak{a}} \xrightarrow{\simeq} \mathbf{L}\Lambda^{\mathfrak{a}} \circ \mathrm{id},$$
$$\mathbf{R}\Gamma_{\mathfrak{a}} \circ \mathbf{R}\Gamma_{\mathfrak{a}} \xrightarrow{\simeq} \mathrm{id} \circ \mathbf{R}\Gamma_{\mathfrak{a}}, \qquad \mathrm{id} \circ \mathbf{L}\Lambda^{\mathfrak{a}} \xrightarrow{\simeq} \mathbf{L}\Lambda^{\mathfrak{a}} \circ \mathbf{L}\Lambda^{\mathfrak{a}}.$$

The following notion of support is due to Foxby [14].

**Definition 2.3.** Let $X \in \mathcal{D}(R)$. The *small support* of $X$ is

$$\mathrm{supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R} X \not\simeq 0\},$$

where $\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

**Fact 2.4.** Let $X \in \mathcal{D}(R)$. Then we know that $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ if and only if $X \in \mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ if and only if each homology module $\mathrm{H}_i(X)$ is $\mathfrak{a}$-torsion, by [38, Proposition 5.4; 32, Corollary 4.32].

The next fact and definition take their cues from work of Hartshorne [21], Kawasaki [22, 23] and Melkersson [29].

[c]This is based on the fact that, for a finitely generated free $R$-module $L$, induction on the rank of $L$ shows that the natural isomorphism $\widehat{R}^{\mathfrak{a}} \otimes_R L \cong \widehat{L}^{\mathfrak{a}}$ is $\widehat{R}^{\mathfrak{a}}$-linear.

**Fact 2.5 ([38, Theorem 1.3]).** For $X \in \mathcal{D}_{\mathrm{b}}(R)$, the next conditions are equivalent.

(i) One has $K^R(\underline{y}) \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ for some (equivalently for every) generating sequence $\underline{y}$ of $\mathfrak{a}$.
(ii) One has $X \otimes^{\mathbf{L}}_R R/\mathfrak{a} \in \mathcal{D}^{\mathrm{f}}(R)$.
(iii) One has $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X) \in \mathcal{D}^{\mathrm{f}}(R)$.
(iv) One has $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(X) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$.

**Definition 2.6.** An $R$-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ is $\mathfrak{a}$-*adically finite* if it satisfies the equivalent conditions of Fact 2.5 and $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

**Remark 2.7.** Because of Fact 2.4, an $R$-module $M$ is $\mathfrak{a}$-adically finite if and only if it is $\mathfrak{a}$-torsion and has $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ finitely generated for all $i$.

**Example 2.8.** Let $X \in \mathcal{D}_{\mathrm{b}}(R)$ be given.

(a) If $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then we have $\mathrm{supp}_R(X) = \mathrm{V}(\mathfrak{b})$ for some ideal $\mathfrak{b}$, and it follows that $X$ is $\mathfrak{a}$-adically finite whenever $\mathfrak{a} \subseteq \mathfrak{b}$. (The case $\mathfrak{a} = 0$ is from [38, Proposition 7.8(a)], and the general case follows readily.)
(b) $K$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ are $\mathfrak{a}$-adically finite, by [38, Fact 3.4 and Theorem 7.10].
(c) The homology modules of $X$ are artinian if and only if there is an ideal $\mathfrak{a}$ of finite colength (i.e. such that $R/\mathfrak{a}$ is artinian) such that $X$ is $\mathfrak{a}$-adically finite, by [37, Proposition 5.11].

We continue with a few semidualizing definitions.

**Definition 2.9.** An $R$-complex $C \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ is *semidualizing* if the natural homothety morphism $\chi^R_C : R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathcal{D}(R)$. The set of shift-isomorphism classes of semidualizing $R$-complexes is denoted $\mathfrak{S}(R)$. A *tilting* $R$-complex[d] is a semidualizing $R$-complex of finite projective dimension, and a *dualizing* $R$-complex is a semidualizing $R$-complex of finite injective dimension.

We end this section with a few useful notes about completions.

**Lemma 2.10.** *Let $\psi : R \to \widehat{R}^{\mathfrak{a}}$ be the natural map.*

(a) *There is a bijection* $\mathrm{m\text{-}Spec}(R) \cap \mathrm{V}(\mathfrak{a}) \to \mathrm{m\text{-}Spec}(\widehat{R}^{\mathfrak{a}})$ *given by* $\mathfrak{m} \mapsto \mathfrak{m}\widehat{R}^{\mathfrak{a}}$. *The inverse of this bijection is given by contraction along* $\psi$.
(b) *There is a bijection* $\mathrm{V}(\mathfrak{a}) \to \mathrm{V}(\mathfrak{a}\widehat{R}^{\mathfrak{a}})$ *given by* $\mathfrak{p} \mapsto \mathfrak{p}\widehat{R}^{\mathfrak{a}}$. *The inverse of this bijection is given by contraction along* $\psi$.
(c) *If $R$ is locally Gorenstein, then so is $\widehat{R}^{\mathfrak{a}}$.*

**Proof.** (a)–(b) The induced map $R/\mathfrak{a} \to \widehat{R}^{\mathfrak{a}}/\mathfrak{a}\widehat{R}^{\mathfrak{a}}$ is an isomorphism. Since $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$ is contained in the Jacobson radical of $\widehat{R}^{\mathfrak{a}}$, the result now follows readily.

[d] These are called "invertible" in [6].

(c) Assume that $R$ is locally Gorenstein, and let $\mathfrak{M} \in \text{m-Spec}(\widehat{R}^{\mathfrak{a}}) \subseteq \mathrm{V}(\mathfrak{a}\widehat{R}^{\mathfrak{a}})$ be given. By part (a), the contraction $\mathfrak{m}$ of $\mathfrak{M}$ in $R$ is maximal and satisfies $\mathfrak{m}\widehat{R}^{\mathfrak{a}} = \mathfrak{M}$. It follows that the closed fiber of the induced flat, local ring homomorphism $R_{\mathfrak{m}} \to \widehat{R}^{\mathfrak{a}}_{\mathfrak{M}}$ is a field. Thus, the assumption that $R_{\mathfrak{m}}$ is Gorenstein implies that $\widehat{R}^{\mathfrak{a}}_{\mathfrak{M}}$ is Gorenstein as well. □

## 3. Homothety Morphisms

This section is devoted to some technical lemmas that we use to show that $\mathfrak{a}$-adic semidualizing complexes are well-defined.

**Lemma 3.1.** *Let $M \in \mathcal{D}_-(R)$ with* $\operatorname{supp}_R(M) \subseteq V(\mathfrak{a})$. *Then $M$ has a bounded above semi-injective resolution $M \xrightarrow{\simeq} J$ over $R$ consisting of $\widehat{R}^{\mathfrak{a}}$-module homomorphisms of injective $\widehat{R}^{\mathfrak{a}}$-modules.*

**Proof.** From [38, Proposition 3.8 and Corollary 3.9] we know that $M$ has a bounded above semi-injective resolution $M \xrightarrow{\simeq} J$ over $R$ consisting of $\mathfrak{a}$-torsion injective $R$-modules with $\operatorname{Ass}_R(J_i) \subseteq \operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ for each $i$. The torsion condition implies that each differential $\partial_i^J$ is $\widehat{R}^{\mathfrak{a}}$-linear and the natural map $J \to \widehat{R}^{\mathfrak{a}} \otimes_R J$ is an isomorphism; see [25, Fact 2.1 and Lemma 2.2]. The associated prime condition implies that each prime ideal $\mathfrak{p} \in \operatorname{Ass}_R(J_i)$ satisfies $\widehat{R}^{\mathfrak{a}}/\mathfrak{p}\widehat{R}^{\mathfrak{a}} \cong R/\mathfrak{p}$, so we have $\widehat{R}^{\mathfrak{a}} \otimes_R \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})$. We conclude from [12, Theorem 1] that each $J_i \cong \widehat{R}^{\mathfrak{a}} \otimes_R J_i$ is injective over $\widehat{R}^{\mathfrak{a}}$, as desired. □

**Remark 3.2.** One might be tempted to prove the preceding result for arbitrary (that is, unbounded) complexes as follows. Let $X \xrightarrow{\simeq} J$ be a semi-injective resolution. The condition $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ says that the natural morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \to X$ is an isomorphism in $\mathcal{D}(R)$, so we have $X \simeq \Gamma_{\mathfrak{a}}(J)$. The complex $\Gamma_{\mathfrak{a}}(J)$ consists of $\mathfrak{a}$-torsion injective $R$-modules, so the isomorphism gives a resolution of the desired form, as in the proof of the preceding result. The problem with this line of reasoning is that $\Gamma_{\mathfrak{a}}(J)$ can fail to be semi-injective; see [35, Example 3.1].

**Definition 3.3.** Let $M \in \mathcal{D}_-(R)$ with $\operatorname{supp}_R(M) \subseteq V(\mathfrak{a})$. Let $M \xrightarrow{\simeq} J$ be a semi-injective resolution as in Lemma 3.1. This yields a well-defined chain map $\chi_J^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(J, J)$ given by $\chi_J^{\widehat{R}^{\mathfrak{a}}}(r)(j) = rj$. This in turn gives rise to a well-defined "homothety morphism" $\chi_M^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\operatorname{Hom}_R(M, M)$ in $\mathcal{D}(R)$.

The rest of this section is devoted to a lemma for use in the proof of Theorem 1.2 from the introduction. A subtlety of the result is worth noting here: in part (a) we only have an isomorphism over $R$; however, we are able to translate it to information about $\widehat{R}^{\mathfrak{a}}$-isomorphisms.

**Lemma 3.4.** *Let $M \in \mathcal{D}_-(R)$ be such that* $\operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$.

(a) *One has* $\mathbf{R}\operatorname{Hom}_R(M, M) \simeq \mathbf{R}\operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))$ *in* $\mathcal{D}(R)$.

(b) *There is an isomorphism* $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_R(M,M)$ *in* $\mathcal{D}(R)$ *if and only if there is an isomorphism* $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))$ *in* $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$.

(c) *The morphism* $\chi_M^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_R(M,M)$ *is an isomorphism in* $\mathcal{D}(R)$ *if and only if the morphism* $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)} : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))$ *is an isomorphism in* $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$.

**Proof.** By Lemma 3.1, the $R$-complex $M$ has a bounded above semi-injective resolution $M \xrightarrow{\simeq} I$ consisting of injective $\mathfrak{a}$-torsion $\widehat{R}^{\mathfrak{a}}$-modules. This explains the first three steps in the next display.

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(M,M) &\simeq \mathrm{Hom}_R(J,J) \\
&= \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(J)) \\
&= \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\Gamma_{\mathfrak{a}}(J), \Gamma_{\mathfrak{a}}(J)) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)).
\end{aligned}$$

The fourth step follows from the fact that $J$ is a bounded above semi-injective resolution of $M$, because this implies that $\Gamma_{\mathfrak{a}}(J)$ is a bounded above complex of injective $\widehat{R}^{\mathfrak{a}}$-modules, so it is a semi-injective resolution of $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)$ over $\widehat{R}^{\mathfrak{a}}$.

(a) In [35, Theorem 4.7] we show that there is an isomorphism

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)) \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) \tag{3.1}$$

in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. With the isomorphisms described above, this explains the isomorphism in part (a) of the theorem.

(b) The isomorphisms from the first paragraph of this proof provide the first step in the next sequence in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$.

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, \mathbf{R}\mathrm{Hom}_R(M,M)) &\simeq \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M))) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R Q(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}, \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M))).
\end{aligned}$$

The second step here is Hom-tensor adjointness, where $Q$ is the forgetful functor $\mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$. The third and fourth steps are from Fact 2.1, and the last one is from Hom-cancellation. From this sequence, we have $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_R(M,M)$ in $\mathcal{D}(R)$ if and only if $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M), \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M))$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, i.e. if and only if $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, by (3.1).

(c) The isomorphisms from the first paragraph of this proof also yield the next commutative diagram in $\mathcal{D}(R)$.

$$\begin{array}{ccc}\widehat{R}^{\mathfrak{a}} & \xrightarrow{\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)}} & \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M),\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)) \\ {\scriptstyle \chi^{\widehat{R}^{\mathfrak{a}}}_{M}}\Big\downarrow & \swarrow {\scriptstyle \simeq} & \\ \mathbf{R}\mathrm{Hom}_{R}(M,M). & & \end{array}$$

In particular, the morphism $\chi^{\widehat{R}^{\mathfrak{a}}}_{M}$ is an isomorphism in $\mathcal{D}(R)$ if and only if $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)}$ is so, that is, if and only if $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)}$ is an isomorphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, since $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)}$ is a morphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. Thus, to explain the desired bi-implication, it suffices to show that the homothety morphisms $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)} : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M),\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M))$ and $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)} : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M),\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ are isomorphisms simultaneously. To this end, first consider the natural $\widehat{R}^{\mathfrak{a}}$-isomorphisms

$$\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M) \simeq \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(M)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)),$$

from [35, Lemmas 4.4 and 4.5]. It follows that $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)}$ is an isomorphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ if and only if $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))}$ is so. Similarly, the isomorphism

$$\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \simeq \mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)),$$

from [35, Theorem 4.3] shows that $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)}$ is an isomorphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ if and only if $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))}$ is so. Thus, it remains to show that $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))}$ is an isomorphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ if and only if $\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))}$ is so. The fact that these are isomorphisms simultaneously follows from the next commutative diagram in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, wherein we set $N := \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$:

$$\begin{array}{ccc}\widehat{R}^{\mathfrak{a}} & \xrightarrow{\chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N)}} & \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N),\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N)) \\ {\scriptstyle \chi^{\widehat{R}^{\mathfrak{a}}}_{\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N)}}\Big\downarrow & & \simeq\Big\downarrow{\scriptstyle (\varepsilon^{N}_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}})_{*}} \\ \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N),\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N)) & & \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N),N) \\ {\scriptstyle (\vartheta^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}_{N})^{*}}\Big\downarrow\simeq & & \simeq\Big\downarrow{\scriptstyle (\vartheta^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}_{N})_{*}} \\ \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(N,\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N)) & \xrightarrow[\simeq]{(\varepsilon^{N}_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}})^{*}} & \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N),\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N)). \end{array}$$

The isomorphisms in this diagram are from [1, Theorem (0.3)*].[e] □

[e]See also [32, Thoerem 6.12]. In addition, we have [32, Remark 6.14] for a discussion of some aspects of this result, and [33] for a correction.

## 4. Adic Semidualizing Complexes

This section consists of examples and fundamental properties of $\mathfrak{a}$-adic semidualizing complexes, including the proof of Theorem 1.2 from the introduction.

**Definition 4.1.** An $R$-complex $M$ is $\mathfrak{a}$-*adically semidualizing* if $M$ is $\mathfrak{a}$-adically finite (see Definition 2.6) and the homothety morphism $\chi_M^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_R(M, M)$ from Definition 3.3 is an isomorphism in $\mathcal{D}(R)$. The set of shift-isomorphism classes in $\mathcal{D}(R)$ of $\mathfrak{a}$-adically semidualizing complexes is denoted $\mathfrak{S}^{\mathfrak{a}}(R)$.

**Remark 4.2.** An $R$-module $M$ is $\mathfrak{a}$-adically semidualizing as an $R$-complex if it is $\mathfrak{a}$-adically finite[f], the natural homothety map $\widehat{R}^{\mathfrak{a}} \to \mathrm{Hom}_R(M, M)$, defined as in Defintion 3.3, is an isomorphism, and $\mathrm{Ext}^i_R(M, M) = 0$ for all $i \geq 1$.

**Remark 4.3.** If $M$ is an $\mathfrak{a}$-adically semidualizing $R$-complex then $\mathrm{supp}_R(M) = \mathrm{V}(\mathfrak{a})$, by [38, Proposition 7.17].

The next two propositions show that Definition 4.1 yields the semidualizing complexes and quasi-dualizing modules as special cases.

**Proposition 4.4.** *An $R$-complex $M$ is semidualizing if and only if it is* 0-*adically semidualizing, that is, we have* $\mathfrak{S}(R) = \mathfrak{S}^0(R)$.

**Proof.** The $R$-complex $M$ is 0-adically finite if and only if it is in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$; see, e.g. [38, Proposition 7.8(a)]. Because of the isomorphism $\widehat{R}^0 \cong R$, we see that the homothety morphisms $\chi_M^{\widehat{R}^0}$ and $\chi_M^R$ are isomorphisms simultaneously. Thus, the result follows by definition. □

**Proposition 4.5.** *Assume that* $(R, \mathfrak{m})$ *is local.*

(a) *An $R$-complex $M \in \mathcal{D}_b(R)$ is $\mathfrak{m}$-adically semidualizing if and only if each homology module $\mathrm{H}_i(M)$ is artinian and the homothety morphism $\chi_M^{\widehat{R}^{\mathfrak{m}}} : \widehat{R}^{\mathfrak{m}} \to \mathbf{R}\mathrm{Hom}_R(M, M)$ is an isomorphism in $\mathcal{D}(R)$.*
(b) *An $R$-module $T$ is quasi-dualizing if and only if it is $\mathfrak{m}$-adically semidualizing.*
(c) *The injective hull $E := E_R(R/\mathfrak{m})$ is $\mathfrak{m}$-adically semidualizing.*

**Proof.** Since each $\mathrm{H}_i(M)$ is artinian if and only if $M \in \mathcal{D}_{\mathrm{b}}(R)$ is $\mathfrak{m}$-adically finite by [38, Proposition 7.8(b)], the result follows by definition and the standard isomorphism $\widehat{R}^{\mathfrak{m}} \xrightarrow{\cong} \mathrm{Hom}_R(E, E)$. □

In light of Fact 2.4 and Remark 2.7, the next result contains Theorem 1.2 from the introduction.

**Theorem 4.6.** *Let $M \in \mathcal{D}_b(R)$. The following conditions are equivalent*:

(i) *$M$ is $\mathfrak{a}$-adically semidualizing over $R$*;

[f] See Remark 2.7.

(ii) *$M$ is $\mathfrak{a}$-adically finite, and we have $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_R(M,M)$ in $\mathcal{D}(R)$; and*
(iii) $\mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ *and $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is semidualizing over $\widehat{R}^{\mathfrak{a}}$.*

**Proof.** The implication (i) $\Rightarrow$ (ii) is by definition.

(ii) $\Rightarrow$ (iii) Assume that $M$ is $\mathfrak{a}$-adically finite and $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_R(M,M)$ in $\mathcal{D}(R)$. By definition, this implies that $\mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$. Also, we have $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M), \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ by Lemma 3.4(b), so [6, Proposition 3.1] implies that $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is semidualizing over $\widehat{R}^{\mathfrak{a}}$.

(iii) $\Rightarrow$ (i) This is verified like the previous implication, using Lemma 3.4(c). □

The next result and its corollary show how to build examples of adic semidualizing complexes.

**Theorem 4.7.** *If $M$ is $\mathfrak{a}$-adically semidualizing complex over $R$ and $\mathfrak{b}$ is an ideal of $R$, then $\mathbf{R}\Gamma_{\mathfrak{b}}(M) \simeq \mathbf{R}\Gamma_{\mathfrak{a}+\mathfrak{b}}(M)$ is $\mathfrak{a}+\mathfrak{b}$-adically semidualizing. In particular, if $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathbf{R}\Gamma_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-adically semidualizing.*

**Proof.** Our assumptions imply that $\mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$, so the natural morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \to M$ is an isomorphism in $\mathcal{D}(R)$ by Fact 2.4. It follows that we have the following isomorphisms in $\mathcal{D}(R)$:

$$\mathbf{R}\Gamma_{\mathfrak{b}}(M) \simeq \mathbf{R}\Gamma_{\mathfrak{b}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}+\mathfrak{b}}(M).$$

Thus, we replace $\mathfrak{b}$ with $\mathfrak{a}+\mathfrak{b}$ to assume that $\mathfrak{a} \subseteq \mathfrak{b}$.

By [38, Theorem 7.10], the fact that $M$ is $\mathfrak{a}$-adically semidualizing implies that $\mathbf{R}\Gamma_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-adically finite. Thus, it suffices by Theorem 4.6 to show that $\widehat{R}^{\mathfrak{b}} \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{b}}(M), \mathbf{R}\Gamma_{\mathfrak{b}}(M))$ in $\mathcal{D}(R)$.

Since $\widehat{R}^{\mathfrak{a}}$ is flat over $R$, the first isomorphism in the next sequence is by definition:

$$\mathbf{L}\Lambda^{\mathfrak{b}}(\widehat{R}^{\mathfrak{a}}) \simeq \Lambda^{\mathfrak{b}}(\widehat{R}^{\mathfrak{a}}) \simeq \widehat{R}^{\mathfrak{b}}.$$

The second isomorphism follows from the containment $\mathfrak{a} \subseteq \mathfrak{b}$ since $\Lambda^{\mathfrak{b}}(-) = \widehat{(-)}^{\mathfrak{b}}$. This explains the first isomorphism in $\mathcal{D}(R)$ in the next sequence.

$$\begin{aligned}
\widehat{R}^{\mathfrak{b}} &\simeq \mathbf{L}\Lambda^{\mathfrak{b}}(\widehat{R}^{\mathfrak{a}}) \\
&\simeq \mathbf{L}\Lambda^{\mathfrak{b}}(\mathbf{R}\mathrm{Hom}_R(M,M)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{b}}(R), \mathbf{R}\mathrm{Hom}_R(M,M)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(M \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{b}}(R), M) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{b}}(M), M) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{b}}(M), \mathbf{R}\Gamma_{\mathfrak{b}}(M)).
\end{aligned}$$

The second isomorphism follows from the isomorphism $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_R(M, M)$. The third and fifth isomorphisms are by Fact 2.1, and the fourth one is Hom-tensor adjointness. The last isomorphism is a consequence of [1, Theorem (0.3)*]. □

**Corollary 4.8.** *If $C$ is a semidualizing $R$-complex, then the complex $\mathbf{R}\Gamma_{\mathfrak{a}}(C)$ is $\mathfrak{a}$-adically semidualizing. Hence, the complex $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-adically semidualizing.*

**Proof.** Since "semidualizing" is equivalent to "0-adically semidualizing", by Proposition 4.4, the first conclusion follows from Theorem 4.7. The second conclusion is the special case $C = R$. □

**Remark 4.9.** Alternately, one can obtain Corollary 4.8 from MGM equivalence 2.2, as follows. By [38, Theorem 7.10], we know that $\mathbf{R}\Gamma_{\mathfrak{a}}(C)$ is $\mathfrak{a}$-adically finite, so it suffices by Theorem 4.6 to show that $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(C), \mathbf{R}\Gamma_{\mathfrak{a}}(C))$ in $\mathcal{D}(R)$. MGM equivalence provides the first isomorphism in the following sequence:

$$\mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(C)) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(C) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R C.$$

The second isomorphism is from Fact 2.1. This explains the second isomorphism in the next sequence:

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(C), \mathbf{R}\Gamma_{\mathfrak{a}}(C)) &\simeq \mathbf{R}\mathrm{Hom}_R(C, \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(C))) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(C, \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R C) \\
&\simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, C) \\
&\simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R R \\
&\simeq \widehat{R}^{\mathfrak{a}}.
\end{aligned}$$

The first isomorphism follows from Fact 2.1 with Hom-tensor adjointness. The third isomorphism is tensor-evaluation [2, Lemma 4.4(F)], the fourth one is from the assumption $\mathbf{R}\mathrm{Hom}_R(C, C) \simeq R$, and the fifth one is tensor-cancellation. From this perspective, the fact that $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-adically semidualizing is even easier:

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\Gamma_{\mathfrak{a}}(R)) &\simeq \mathbf{R}\mathrm{Hom}_R(R, \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(R))) \\
&\simeq \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(R)) \\
&\simeq \mathbf{L}\Lambda^{\mathfrak{a}}(R) \\
&\simeq \widehat{R}^{\mathfrak{a}}.
\end{aligned}$$

## 5. Transfer of the Adic Semidualizing Property

This section focuses on some transfer properties for adic semidualizing complexes, including Theorems 1.1 and 1.3 from the introduction. In particular, it furthers the theme from Theorem 4.6, which describes some of the interplay between the semidualizing $\widehat{R}^{\mathfrak{a}}$-complexes and the $\mathfrak{a}$-adically semidualizing $R$-complexes.

**Notation 5.1.** In this section, let $\varphi : R \to S$ be a homomorphism of commutative noetherian rings with $\mathfrak{a}S \neq S$, and consider the forgetful functor $Q : \mathcal{D}(S) \to \mathcal{D}(R)$.

**Restriction of Scalars.** Note that each result of this subsection and the next one holds for the natural flat homomorphism $R \to \widehat{R}^{\mathfrak{a}}$, moreover, for the map $R \to \widehat{R}^{\mathfrak{b}}$ for any ideal $\mathfrak{b} \subseteq \mathfrak{a}$.

**Proposition 5.2.** *Assume that $\varphi$ is flat and that the induced map $\widehat{R}^{\mathfrak{a}} \to \widehat{S}^{\mathfrak{a}S}$ is an isomorphism. Then an $S$-complex $Y \in \mathcal{D}(S)$ is $\mathfrak{a}S$-adically semidualizing over $S$ if and only if $Q(Y)$ is $\mathfrak{a}$-adically semidualizing over $R$.*

**Proof.** From [37, Lemma 5.3], we know that $\operatorname{supp}_R(Q(Y)) \subseteq \mathrm{V}(\mathfrak{a})$ if and only if $\operatorname{supp}_S(Y) \subseteq \mathrm{V}(\mathfrak{a}S)$. Thus, we assume for the rest of this proof that $\operatorname{supp}_S(Y) \subseteq \mathrm{V}(\mathfrak{a}S)$. It follows that $Y$ is a $\mathfrak{a}S$-adically semidualizing over $S$ if and only if $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(Y)$ is semidualizing over $\widehat{S}^{\mathfrak{a}S} \cong \widehat{R}^{\mathfrak{a}}$, by Theorem 4.6; and $Q(Y)$ is $\mathfrak{a}$-adically semidualizing over $R$ if and only if $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Q(Y))$ is semidualizing over $\widehat{R}^{\mathfrak{a}}$. Thus, it suffices to show that we have $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(Y) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Q(Y))$ in $\mathcal{D}(\widehat{S}^{\mathfrak{a}S})$.[g]

Let $F \xrightarrow{\simeq} Y$ be a semi-flat resolution over $S$. Since $S$ is flat over $R$, this also yields a semi-flat resolution $Q(F) \xrightarrow{\simeq} Q(Y)$. Completing an $S$-complex with respect to $\mathfrak{a}$ is the same as completing it with respect to $\mathfrak{a}S$, so we have

$$\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Q(Y)) \simeq \Lambda^{\mathfrak{a}}(Q(F)) \cong \Lambda^{\mathfrak{a}S}(F) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(Y)$$

in $\mathcal{D}(\widehat{S}^{\mathfrak{a}S})$, as desired. □

**Remark 5.3.** It is worth noting that, even when the map $\varphi$ is flat and local, the hypothesis $\widehat{R}^{\mathfrak{a}} \cong \widehat{S}^{\mathfrak{a}S}$ is necessary for each implication in the previous result. Indeed, using the natural maps $\varphi$ where $S = R[\![X]\!]$ or $R[\![X]\!]/(X^2)$, one implication fails for $Y = S$, and the other implication fails with $Y = R$.

**Extension of Scalars.** Our next Theorem is akin to [8, Theorem 5.6; 16, Theorem 4.5], which describe finite flat dimension base change for semidualizing complexes. First, we prove two lemmas. Recall that the homomorphism $\varphi$ is *locally of finite flat dimension* if, for every prime $\mathfrak{P} \in \operatorname{Spec}(S)$ the induced map $\varphi_{\mathfrak{P}} : R_{\mathfrak{p}} \to S_{\mathfrak{P}}$ has finite flat dimension where $\mathfrak{p}$ is the contraction of $\mathfrak{P}$ in $R$. The main point for introducing this notion is the following fact: if $\varphi$ is of finite flat dimension, (more generally, if it is locally of finite flat dimension–see the next lemma) then the induced map $\widehat{R}^{\mathfrak{a}} \to \widehat{S}^{\mathfrak{a}S}$ is locally of finite flat dimension, though it is not clear that it has finite flat dimension; see [4, Theorem 6.11(c)].

**Lemma 5.4.** (a) *If* $\operatorname{fd}_R(S) < \infty$, *then $\varphi$ is locally of finite flat dimension.*

[g]Technically, we should use the forgetful functor $\mathcal{D}(\widehat{S}^{\mathfrak{a}S}) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$ here. However, since our isomorphism assumption implies that this is an equivalence, we avoid the extra notation.

(b) *If for every maximal ideal* $\mathfrak{M} \in \text{m-Spec}(S)$ *the induced map* $\varphi_{\mathfrak{M}}$ *has finite flat dimension, then* $\varphi$ *is locally of finite flat dimension.*

**Proof.** (a) Let $\mathfrak{P} \in \text{Spec}(S)$ and let $\mathfrak{p} \in \text{Spec}(R)$ denote the contraction of $\mathfrak{P}$ in $R$. We have $\text{fd}_{R_\mathfrak{p}}(S_\mathfrak{p}) \leq \text{fd}_R(S) < \infty$. Since the induced local map $\varphi_{\mathfrak{P}} : R_\mathfrak{p} \to S_\mathfrak{P}$ is the composition of the natural maps $R_\mathfrak{p} \to S_\mathfrak{p} \to S_\mathfrak{P}$, each of which finite flat dimension, the map $\varphi_\mathfrak{P}$ has finite flat dimension as well.

(b) Assume that for every maximal ideal $\mathfrak{M} \in \text{m-Spec}(S)$ the induced map $\varphi_\mathfrak{M}$ has finite flat dimension. Given a prime $\mathfrak{P} \in \text{Spec}(S)$, let $\mathfrak{M} \in \text{m-Spec}(S)$ be such that $\mathfrak{P} \subseteq \mathfrak{M}$. The induced map $\varphi_\mathfrak{M}$ has finite flat dimension by assumption, so it is locally of finite flat dimension by part (a). Under the prime correspondence for localization, the map $\varphi_\mathfrak{P}$ corresponds to the map $(\varphi_\mathfrak{M})_{\mathfrak{P}_\mathfrak{M}}$, so it has finite flat dimension, as desired. □

For perspective and use in the next results, note that [37, Proposition 5.6(a)] shows that $\text{supp}_R(S) = \text{Im}(\varphi^*)$ where $\varphi^* : \text{Spec}(S) \to \text{Spec}(R)$ is the induced map.

**Lemma 5.5.** *Assume that* $\varphi$ *is locally of finite flat dimension, and let* $C \in \mathcal{D}^f(R)$.

(a) *If* $C$ *is a semidualizing* $R$*-complex such that* $S \otimes^{\mathbf{L}}_R C \in \mathcal{D}_b(S)$, *then* $S \otimes^{\mathbf{L}}_R C$ *is semidualizing over* $S$.
(b) *The converse of part* (a) *holds when* $\text{supp}_R(S) \supseteq \text{m-Spec}(R)$.

**Proof.** (a) Assume that $C$ is a semidualizing $R$-complex such that $S \otimes^{\mathbf{L}}_R C \in \mathcal{D}_\text{b}(S)$. Since this implies that $C \in \mathcal{D}^\text{f}_\text{b}(R)$, it follows that we have $S \otimes^{\mathbf{L}}_R C \in \mathcal{D}^\text{f}_\text{b}(S)$. The semidualizing property is local for such complexes [16, Lemma 2.3], so it suffices to show that $(S \otimes^{\mathbf{L}}_R C)_\mathfrak{P}$ is semidualizing over $S_\mathfrak{P}$ for each prime $\mathfrak{P} \in \text{Spec}(S)$. In other words, we need to show that the following $S_\mathfrak{P}$-complex is semidualizing:

$$S_\mathfrak{P} \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R C) \simeq S_\mathfrak{P} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (R_\mathfrak{p} \otimes^{\mathbf{L}}_R C).$$

This is so by [16, Theorem 4.5], because the maps $R \to R_\mathfrak{p} \to S_\mathfrak{P}$ each have finite flat dimension, where $\mathfrak{p}$ is the contraction of $\mathfrak{P}$ in $R$.

(b) Assume that $S \otimes^{\mathbf{L}}_R C$ is semidualizing over $S$, and that we have $\text{supp}_R(S) \supseteq \text{m-Spec}(R)$. In particular, this implies that $S \otimes^{\mathbf{L}}_R C \in \mathcal{D}_\text{b}(R)$.

**Claim.** $C \in \mathcal{D}_\text{b}(R)$.[h] *For this, we use a bit of bookkeeping notation from Foxby* [13]: *given an* $R$*-complex* $Z$, *set*

$$\sup(Z) := \sup\{i \in \mathbb{Z} \,|\, \text{H}_i(Z) \neq 0\}, \quad \inf(Z) := \inf\{i \in \mathbb{Z} \,|\, \text{H}_i(Z) \neq 0\}.$$

With the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. Thus, to prove the claim, we need to show that $-\infty < \inf(C)$ and $\sup(C) < \infty$. Let $\mathfrak{m} \in \text{m-Spec}(R) \subseteq \text{supp}_R(S) = \varphi^*(\text{Spec}(S))$. It follows that there is a maximal ideal $\mathfrak{M} \in \text{m-Spec}(S)$ such that

[h] Note that if we had $\text{fd}_R(S) < \infty$, this would follow from [16, Theorem I].

$\mathfrak{m} = \varphi^{-1}(\mathfrak{M})$. The induced map $\varphi_{\mathfrak{M}}$ has finite flat dimension, so [16, Theorem I(c)] explains the first step in the next sequence.

$$\inf(C_{\mathfrak{m}}) = \inf(S_{\mathfrak{M}} \otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} C_{\mathfrak{m}}) = \inf((S \otimes^{\mathbf{L}}_R C)_{\mathfrak{M}}) \geq \inf(S \otimes^{\mathbf{L}}_R C).$$

The other steps are straightforward. It follows that we have

$$\inf(C) = \inf\{\inf(C_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{m-Spec}(R)\} \geq \inf(S \otimes^{\mathbf{L}}_R C) > -\infty.$$

For $\sup(C)$, we argue similarly:

$$\sup(C_{\mathfrak{m}}) \leq \sup(S_{\mathfrak{M}} \otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} C_{\mathfrak{m}}) = \sup((S \otimes^{\mathbf{L}}_R C)_{\mathfrak{M}}) \leq \sup(S \otimes^{\mathbf{L}}_R C),$$

$$\sup(C) = \sup\{\sup(C_{\mathfrak{m}}) \mid \mathfrak{m} \in \text{m-Spec}(R)\} \leq \sup(S \otimes^{\mathbf{L}}_R C) < \infty.$$

This establishes the Claim.

To complete the proof, it suffices to show that $C_{\mathfrak{m}}$ is semidualizing over $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{m-Spec}(R)$. Let $\mathfrak{m} \in \text{m-Spec}(R)$ be given, and let $\mathfrak{M} \in \text{m-Spec}(S)$ lie over $\mathfrak{m}$. By assumption, the complex $S \otimes^{\mathbf{L}}_R C$ is semidualizing over $S$, so the localization $(S \otimes^{\mathbf{L}}_R C)_{\mathfrak{M}}$ is semidualizing over $S_{\mathfrak{M}}$. That is, the complex

$$S_{\mathfrak{M}} \otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} (R_{\mathfrak{m}} \otimes^{\mathbf{L}}_R C) \simeq S_{\mathfrak{M}} \otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} C_{\mathfrak{m}}$$

is semidualizing over $S_{\mathfrak{M}}$. The induced map $\varphi_{\mathfrak{M}} : R_{\mathfrak{m}} \to S_{\mathfrak{M}}$ is local and has finite flat dimension, so $C_{\mathfrak{m}}$ is semidualizing over $R_{\mathfrak{m}}$, by [16, Theorem 4.5], as desired. □

In the next result, note that $S \otimes^{\mathbf{L}}_R M \in \mathcal{D}_{\mathrm{b}}(S)$ holds automatically if $\mathrm{fd}_R(S) < \infty$.

**Theorem 5.6.** *Assume that $\varphi$ is (locally) of finite flat dimension, and let $M \in \mathcal{D}_b(R)$ be given.*

(a) *If $M$ is $\mathfrak{a}$-adically semidualizing with $S \otimes^{\mathbf{L}}_R M \in \mathcal{D}_b(S)$, then $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically semidualizing over $S$.*
(b) *The converse of part (a) holds if $\mathrm{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \cap \text{m-Spec}(R)$ and $M$ is $\mathfrak{a}$-adically finite over $R$, e.g. if $\varphi$ is local and $M$ is $\mathfrak{a}$-adically finite over $R$.*
(c) *In particular, the converse of part (a) holds if $\mathrm{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcup \mathrm{supp}_R(X)$ and $\varphi$ is flat, e.g. $\varphi$ is faithfully flat.*

**Proof.** (a) Assume that $M$ is $\mathfrak{a}$-adically semidualizing. In particular, $M$ is $\mathfrak{a}$-adically finite over $R$, so [37, Theorem 5.10] implies that $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically finite over $S$, and by [35, Theorem 7.3] we have an isomorphism in $\mathcal{D}(\widehat{S}^{\mathfrak{a}S})$

$$\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_R M).$$

The fact that $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically finite over $S$ tells us that the second displayed complex is in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{S}^{\mathfrak{a}S})$, hence so is the first. Also, Theorem 4.6 implies that $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is semidualizing over $\widehat{R}^{\mathfrak{a}}$. Since the induced map $\widehat{\varphi}^{\mathfrak{a}} : \widehat{R}^{\mathfrak{a}} \to \widehat{S}^{\mathfrak{a}S}$ is locally of finite flat dimension, Lemma 5.5(a) implies that the first displayed complex is semidualizing over $\widehat{S}^{\mathfrak{a}S}$, hence so is the second one. Another application of Theorem 4.6 tells us that $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically semidualizing over $S$, as desired.

(b) **Claim 1.** *if $\varphi$ is local, then we have* $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \cap \text{m-Spec}(R)$. *Indeed, if $\varphi$ is local, then the maximal ideal $\mathfrak{m}$ of $R$ satisfies $\mathfrak{m} \in \varphi^*(\operatorname{Spec}(S))$; hence, we have the second step in the next display.*

$$\mathrm{V}(\mathfrak{a}) \cap \text{m-Spec}(R) = \{\mathfrak{m}\} \subseteq \varphi^*(\operatorname{Spec}(S)) = \operatorname{supp}_R(S).$$

*The first step is from the assumption $\mathfrak{a} \neq R$, since $R$ is local here. The last step is from* [37, Proposition 5.6(a)]. *The establishes Claim* 1.

Now, to prove part (b), assume that $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically semidualizing over $S$, we have $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \cap \text{m-Spec}(R)$, and $M$ is $\mathfrak{a}$-adically finite over $R$. In particular, we have the isomorphism displayed in the previous paragraph. Another application of Theorem 4.6 tells us that $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_R M)$ is semidualizing over $\widehat{S}^{\mathfrak{a}S}$, so the isomorphism implies that $\widehat{S}^{\mathfrak{a}S} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is semidualizing over $\widehat{S}^{\mathfrak{a}S}$ a well.

**Claim 2.** $(\widehat{\varphi}^{\mathfrak{a}})^*(\operatorname{Spec}(\widehat{S}^{\mathfrak{a}S})) \supseteq \text{m-Spec}(\widehat{R}^{\mathfrak{a}})$. *Indeed, let $\mathfrak{m} \in \text{m-Spec}(\widehat{R}^{\mathfrak{a}})$. It follows from Lemma* 2.10(a) *that there is a maximal ideal $\mathfrak{m}_0 \in \text{m-Spec}(R) \cap \mathrm{V}(\mathfrak{a})$ such that $\mathfrak{m} = \mathfrak{m}_0\widehat{R}^{\mathfrak{a}}$. By assumption, there is a maximal ideal $\mathfrak{M}_0 \in \text{m-Spec}(S)$ such that $\mathfrak{m}_0 = \varphi^{-1}(\mathfrak{M}_0)$. The condition $\mathfrak{m}_0 \supseteq \mathfrak{a}$ implies that $\mathfrak{M}_0 \supseteq \mathfrak{m}_0S \supseteq \mathfrak{a}S$. An application of Lemma* 2.10(a) *to $S$ shows that the ideal $\mathfrak{M} := \mathfrak{M}_0\widehat{S}^{\mathfrak{a}S} \supseteq \mathfrak{a}\widehat{S}^{\mathfrak{a}S}$ is maximal in $\widehat{S}^{\mathfrak{a}S}$ and contracts to $\mathfrak{M}_0$ in $S$. In particular, the prime ideal $\mathfrak{P} := (\widehat{\varphi}^{\mathfrak{a}})^{-1}(\mathfrak{M}) \supseteq \mathfrak{a}\widehat{R}^{\mathfrak{a}}$ contracts to $\mathfrak{m}_0$ in $R$. Since $\mathfrak{P}$ and $\mathfrak{m}$ both are in $\mathrm{V}(\mathfrak{a}\widehat{R}^{\mathfrak{a}})$ and contract to $\mathfrak{m}_0$ in $R$, Lemma* 2.10(b) *implies that $\mathfrak{P} = \mathfrak{m}$. This establishes Claim* 2.

Now we complete the proof of part (b). Recall that the induced map $\widehat{R}^{\mathfrak{a}} \to \widehat{S}^{\mathfrak{a}S}$ is locally of finite flat dimension. Since $M$ is $\mathfrak{a}$-adically finite over $R$, we have $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $\operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$. Thus, Lemma 5.5(b) conspires with Claim 2 to imply that $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is semidualizing over $\widehat{R}^{\mathfrak{a}}$, and Theorem 4.6 implies that $M$ is $\mathfrak{a}$-adically semidualizing over $R$, as desired.

(c) Assume that $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \cup \operatorname{supp}_R(X)$ and $\varphi$ is flat. If $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically semidualizing over $S$, then $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically finite over $S$, so [37, Theorem 5.10] implies that $M$ is $\mathfrak{a}$-adically finite over $R$. Thus, the desired converse follows from part (b). □

Here is a local-global principal for the adic semidualizing property.

**Theorem 5.7.** *Let $M \in \mathcal{D}_b(R)$ be $\mathfrak{a}$-adically finite. The following are equivalent*:

(i) *$M$ is $\mathfrak{a}$-adically semidualizing.*
(ii) *For each multiplicatively closed subset $U \subseteq R$ such that $\mathfrak{a}U^{-1}R \neq U^{-1}R$, the $U^{-1}R$-complex $U^{-1}M \simeq (U^{-1}R) \otimes^{\mathbf{L}}_R M$ is $U^{-1}\mathfrak{a}$-adically semidualizing.*
(iii) *For all $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$, the $R_{\mathfrak{p}}$-complex $M_{\mathfrak{p}} \simeq R_{\mathfrak{p}} \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}_{\mathfrak{p}}$-adically semidualizing.*
(iv) *For all $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \cap \text{m-Spec}(R)$, the $R_{\mathfrak{m}}$-complex $M_{\mathfrak{m}} \simeq R_{\mathfrak{m}} \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}_{\mathfrak{m}}$-adically semidualizing.*

**Proof.** In view of Theorem 5.6(a), it suffices to prove the implication (iv) $\Rightarrow$ (i). Assume that for each $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \cap \text{m-Spec}(R)$, the $R_{\mathfrak{m}}$-complex $M_{\mathfrak{m}}$ is $\mathfrak{a}_{\mathfrak{m}}$-adically semidualizing. Since $M$ is also assumed to be $\mathfrak{a}$-adically finite, to prove that is it $\mathfrak{a}$-adically semidualizing, it suffices by Theorem 4.6 to show that $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is semidualizing over $\widehat{R}^{\mathfrak{a}}$. By assumption, we have $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$, so to show that this complex is semidualizing over $\widehat{R}^{\mathfrak{a}}$, it suffices to show that for each $\mathfrak{M} \in \text{m-Spec}(\widehat{R}^{\mathfrak{a}})$ the localization $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)_{\mathfrak{M}}$ is semidualizing over $(\widehat{R}^{\mathfrak{a}})_{\mathfrak{M}}$; see [16, Lemma 2.3].

Let $\mathfrak{M} \in \text{m-Spec}(\widehat{R}^{\mathfrak{a}})$ be given, and let $\mathfrak{m}$ be the contraction of $\mathfrak{M}$ in $R$. By assumption, the $R_{\mathfrak{m}}$-complex $M_{\mathfrak{m}}$ is $\mathfrak{a}_{\mathfrak{m}}$-adically semidualizing. Thus, Theorem 4.6 implies that the complex $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}_{\mathfrak{m}}}(M_{\mathfrak{m}})$ is semidualizing over $\widehat{R_{\mathfrak{m}}}^{\mathfrak{a}_{\mathfrak{m}}}$.

According to [18, Corollaire 0.7.6.14], the natural map $\widehat{R}^{\mathfrak{a}} \to \widehat{R_{\mathfrak{m}}}^{\mathfrak{a}_{\mathfrak{m}}}$ is flat. The ring $\widehat{R_{\mathfrak{m}}}^{\mathfrak{a}_{\mathfrak{m}}}$ is local with maximal ideal $\mathfrak{m}\widehat{R_{\mathfrak{m}}}^{\mathfrak{a}_{\mathfrak{m}}}$, and the contraction of this maximal ideal in $\widehat{R}^{\mathfrak{a}}$ is $\mathfrak{m}\widehat{R}^{\mathfrak{a}} = \mathfrak{M}$. It follows that the induced map $(\widehat{R}^{\mathfrak{a}})_{\mathfrak{M}} \to \widehat{R_{\mathfrak{m}}}^{\mathfrak{a}_{\mathfrak{m}}}$ is flat and local. As we have already seen, the next complex is semidualizing over $\widehat{R_{\mathfrak{m}}}^{\mathfrak{a}_{\mathfrak{m}}}$.

$$\begin{aligned}
\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}_{\mathfrak{m}}}(M_{\mathfrak{m}}) &\simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}R_{\mathfrak{m}}}(R_{\mathfrak{m}} \otimes^{\mathbf{L}}_{R} M) \\
&\simeq \widehat{R_{\mathfrak{m}}}^{\mathfrak{a}R_{\mathfrak{m}}} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \\
&\simeq \widehat{R_{\mathfrak{m}}}^{\mathfrak{a}R_{\mathfrak{m}}} \otimes^{\mathbf{L}}_{(\widehat{R}^{\mathfrak{a}})_{\mathfrak{M}}} ((\widehat{R}^{\mathfrak{a}})_{\mathfrak{M}} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) \\
&\simeq \widehat{R_{\mathfrak{m}}}^{\mathfrak{a}_{\mathfrak{m}}} \otimes^{\mathbf{L}}_{(\widehat{R}^{\mathfrak{a}})_{\mathfrak{M}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)_{\mathfrak{M}}.
\end{aligned}$$

The second isomorphism here is by [35, Theorem 7.3] with $S = R_{\mathfrak{m}}$. Since we have $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)_{\mathfrak{M}} \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R_{\mathfrak{m}}}^{\mathfrak{a}_{\mathfrak{m}}})$, it follows from [8, Theorem 5.6] that $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)_{\mathfrak{M}}$ is semidualizing over $\widehat{R_{\mathfrak{m}}}^{\mathfrak{a}_{\mathfrak{m}}}$ as desired. $\square$

**Extended Derived Local Cohomology.** We next consider the behavior of adic semidualizing complexes with respect to the functor $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$.

**Proposition 5.8.** *An $R$-complex $M \in \mathcal{D}_b(R)$ is $\mathfrak{a}$-adically semidualizing over $R$ if and only if* $\mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ *and* $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)$ *is $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-adically semidualizing over $\widehat{R}^{\mathfrak{a}}$.*

**Proof.** For the forward implication, assume that $M$ is $\mathfrak{a}$-adically semidualizing over $R$. In particular, we have $\mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$, hence $M \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(M)$ by Fact 2.4. From Fact 2.1 we therefore have isomorphisms

$$\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} \mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_{R} M$$

in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. Thus, Theorem 5.6(a) implies that $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)$ is $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-adically semidualizing over $\widehat{R}^{\mathfrak{a}}$, as desired.

For the converse, assume that $\mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ and that $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)$ is $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-adically semidualizing over $\widehat{R}^{\mathfrak{a}}$. The condition $\mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ implies that we have $M \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq Q(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M))$ in $\mathcal{D}(R)$, by Facts 2.1 and 2.4, respectively. Proposition 5.2 implies that $M$ is $\mathfrak{a}$-adically semidualizing over $R$, as desired. $\square$

**Remark 5.9.** Some of our results become trivial when $R$ is $\mathfrak{a}$-adically complete. For instance, if $R$ is $\mathfrak{a}$-adically complete, then the conclusion of the previous result says that $M$ is $\mathfrak{a}$-adically semidualizing if and only if $\operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ and $M$ is $\mathfrak{a}$-adically semidualizing. Indeed, the completeness assumption implies $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} = \mathbf{R}\Gamma_{\mathfrak{a}}$; so if $\operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$, e.g. if $M$ is $\mathfrak{a}$-adically semidualizing, then this says that $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq M$ by Fact 2.4. Similar comments apply to our next result.

On the other hand, other results of this section have cleaner (and nontrivial) statements when one assumes that $R$ is $\mathfrak{a}$-adically complete. We include a few of these explicitly below.

Our next result contains part of Theorem 1.3 from the introduction.

**Theorem 5.10.** *The functor $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ induces a bijection $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}})$ with inverse induced by the forgetful functor $Q : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$. Also, the bijection $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}} : \mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}})$ is the same as the base-change map $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R - : \mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}})$ from Theorem* 5.6.

**Proof.** Propositions 5.2 and 5.8 show that $Q$ and $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ induce well-defined maps $\mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}) \to \mathfrak{S}^{\mathfrak{a}}(R)$ and $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}})$. Also, $Q$ and $\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}$ induce inverse equivalences between $\mathcal{D}(R)_{\mathfrak{a}\text{-tor}}$ and $\mathcal{D}(\widehat{R}^{\mathfrak{a}})_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}\text{-tor}}$, by [35, Theorem 4.11]; as these contain the $\mathfrak{a}$-adic semidualizing $R$-complexes and the $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-adic semidualizing $\widehat{R}^{\mathfrak{a}}$-complexes, respectively, the maps $\mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}) \to \mathfrak{S}^{\mathfrak{a}}(R)$ and $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}})$ are inverse bijections. Lastly, the proof of Proposition 5.8 shows that the maps $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}})$ induced by $\mathbf{R}\Gamma_{\mathfrak{a}}$ and $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R -$ are equal. $\square$

**Corollary 5.11.** *Assume that $\varphi$ is flat and that the induced map $\widehat{R}^{\mathfrak{a}} \to \widehat{S}^{\mathfrak{a}S}$ is an isomorphism. Then the forgetful functor $Q$ induces a bijection $\mathfrak{S}^{\mathfrak{a}S}(S) \to \mathfrak{S}^{\mathfrak{a}}(R)$ with inverse induced by the functor $S \otimes^{\mathbf{L}}_R -$.*

**Proof.** Proposition 5.2 and Theorem 5.6(a) show that the functors $Q$ and $S \otimes^{\mathbf{L}}_R -$ induce well-defined functions $\mathfrak{S}^{\mathfrak{a}S}(S) \to \mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}^{\mathfrak{a}S}(S)$. Also, the next diagrams commute, where the unspecified maps are given by the respective forgetful functors.

$$\begin{array}{ccc} \mathfrak{S}^{\mathfrak{a}}(R) & \xrightarrow{S\otimes^{\mathbf{L}}_R -} & \mathfrak{S}^{\mathfrak{a}S}(S) \\ {\scriptstyle \widehat{R}^{\mathfrak{a}}\otimes^{\mathbf{L}}_R -}\downarrow \simeq & & \simeq \downarrow {\scriptstyle \widehat{S}^{\mathfrak{a}S}\otimes^{\mathbf{L}}_S -} \\ \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}) & \xrightarrow[\simeq]{\widehat{S}^{\mathfrak{a}S}\otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} -} & \mathfrak{S}^{\mathfrak{a}\widehat{S}^{\mathfrak{a}S}}(\widehat{S}^{\mathfrak{a}S}) \end{array} \qquad \begin{array}{ccc} \mathfrak{S}^{\mathfrak{a}S}(S) & \xrightarrow{Q} & \mathfrak{S}^{\mathfrak{a}}(R) \\ \simeq \uparrow & & \simeq \uparrow \\ \mathfrak{S}^{\mathfrak{a}\widehat{S}^{\mathfrak{a}S}}(\widehat{S}^{\mathfrak{a}S}) & \xrightarrow[\simeq]{} & \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}). \end{array}$$

The vertical bijections are from Theorem 5.10, and the horizontal ones are from our completion assumption. It follows that the upper horizontal maps are bijections as well. Furthermore, since three of the four forgetful maps are the inverses of the corresponding base change maps, one uses the diagrams to show that the upper horizontal maps compose to the respective identities, as desired. $\square$

For perspective in the next result, recall that [37, Theorem 6.1] shows that every $\mathfrak{a}$-adically finite $R$-complex $X$ satisfies $\mathrm{fd}_R(X) = \mathrm{pd}_R(X)$.

**Corollary 5.12.** *Assume $\varphi$ is flat and the induced map $\widehat{R}^{\mathfrak{a}} \to \widehat{S}^{\mathfrak{a}S}$ is bijective.*

(a) *Given an $\mathfrak{a}$-adic semidualizing complex $M \in \mathfrak{S}^{\mathfrak{a}}(R)$, we have*

$$\mathrm{id}_R(M) = \mathrm{id}_S(S \otimes^{\mathbf{L}}_R M), \quad \mathrm{fd}_R(M) = \mathrm{fd}_S(S \otimes^{\mathbf{L}}_R M).$$

(b) *Given an $\mathfrak{a}S$-adic semidualizing complex $N \in \mathfrak{S}^{\mathfrak{a}S}(S)$, we have*

$$\mathrm{id}_R(Q(N)) = \mathrm{id}_S(N), \quad \mathrm{fd}_R(Q(N)) = \mathrm{fd}_S(N).$$

**Proof.** Note that the fact that the map $\widehat{R}^{\mathfrak{a}} \to \widehat{S}^{\mathfrak{a}S}$ is an isomorphism implies that the same is true of the lower horizontal map in the next commutative diagram

$$\begin{array}{ccc} R/\mathfrak{a} & \longrightarrow & S/\mathfrak{a}S \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \widehat{R}^{\mathfrak{a}}/\mathfrak{a}\widehat{R}^{\mathfrak{a}} & \xrightarrow{\cong} & \widehat{S}^{\mathfrak{a}S}/\mathfrak{a}\widehat{S}^{\mathfrak{a}S}. \end{array}$$

It follows that the upper horizontal map is also an isomorphism. Consequently, for each $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$, the induced map $R/\mathfrak{p} \to S/\mathfrak{p}S$ is an isomorphism, hence so is the map $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$.

Let $M \in \mathfrak{S}^{\mathfrak{a}}(R)$ and $N \in \mathfrak{S}^{\mathfrak{a}S}(S)$ be given. We prove the injective dimension formulas; the flat dimension formulas are verified similarly.

The inequality $\mathrm{id}_R(Q(N)) \leq \mathrm{id}_S(N)$ holds because the flatness of $\varphi$ implies that any (bounded) semi-injective resolution of $N$ over $S$ restricts to a (bounded) semi-injective resolution of $Q(N)$ over $R$.

Next, we verify the inequality $\mathrm{id}_S(S \otimes^{\mathbf{L}}_R M) \leq \mathrm{id}_R(M)$. For this argument, assume without loss of generality that $\mathrm{id}_R(M) < \infty$. Then the minimal semi-injective resolution $M \xrightarrow{\simeq} J$ over $R$ is bounded with minimal length, since it is a direct summand of every semi-injective resolution of $M$. Furthermore, each module $J_i$ is a direct sum of $R$-modules of the form $E_R(R/\mathfrak{p})$ with $\mathfrak{p} \in \mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$; see [38, Proposition 3.8]. Since $\varphi$ is flat, we have $S \otimes^{\mathbf{L}}_R M \simeq S \otimes_R J$ in $\mathcal{D}(S)$, so it suffices to show that each module $S \otimes_R J_i$ is injective over $S$. This follows from [12, Theorem 1]: for each $\mathfrak{p} \in \mathrm{Ass}_R(J_i) \subseteq \mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$, the map $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$ is an isomorphism, so the $S$-module $S \otimes_R J_i$ is injective.

By Corollary 5.11, we have $Q(S \otimes^{\mathbf{L}}_R M) \simeq M$, so the previous two paragraphs (with $N = S \otimes^{\mathbf{L}}_R M$) imply that

$$\mathrm{id}_R(M) = \mathrm{id}_R(Q(S \otimes^{\mathbf{L}}_R M)) \leq \mathrm{id}_S(S \otimes^{\mathbf{L}}_R M) \leq \mathrm{id}_R(M),$$

so we have the first formula in part (a). The first formula in part (b) follows similarly using the isomorphism $N \simeq S \otimes^{\mathbf{L}}_R (Q(N))$ from Corollary 5.11. $\square$

For convenience, we state the special case $S = \widehat{R}^{\mathfrak{a}}$ of the previous result next.

**Corollary 5.13.** *Consider the forgetful functor $Q : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$.*

(a) *Given an $\mathfrak{a}$-adic semidualizing complex $M \in \mathfrak{S}^{\mathfrak{a}}(R)$, we have*

$$\mathrm{id}_R(M) = \mathrm{id}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)), \quad \mathrm{fd}_R(M) = \mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)).$$

(b) *Given an $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-adic semidualizing complex $N \in \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}})$, we have*

$$\mathrm{id}_R(Q(N)) = \mathrm{id}_{\widehat{R}^{\mathfrak{a}}}(N), \quad \mathrm{fd}_R(Q(N)) = \mathrm{fd}_{\widehat{R}^{\mathfrak{a}}}(N).$$

**Proof.** By Theorem 5.10, we have $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R M \simeq \mathbf{R}\widehat{\Gamma}_{\mathfrak{a}}(M)$, so the desired conclusions follow from Corollary 5.12. □

**Extended Derived Local Homology.** We now investigate the interaction between the adic semidualizing property and the functor $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$, building on Theorem 4.6. Our next result contains the rest of Theorem 1.3 from the introduction.

**Theorem 5.14.** *Consider the forgetful functor $Q : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$. Then $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ induces a bijection $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}(\widehat{R}^{\mathfrak{a}})$ with inverse induced by $Q \circ \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \simeq \mathbf{R}\Gamma_{\mathfrak{a}} \circ Q$.*

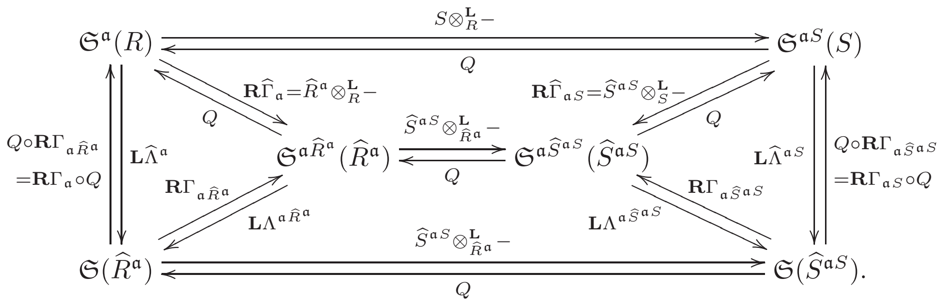
**Proof.** The isomorphism $Q \circ \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}} \simeq \mathbf{R}\Gamma_{\mathfrak{a}} \circ Q$ is from [35, Corollary 4.12].

By Theorem 4.6, the functor $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ induces a well-defined function $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}(\widehat{R}^{\mathfrak{a}})$. On the other hand, Corollary 4.8 and Proposition 5.2 show that the functors $\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}$ and $Q$ induce well-defined functions $\mathfrak{S}(\widehat{R}^{\mathfrak{a}}) \to \mathfrak{S}^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}) \to \mathfrak{S}^{\mathfrak{a}}(R)$.

Furthermore, from [35, Theorem 6.3(b)], the functor $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ induces an equivalence between the category of $\mathfrak{a}$-adically finite $R$-complexes and the category $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$, with quasi-inverse induced by $Q \circ \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}$. Since the $\mathfrak{a}$-adically semidualizing $R$-complexes are $\mathfrak{a}$-adically finite over $R$, and the semidualizing $\widehat{R}^{\mathfrak{a}}$-complexes are in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$, the maps from the previous paragraph are inverse bijections. □

**Corollary 5.15.** *Assume that $R$ is $\mathfrak{a}$-adically complete. The functor $\mathbf{L}\Lambda^{\mathfrak{a}}$ induces a bijection $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}(R)$ with inverse induced by $\mathbf{R}\Gamma_{\mathfrak{a}}$.*

**Remark 5.16.** Assume that $\varphi$ is flat and that the induced map $\widehat{R}^{\mathfrak{a}} \to \widehat{S}^{\mathfrak{a}S}$ is an isomorphism. The following diagram displays the bijections described in Theorem 5.10, Corollary 5.11 and Theorem 5.14; in it, each pair of arrows is an inverse pair, and each cell commutes (in every composable combination). In a feeble attempt to keep the notation from getting out of hand, we use $Q$ for each of the forgetful functors.



We end this subsection with connections to tilting and dualizing complexes.

**Corollary 5.17.** *Consider the forgetful functor* $Q : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$. *Let* $M \in \mathfrak{S}^{a}(R)$ *and* $C \in \mathfrak{S}(\widehat{R}^{\mathfrak{a}})$ *be given.*

(a) *We have* $\operatorname{fd}_R(M) < \infty$ *if and only if* $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ *is a tilting* $\widehat{R}^{\mathfrak{a}}$*-complex.*
(b) *We have* $\operatorname{id}_R(M) < \infty$ *if and only if* $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ *is a dualizing* $\widehat{R}^{\mathfrak{a}}$*-complex.*
(c) *The complex* $C$ *is tilting over* $\widehat{R}^{\mathfrak{a}}$ *if and only if* $\operatorname{fd}_R(Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(C))) < \infty$.
(d) *The complex* $C$ *is dualizing over* $\widehat{R}^{\mathfrak{a}}$ *if and only if* $\operatorname{id}_R(Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(C))) < \infty$.

**Proof.** By Theorem 5.14, the complex $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is semidualizing over $\widehat{R}^{\mathfrak{a}}$ such that $M \simeq Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M))) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Q(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)))$, and the complex $Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(C)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Q(C))$ is $\mathfrak{a}$-adically semidualizing over $R$ such that $C \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(C)))$.

In this paragraph, we assume that $\operatorname{fd}_R(M) < \infty$ and show that $\operatorname{pd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) < \infty$. From [35, Proposition 4.5(b)], we have $\operatorname{fd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) \le \operatorname{fd}_R(M) < \infty$. Since $M$ is $\mathfrak{a}$-adically finite, the complex $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is homologically finite over $\widehat{R}^{\mathfrak{a}}$, and it follows that $\operatorname{pd}_{\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) < \infty$, as desired.

Next, we assume that $\operatorname{pd}_{\widehat{R}^{\mathfrak{a}}}(C) < \infty$ and show that $\operatorname{fd}_R(Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(C))) < \infty$. Since $\widehat{R}^{\mathfrak{a}}$ is flat over $R$, we have $\operatorname{fd}_R(Q(C)) \le \operatorname{pd}_{\widehat{R}^{\mathfrak{a}}}(C) < \infty$. Thus, the following $R$-complexes have finite flat dimension over $R$:

$$Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(C)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Q(C)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R Q(C),$$

since $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ and $Q(C)$ both have finite flat dimension.

As in the proof of Corollary 5.12, parts (a) and (c) now follow. Parts (b) and (d) are verified similarly. □

**Corollary 5.18.** *Assume that* $R$ *is* $\mathfrak{a}$*-adically complete, and let* $M \in \mathfrak{S}^{a}(R)$ *and* $C \in \mathfrak{S}(R)$ *be given.*

(a) *We have* $\operatorname{fd}_R(M) < \infty$ *if and only if* $\mathbf{L}\Lambda^{\mathfrak{a}}(M)$ *is a tilting* $R$*-complex.*
(b) *We have* $\operatorname{id}_R(M) < \infty$ *if and only if* $\mathbf{L}\Lambda^{\mathfrak{a}}(M)$ *is a dualizing* $R$*-complex.*
(c) *The complex* $C$ *is tilting over* $R$ *if and only if* $\operatorname{fd}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(C)) < \infty$.
(d) *The complex* $C$ *is dualizing over* $R$ *if and only if* $\operatorname{id}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(C)) < \infty$.

**Semidualizing DG *K*-Modules.** Given an $\mathfrak{a}$-adic semidualizing $R$-complex $M$, a crucial point in Theorem 5.14 is that we have $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. By Fact 2.5, we also have $K \otimes^{\mathbf{L}}_R M \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$; when translated to the language of "DG $K$-modules", this says $K \otimes^{\mathbf{L}}_R M \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. In short, this means that, when one considers the exterior algebra structure on $K$, the complex $K \otimes^{\mathbf{L}}_R M \simeq K \otimes_R M$ inherits a $K$-module structure from the left; the DG structure means that this scalar multiplication respects the differentials in these complexes.

In this setting, one forms the derived category $\mathcal{D}(R)$ from the category of DG $K$-modules like one forms $\mathcal{D}(R)$ from the category of $R$-complexes. A DG $K$-module

$N$ is in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(K)$ provided that $\bigoplus_{i\in\mathbb{Z}} \mathrm{H}_i(N)$ is finitely generated over $R$, that is, when $N$ is in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. And $N$ is a *semidualizing* DG $K$-module when it is in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(K)$ and the natural homothety morphism $K \to \mathbf{R}\mathrm{Hom}_K(N,N)$ is an isomorphism in $\mathcal{D}(K)$. The set of shift-isomorphism classes of semidualizing DG $K$-modules is denoted $\mathfrak{S}(K)$. See [9, 30, 31] for more about these objects, including applications to the study of $\mathfrak{S}(R)$.

We now reach the point of this discussion: in the same way that the condition $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$ makes it reasonable for us to have $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M) \in \mathfrak{S}(\widehat{R}^{\mathfrak{a}})$, the condition $K \otimes^{\mathbf{L}}_R M \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(K)$ makes it reasonable for us to have $K \otimes^{\mathbf{L}}_R M \in \mathfrak{S}(K)$, as we see in the next result.

**Corollary 5.19.** *The functor $K \otimes^{\mathbf{L}}_R -$ induces a bijection $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}(K)$.*

**Proof.** Set $\widehat{K} := \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R K$, which is the Koszul complex over $\widehat{R}^{\mathfrak{a}}$ on a finite generating sequence for $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$.

Theorem 5.14, implies that the functor $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(-) : \mathcal{D}(R) \to \mathcal{D}(\widehat{R}^{\mathfrak{a}})$ induces a bijection $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}(\widehat{R}^{\mathfrak{a}})$. From [30, Corollary 3.10], we know that the functor $\widehat{K} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} - : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(\widehat{K})$ induces a bijection $\mathfrak{S}(\widehat{R}^{\mathfrak{a}}) \to \mathfrak{S}(\widehat{K})$. Also, since the natural map $K \to \widehat{K}$ is a quasiisomorphism of DG algebras, the forgetful functor $Q : \mathcal{D}(\widehat{K}) \to \mathcal{D}(K)$ induces a bijection $\mathfrak{S}(\widehat{K}) \to \mathfrak{S}(K)$. Thus, it remains to show that the composition of these bijections $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}(\widehat{R}^{\mathfrak{a}}) \to \mathfrak{S}(\widehat{K}) \to \mathfrak{S}(K)$ is given by $K \otimes^{\mathbf{L}}_R -$.

Let $M \in \mathfrak{S}^{\mathfrak{a}}(R)$. We need to show that $Q(\widehat{K} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) \simeq K \otimes^{\mathbf{L}}_R M$ in $\mathcal{D}(K)$. This is accomplished in the next sequence, wherein $Q' : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$ denotes the forgetful functor.

$$\begin{aligned} Q(\widehat{K} \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) &\simeq Q((K \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}) \otimes^{\mathbf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) \\ &\simeq K \otimes^{\mathbf{L}}_R Q'(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)) \\ &\simeq K \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(M) \\ &\simeq K \otimes^{\mathbf{L}}_R M. \end{aligned}$$

The first two isomorphisms here are straightforward, and the third one is from Fact 2.1. For the fourth one, note that [35, Lemma 2.8] shows that the natural morphism $K \otimes^{\mathbf{L}}_R M \to K \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(M)$ is an isomorphism in $\mathcal{D}(R)$. Since it is also a morphism in $\mathcal{D}(K)$, it is also an isomorphism in $\mathcal{D}(K)$, as desired. □

**Remark 5.20.** Unlike in our previous results, it is not clear how to give a functorial description of the inverse of the bijection $\mathfrak{S}^{\mathfrak{a}}(R) \to \mathfrak{S}(K)$ from Corollary 5.19. The problem is that [30, Corollary 3.10] uses a lifting property to show that the map $\mathfrak{S}(\widehat{R}^{\mathfrak{a}}) \to \mathfrak{S}(\widehat{K})$ is bijective, but it does not give a functorial description of the inverse of this map, nor is it clear that such a description exists.

## 6. Dualizing Complexes and Gorenstein Rings

We begin this section by proving Theorem 1.1 from the introduction.

**Theorem 6.1.** (a) *The ring* $\widehat{R}^{\mathfrak{a}}$ *has a dualizing complex if and only if* $R$ *has an* $\mathfrak{a}$*-adically semidualizing complex of finite injective dimension.*

(b) *If* $(R,\mathfrak{m},k)$ *is local, then the* $\widehat{R}^{\mathfrak{m}}$*-complex* $\mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(E_R(k))$ *is dualizing for* $\widehat{R}^{\mathfrak{m}}$*.*

**Proof.** (a) For one implication, if $\widehat{R}^{\mathfrak{a}}$ has a dualizing complex $D$, then Theorem 5.14 and Corollary 5.17(d) imply that $M := Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(D))$ is an $\mathfrak{a}$-adically semidualizing $R$-complex of finite injective dimension. Conversely, if $M$ is an $\mathfrak{a}$-adically semidualizing $R$-complex with $\mathrm{id}_R(M) < \infty$, then Corollary 5.17(b) implies that the complex $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(M)$ is dualizing over $\widehat{R}^{\mathfrak{a}}$.

(b) When $(R,\mathfrak{m},k)$ is local, the injective hull $E := E_R(k)$ is $\mathfrak{m}$-adically semidualizing by Proposition 4.5. Since it also has finite injective dimension over $R$, the desired conclusion follows from Corollary 5.17(b) as in the previous paragraph.

Alternately, one can prove this using Grothendieck's local duality, appropriately extended. Indeed, in the following display, the first isomorphism in $\mathcal{D}(\widehat{R}^{\mathfrak{m}})$ is from [35, Theorem 5.1].

$$\begin{aligned}\mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(E) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\widehat{\Gamma}_{\mathfrak{m}}(R),E)\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{m}}}(\mathbf{R}\widehat{\Gamma}_{\mathfrak{m}}(R),E)\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{m}}}(\mathbf{R}\Gamma_{\mathfrak{m}\widehat{R}^{\mathfrak{m}}}(\widehat{R}^{\mathfrak{m}}),E).\end{aligned}$$

The second isomorphism is verified as in the proof of [35, Theorem 5.1], and the third one is from [35, Lemmas 4.4 and 4.5]. Now, local duality over $\widehat{R}^{\mathfrak{m}}$ allows us to conclude that the last complex in this display is dualizing for $\widehat{R}^{\mathfrak{m}}$. □

**Corollary 6.2.** (a) *An* $\mathfrak{a}$*-adically complete ring has a dualizing complex if and only if it has an* $\mathfrak{a}$*-adically semidualizing complex of finite injective dimension.*

(b) *If* $(R,\mathfrak{m},k)$ *is local and* $\mathfrak{m}$*-adically complete, then the* $R$*-complex* $\mathbf{L}\Lambda^{\mathfrak{m}}(E_R(k))$ *is dualizing for* $R$*.*

**Remark 6.3.** It is important to note that Theorem 6.1(a) cannot be used to construct dualizing complexes for rings that do not have them, obviously. The point is that the condition of $R$ having an $\mathfrak{a}$-adic semidualizing complex of finite injective dimension can be quite restrictive, in general.

Our alternate proof of Theorem 6.1(b) uses the fact that $\widehat{R}^{\mathfrak{m}}$ has a dualizing complex, since that is part of local duality. On the other hand, the first proof we give for this result does not use this fact, so it gives a new proof of the existence of a dualizing complex for $\widehat{R}^{\mathfrak{m}}$.

Also, from [35, Theorem 5.1] we have the next isomorphism in $\mathcal{D}(\widehat{R}^{\mathfrak{m}})$

$$\mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(E) \simeq \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{m}},\mathbf{L}\Lambda^{\mathfrak{m}}(E)),$$

so this gives another strange description of a dualizing complex for $\widehat{R}^{\mathfrak{m}}$.

This result also shows a stark distinction between the functors $\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}$ and $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R -$. Indeed, the complex $\mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(E)$ is dualizing for $\widehat{R}^{\mathfrak{m}}$; in particular, it is homologically finite over $\widehat{R}^{\mathfrak{m}}$. On the other hand, we have $\widehat{R}^{\mathfrak{m}} \otimes^{\mathbf{L}}_R E \simeq E_{\widehat{R}^{\mathfrak{m}}}(k)$, which is only homologically finite over $\widehat{R}^{\mathfrak{m}}$ if $R$ is artinian. Even when $R$ is $\mathfrak{m}$-adically complete, this shows how strange the functor $\mathbf{L}\Lambda^{\mathfrak{a}}$ is, for instance, since $E$ is a module, but $\mathbf{L}\Lambda^{\mathfrak{m}}(E)$ is a dualizing complex for $R$, by Corollary 6.2(b).

We now turn our attention to a uniqueness result for Gorenstein rings. The next result and its corollary should be compared to [8, Corollary 8.6], which says that the semidualizing complexes over a local Gorenstein ring $R$ are exactly the complexes of the form $\Sigma^n R$ for some $n$.

**Corollary 6.4.** *Assume that $\widehat{R}^{\mathfrak{a}}$ is locally Gorenstein, e.g. that $R$ is locally Gorenstein. Consider the forgetful functor $Q : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \to \mathcal{D}(R)$.*

(a) *The $\mathfrak{a}$-adically semidualizing $R$-complexes are precisely the $R$-complexes of the form $Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(L))$ for some tilting $\widehat{R}^{\mathfrak{a}}$-complex $L$.*
(b) *Assume that $(R, \mathfrak{m}, k)$ is local. Then the $\mathfrak{a}$-adically semidualizing $R$-complexes are precisely the $R$-complexes of the form $\Sigma^n \mathbf{R}\Gamma_{\mathfrak{a}}(R)$ for some $n \in \mathbb{Z}$. In particular, the $\mathfrak{m}$-adically semidualizing complexes are precisely the $R$-complexes of the form $\Sigma^n E_R(k)$ for some $n \in \mathbb{Z}$.*

**Proof.** Lemma 2.10(c) shows that if $R$ is locally Gorenstein, then so is $\widehat{R}^{\mathfrak{a}}$.

(a) In view of Theorem 5.14, it suffices to show that every semidualizing $\widehat{R}^{\mathfrak{a}}$-complex $C$ is tilting. For each $\mathfrak{P} \in \operatorname{Spec}(\widehat{R}^{\mathfrak{a}})$, the $\widehat{R}^{\mathfrak{a}}_{\mathfrak{P}}$ complex $C_{\mathfrak{P}}$ is semidualizing, hence it is isomorphic to $\Sigma^n \widehat{R}^{\mathfrak{a}}_{\mathfrak{P}}$ for some $n$ by [8, Corollary 8.6]. It follows from [17, Proposition 4.4] that $C$ is tilting over $\widehat{R}^{\mathfrak{a}}$, as desired.

(b) In the following sequence of isomorphisms in $\mathcal{D}(R)$, the first isomorphism is from Fact 2.1, and the second one is from [35, Corollary 4.12].

$$\begin{aligned} Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}})) &\simeq Q(\mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(R))) \\ &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Q(\mathbf{L}\widehat{\Lambda}^{\mathfrak{a}}(R))) \\ &\simeq \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(R)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R). \end{aligned}$$

The third isomorphism is from Fact 2.1, and the last one is MGM equivalence 2.2.

Since $R$ is local and Gorenstein, the same is true of $\widehat{R}^{\mathfrak{a}}$, so [8, Corollary 8.6] implies that the only semidualizing $\widehat{R}^{\mathfrak{a}}$-complex, up to isomorphism and shift, is $\widehat{R}^{\mathfrak{a}}$. Thus, part (a) and the previous paragraph show that the $\mathfrak{a}$-adically semidualizing $R$-complexes are of the form $\Sigma^n \mathbf{R}\Gamma_{\mathfrak{a}}(R)$. In particular, for the ideal $\mathfrak{a} = \mathfrak{m}$, the $\mathfrak{a}$-adically semidualizing $R$-complexes are of the form $\Sigma^n \mathbf{R}\Gamma_{\mathfrak{m}}(R) \simeq \Sigma^{n-d} E_R(k)$ where $d = \dim(R)$. (This uses the well-known structure of the minimal injective resolution of $R$). □

**Corollary 6.5.** *Assume that $R$ is locally Gorenstein and $\mathfrak{a}$-adically complete. Then the $\mathfrak{a}$-adically semidualizing $R$-complexes are precisely the $R$-complexes of the form $\mathbf{R}\Gamma_{\mathfrak{a}}(L)$ for some tilting $R$-complex $L$.*

**Remark 6.6.** One can combine Theorem 5.14 with other results from the semidualizing literature to obtain further results like Corollary 6.4. For instance, if $R$ is local, then we know from [31, Theorem A] that $\mathfrak{S}(\widehat{R}^{\mathfrak{a}})$ is finite, so we conclude that $\mathfrak{S}^{\mathfrak{a}}(R)$ is finite as well. If $R$ is local and either is Golod or has embedding codepth at most 3, then [7, Corollary 6.4] shows that $|\mathfrak{S}(\widehat{R}^{\mathfrak{a}})| \leq 2$, so we have $|\mathfrak{S}^{\mathfrak{a}}(R)| \leq 2$.

Here is the example promised in [35, Example 6.4].

**Example 6.7.** Let $(R, \mathfrak{m}, k)$ be a local ring. The injective hull $E := E_R(k)$ is $\mathfrak{m}$-adically semidualizing by Proposition 4.5(c). Suppose that there is an $R$-complex $N \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ such that $E \simeq \mathbf{R}\Gamma_{\mathfrak{m}}(N)$. We show that $N$ is dualizing for $R$. It suffices to show that the $\widehat{R}^{\mathfrak{m}}$-complex $\widehat{R}^{\mathfrak{m}} \otimes^{\mathbf{L}}_R N \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(N)$ is dualizing for $\widehat{R}^{\mathfrak{m}}$; see [3, (2.2)] and Fact 2.1. From Theorem 6.1(b), the $\widehat{R}^{\mathfrak{m}}$-complex $\mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(E)$ is dualizing for $\widehat{R}^{\mathfrak{m}}$, so it suffices to show that $\mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(N) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(E)$. To this end, we compute:

$$\mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(E) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(\mathbf{R}\Gamma_{\mathfrak{m}}(N)) \simeq \mathbf{L}\widehat{\Lambda}^{\mathfrak{m}}(N).$$

The assumption $E \simeq \mathbf{R}\Gamma_{\mathfrak{m}}(N)$ explains the first isomorphism in this sequence, and the second one is from [35, Lemma 4.1].

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