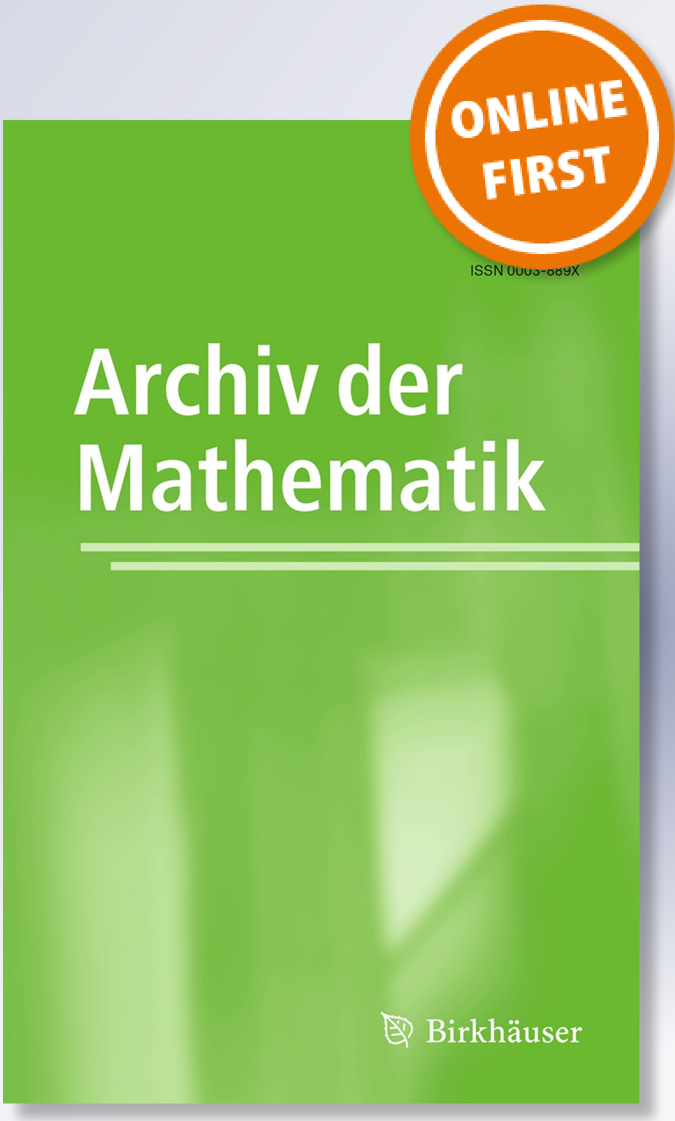
# Ascent properties for pairs of modules

**Sean Sather-Wagstaff**

**Archiv der Mathematik**
Archives Mathématiques Archives of Mathematics

ISSN 0003-889X

Arch. Math.
DOI 10.1007/s00013-014-0688-3



Arch. Math.

DOI 10.1007/s00013-014-0688-3

**Archiv der Mathematik**

# Ascent properties for pairs of modules

Sean Sather-Wagstaff

**Abstract.** Given a flat local ring homomorphism $R \to S$ and two finitely generated $R$-modules $M$ and $N$, we describe conditions under which the modules $\mathrm{Tor}_i^R(M,N)$ and $\mathrm{Ext}_R^i(M,N)$ have $S$-module structures that are compatible with their $R$-module structures.

**Mathematical Subject Classification.** Primary 13B40, 13D07; Secondary 13D02.

**Keywords.** Ascent, Ext, Flat homomorphism, NAK, Tor.

**Convention.** Throughout this paper, the term "ring" is short for "commutative noetherian ring with identity", and "module" means "unital module".

We are interested in ascent of module structures along certain ring homomorphisms, following [1,4,5] where the following result is proved; see [1, Theorems 1.5 and 1.7] and [5, Proposition 1.10 and Theorem 1.13]. Note that the natural maps from $R$ to its completion $\widehat{R}$ and to its henselization $R^{\mathrm{h}}$ satisfy the hypotheses of this result.

**Fact 1.** Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism, and let $N$ be a finitely generated $R$-module. Then the following conditions are equivalent:

(i) $N$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$.
(ii) The natural map $\mathrm{Hom}_R(S,N) \to N$ given by $f \mapsto f(1)$ is an isomorphism.
(iii) The natural map $N \to S \otimes_R N$ given by $n \mapsto 1 \otimes n$ is an isomorphism.
(iv) $\mathrm{Ext}_R^i(S,N) = 0$ for all $i \geqslant 1$.
(v) $\mathrm{Ext}_R^i(S,N)$ is finitely generated over $S$ (or over $R$) for $i{=}1,\ldots,\dim_R(N)$.
(vi) $S \otimes_R N$ is finitely generated over $R$.

The author was supported in part by a grant from the NSA.

Published online: 26 September 2014

Birkhäuser

(vii) The induced map $R/\operatorname{Ann}_R(N) \to S/\operatorname{Ann}_R(N)S$ is an isomorphism.
(viii) For all $\mathfrak{p} \in \operatorname{Min}_R(N)$ (equivalently, for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$), the induced map $R/\mathfrak{p} \to S/\mathfrak{p}S$ is an isomorphism.

To be clear, condition (i) in this fact means that $N$ can be given the structure of an $S$-module (say, with notation $s \cdot n$) such that for all $r \in R$ we have $\varphi(r) \cdot n = rn$, where the second product is the original $R$-module product. Note that [5, Lemma 1.4] shows that such an $S$-module structure is unique.

Condition (viii) in Fact 1 shows that ascent (i.e., condition (i)) is somehow a topological condition on the closed set $\operatorname{Supp}_R(N) \subseteq \operatorname{Spec}(R)$. The point of this note is to exploit this idea to identify conditions on $M$ and $N$ that guarantee ascent of module structures for $\operatorname{Ext}^i_R(M,N)$ and $\operatorname{Tor}^R_i(M,N)$. Of course, one way to guarantee that $\operatorname{Ext}^i_R(M,N)$ and $\operatorname{Tor}^R_i(M,N)$ have compatible $S$-module structures is for $M$ or $N$ to have a compatible $S$-module. For instance, if $M$ has such an $S$-module structure, then so does $M \otimes_R N$ by the formula $s(m \otimes n) := (sm) \otimes n$. However, straightforward examples show that this condition is sufficient but not necessary.

**Example 2.** Let $k$ be a field, and consider the localized polynomial ring $R = k[X,Y]_{(X,Y)}$. The modules $R/XR$ and $R/YR$ do not have $\widehat{R}$-module structures compatible with their $R$-module structures, by Fact 1. However, the modules $\operatorname{Ext}^i_R(R/XR, R/YR)$ and $\operatorname{Tor}^R_i(R/XR, R/YR)$ are finite-dimensional vector spaces over $R/(X,Y)R \cong \widehat{R}/(X,Y)\widehat{R}$, so they do have compatible $\widehat{R}$-module structures.

Our main result is the following.

**Theorem 3.** *Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism, and let $M$ and $N$ be finitely generated $R$-modules. Then the following conditions are equivalent:*

(i) *$M \otimes_R N$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$.*
(ii) *$\operatorname{Tor}^R_i(M,N)$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$ for all $i \geqslant 0$.*
(iii) *$\operatorname{Ext}^i_R(M,N)$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$ for all $i \geqslant 0$.*
(iv) *$\operatorname{Ext}^i_R(M,N)$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$ for $i = 0, \ldots, \dim_R(N) - 1$.*
(v) *$\operatorname{Ext}^i_R(S \otimes_R M, N)$ is finitely generated over $R$ for all $i \geqslant 1$.*
(vi) *The natural map $\operatorname{Ext}^i_R(S \otimes_R M, N) \to \operatorname{Ext}^i_R(M,N)$ is bijective for all $i \geqslant 0$.*
(vii) *For all $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N)$ (equivalently, for all $\mathfrak{p}$ that are minimal elements of $\operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N)$), the induced map $R/\mathfrak{p} \to S/\mathfrak{p}S$ is an isomorphism.*

**Remark 4.** The special case $M = R$ in Theorem 3 recovers much of Fact 1. Of course, we use Fact 1 in the proof of Theorem 3, so we are not claiming that the fact is a corollary of the theorem.

One can combine Fact 1 with Theorem 3 in several ways to give other conditions equivalent to the ones from Theorem 3, like the following:

(ii') The natural map $\operatorname{Hom}_R(S, M\otimes_R N) \to M\otimes_R N$ given by $f\mapsto f(1)$ is an isomorphism.

(vi') $S\otimes_R M\otimes_R N$ is finitely generated over $R$.

We leave other such variations to the interested reader. See, though, Proposition 14.

Regarding the range $i \geqslant 1$ in condition (v), note that the module

$$\operatorname{Ext}^0_R(S\otimes_R M, N) \cong \operatorname{Hom}_R(S\otimes_R M, N) \cong \operatorname{Hom}_R(S, \operatorname{Hom}_R(M,N))$$

is automatically finitely generated; indeed, as $M$ and $N$ are finitely generated over $R$, so is $\operatorname{Hom}_R(M,N)$, hence so is $\operatorname{Hom}_R(S,\operatorname{Hom}_R(M,N))$ by [5, Corollary 1.7].

The proof of Theorem 3 is given in Proof 9 below, after a few preliminaries.

**Notation 5.** Because it is convenient for us, we use some standard notions from the derived category $\mathcal{D}(R)$ of the category of $R$-modules [6–9], with some notation that is summarized in [2]. In particular, $R$-complexes are indexed homologically

$$X = \cdots \to X_i \to X_{i-1} \to \cdots$$

and the supremum of an $R$-complex $X$ is

$$\sup(X) := \sup\{i\in\mathbb{Z} \mid \mathrm{H}_i(X)\neq 0\}.$$

Given two $R$-complexes $X$ and $Y$, the derived Hom-complex and derived tensor product of $X$ and $Y$ are denoted $\mathbf{R}\operatorname{Hom}_R(X,Y)$ and $X\otimes^{\mathbf{L}}_R Y$, respectively.

We use the term "morphism of complexes" (also known as "chain map") for a morphism in the category of $R$-complexes. A quasiisomorphism is a morphism of complexes such that each induced map on homology modules is an isomorphism. The complex $\boldsymbol{\Sigma}^n X$ is obtained by shifting $X$ by $n$ steps to the left.

The next three lemmas are almost certainly standard, but we include proofs since they are relatively straightforward.

**Lemma 6.** *Let $(R,\mathfrak{m},k)$ be a local ring, and let $N$ be a non-zero finitely generated $R$-module. Fix a system of parameters $\mathbf{x} = x_1,\ldots,x_d\in\mathfrak{m}$ for $R$, and let $K$ denote the Koszul complex on $\mathbf{x}$. Then one has $\dim(R) - \sup(K\otimes^{\mathbf{L}}_R N) = \operatorname{depth}_R(N)$.*

*Proof.* Since $\operatorname{rad}((\mathbf{x})R) = \mathfrak{m}$, the first equality in the following sequence can be found in [3, Proposition 2.11(2)].

$$\begin{aligned}\operatorname{depth}_R(N) &= \operatorname{depth}_R((\mathbf{x})R, N)\\ &= \inf\{n\geqslant 1 \mid \operatorname{Ext}^n_R(R/(\mathbf{x})R, N)\neq 0\}\\ &= d - \sup(K\otimes^{\mathbf{L}}_R N)\end{aligned}$$

The second equality is classical, and the third equality is from [3, Theorem 2.1]. □

**Lemma 7.** *Let $(R, \mathfrak{m}, k)$ be a local ring, and let $M$, $N$ be non-zero finitely generated $R$-modules. Then there is an integer $i \leqslant \operatorname{depth}_R(N)$ such that $\operatorname{Ext}^i_R(M,N) \neq 0$.*

*Proof.* Let $K$ denote the Koszul complex on a system of parameters for $R$. Since the homologies of $K \otimes^{\mathbf{L}}_R N$ have finite length and $M$ is a non-zero module, Lemma 6 implies that

$$\sup(\mathbf{R}\operatorname{Hom}_R(M, K \otimes^{\mathbf{L}}_R N)) = \sup(K \otimes^{\mathbf{L}}_R N) = \dim(R) - \operatorname{depth}_R(N).$$

The tensor-evaluation isomorphism

$$\mathbf{R}\operatorname{Hom}_R(M, K \otimes^{\mathbf{L}}_R N) \simeq K \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(M,N)$$

implies that

$$\begin{aligned}\dim(R) - \operatorname{depth}_R(N) &= \sup(K \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(M,N)) \\ &\leqslant \dim(R) + \sup(\mathbf{R}\operatorname{Hom}_R(M,N))\end{aligned}$$

from which we conclude that $\sup(\mathbf{R}\operatorname{Hom}_R(M,N)) \geqslant -\operatorname{depth}_R(N)$. The desired conclusion now follows. □

**Lemma 8.** *Let $R$ be a ring, and let $f \colon X \to Y$ be morphism of $R$-complexes. Assume that $\mathrm{H}_i(X)$ and $\mathrm{H}_i(Y)$ are finitely generated over $R$ for all $i$. Let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence in the Jacobson radical of $R$, and let $K$ be the Koszul complex $K^R(\mathbf{x})$. Then $f$ is a quasiisomorphism if and only if $K \otimes_R f \colon K \otimes_R X \to K \otimes_R Y$ is a quasiisomorphism.*

*Proof.* The forward implication follows from the fact that $K$ is a bounded (below) complex of flat $R$-modules. For the converse, assume that $K \otimes_R f$ is a quasiisomorphism. It follows that the mapping cone

$$\operatorname{Cone}(K \otimes_R f) \cong K \otimes_R \operatorname{Cone}(f)$$

is exact, and it suffices to show that $\operatorname{Cone}(f)$ is exact. This is done by induction on $n$, the length of the sequence $\mathbf{x}$; this reduces immediately to the case $n = 1$. Part of the long exact sequence for $\operatorname{Cone}(f)$ and $K^R(x_1) \otimes_R \operatorname{Cone}(f)$ has the following form.

$$\mathrm{H}_i(\operatorname{Cone}(f)) \xrightarrow{x_1} \mathrm{H}_i(\operatorname{Cone}(f)) \to \underbrace{\mathrm{H}_i(K^R(x_1) \otimes_R \operatorname{Cone}(f))}_{=0}$$

Since the homology module $\mathrm{H}_i(\operatorname{Cone}(f))$ is finitely generated, Nakayama's Lemma implies that $\mathrm{H}_i(\operatorname{Cone}(f)) = 0$. □

**Proof 9.** (Proof of Theorem 3) The implications (ii) $\Longrightarrow$ (i), (iii) $\Longrightarrow$ (iv), and (vi) $\Longrightarrow$ (iii) are routine. The implications (i) $\Longrightarrow$ (vii) $\Longrightarrow$ (ii) and (vii) $\Longrightarrow$ (iii) follow from Fact 1, since $\operatorname{Supp}_R(\operatorname{Ext}^i_R(M,N))$ and $\operatorname{Supp}_R(\operatorname{Tor}^R_i(M,N))$ are contained in $\operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N) = \operatorname{Supp}_R(M \otimes_R N)$.

(iv) $\Longrightarrow$ (vii). Assume that $\operatorname{Ext}^i_R(M,N)$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$ for $i = 0, \ldots, \dim_R(N) - 1$. Fix a prime $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N)$, and suppose that the induced map $R/\mathfrak{p} \to S/\mathfrak{p}S$ is not bijective. Applying Fact 1 to the modules $\operatorname{Ext}^i_R(M,N)$ for $i < \dim_R(N)$, we see that $\mathfrak{p} \notin \operatorname{Supp}_R(\operatorname{Ext}^i_R(M,N))$ for all $i < \dim_R(N)$. It follows that

$$0 = \operatorname{Ext}^i_R(M,N)_\mathfrak{p} \cong \operatorname{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \quad \text{for all } i = 0, \dots, \dim_R(N) - 1.$$

However, since $M_\mathfrak{p} \neq 0 \neq N_\mathfrak{p}$, Lemma 7 implies that there is an integer $i_0 \leqslant \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) \leqslant \dim_{R_\mathfrak{p}}(N_\mathfrak{p}) \leqslant \dim_R(N)$ such that $\operatorname{Ext}^{i_0}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \neq 0$. The displayed vanishing implies that

$$\dim_R(N) \leqslant i_0 \leqslant \dim_{R_\mathfrak{p}}(N_\mathfrak{p}) \leqslant \dim_R(N)$$

so we have $\dim_{R_\mathfrak{p}}(N_\mathfrak{p}) = \dim_R(N)$. It follows that $\mathfrak{p} = \mathfrak{m}$, so the induced map $R/\mathfrak{p} \to S/\mathfrak{p}S$ is an isomorphism by assumption, contradicting our supposition on $\mathfrak{p}$.

(v) $\Longrightarrow$ (vi). Assume that $\operatorname{Ext}^i_R(S \otimes_R M, N)$ is finitely generated over $R$ for all $i \geqslant 1$. Remark 4 implies $\operatorname{Ext}^i_R(S \otimes_R M, N)$ is finitely generated over $R$ for all $i \geqslant 0$. Let $J$ be an $R$-injective resolution of $N$, and let $P$ be an $R$-projective resolution of $M$. To show that the natural map $\operatorname{Ext}^i_R(S \otimes_R M, N) \to \operatorname{Ext}^i_R(M,N)$ is bijective for all $i \geqslant 0$, it suffices to show that the natural chain map

$$f\colon \operatorname{Hom}_R(S \otimes_R P, J) \to \operatorname{Hom}_R(P, J)$$

is a quasiisomorphism.

Let $K = K^R(y_1, \dots, y_e)$ denote the Koszul complex on a minimal generating sequence for $\mathfrak{m}$. Our assumption implies that $\operatorname{Hom}_R(S \otimes_R P, J)$ and $\operatorname{Hom}_R(P, J)$ have finitely generated homology over $R$. Thus, Lemma 8 shows that it suffices to show that the following induced chain map is a quasiisomorphism.

$$K \otimes_R f\colon K \otimes_R \operatorname{Hom}_R(S \otimes_R P, J) \to K \otimes_R \operatorname{Hom}_R(P, J)$$

The chain map $K \otimes_R f$ is compatible with the following (quasi)isomorphisms.

$$\begin{aligned} K \otimes_R \operatorname{Hom}_R(S \otimes_R P, J) &\cong K \otimes_R \operatorname{Hom}_R(S, \operatorname{Hom}_R(P, J)) \\ &\cong \operatorname{Hom}_R(\boldsymbol{\Sigma}^{-e} K \otimes_R S, \operatorname{Hom}_R(P, J)) \\ &\simeq \operatorname{Hom}_R(\boldsymbol{\Sigma}^{-e} K, \operatorname{Hom}_R(P, J)) \\ &\cong K \otimes_R \operatorname{Hom}_R(P, J) \end{aligned}$$

The first step in this sequence is adjointness. The second and fourth steps follow from the fact that $K$ is a self-dual bounded complex of finite-rank free $R$-modules. For the third step, the assumptions on $\varphi$ imply that the chain map $K \to K \otimes_R S$ is a quasiisomorphism (see [5, 2.3]); since $\operatorname{Hom}_R(P, J)$ is a bounded-above complex of injective $R$-modules, the induced chain map

$$\operatorname{Hom}_R(\boldsymbol{\Sigma}^{-e} K \otimes_R S, \operatorname{Hom}_R(P, J)) \xrightarrow{\simeq} \operatorname{Hom}_R(\boldsymbol{\Sigma}^{-e} K, \operatorname{Hom}_R(P, J))$$

is also a quasiisomorphism.

(iii) $\Longrightarrow$ (v). Assume that $\operatorname{Ext}^i_R(M,N)$ has an $S$-module structure compatible with its $R$-module structure via $\varphi$ for all $i \geqslant 0$. To show that

$\operatorname{Ext}^i_R(S \otimes_R M, N)$ is finitely generated over $R$ for all $i \geqslant 1$, we show that $\operatorname{Ext}^i_R(S \otimes_R M, N) \cong \operatorname{Ext}^i_R(M, N)$. To this end, we use the spectral sequence

$$\operatorname{Ext}^p_R(S, \operatorname{Ext}^q_R(M, N)) \implies \operatorname{Ext}^{p+q}_R(S \otimes_R M, N).$$

If you like, this spectral sequence comes from the $R$-flatness of $S$ and the adjointness isomorphism $\mathbf{R}\operatorname{Hom}_R(S, \mathbf{R}\operatorname{Hom}_R(M, N)) \simeq \mathbf{R}\operatorname{Hom}_R(S \otimes^{\mathbf{L}}_R M, N)$ in $\mathcal{D}(R)$. As each $\operatorname{Ext}^q_R(M, N)$ has a compatible $S$-module structure, Fact 1 implies that

$$\operatorname{Hom}_R(S, \operatorname{Ext}^q_R(M, N)) \cong \operatorname{Ext}^q_R(M, N)$$

and that $\operatorname{Ext}^p_R(S, \operatorname{Ext}^q_R(M, N)) = 0$ for all $p \geqslant 1$. Hence, the spectral sequence degenerates, implying that

$$\operatorname{Ext}^q_R(S \otimes_R M, N) \cong \operatorname{Ext}^0_R(S, \operatorname{Ext}^q_R(M, N)) \cong \operatorname{Ext}^q_R(M, N)$$

as desired. □

The following example shows that the range $i = 0, \ldots, \dim_R(N) - 1$ is optimal in condition (iv) of Theorem 3.

**Example 10.** Let $k$ be a field, and consider the localized polynomial ring $R = k[X_1, \ldots, X_d]_{(X_1, \ldots, X_d)}$ with $d \geqslant 1$. Choose $j$ such that $0 \leqslant j < d$, and set $M = R/(X_1, \ldots, X_j)$ and $N = R/(X_{j+1}, \ldots, X_{d-1})$. (If $j = d - 1$, this means that $N = R$.) Note that $N$ does not have a compatible $\widehat{R}$-module structure by Fact 1.

The sequence $X_1, \ldots, X_j$ is $R$-regular, so the Koszul complex $K^R(X_1, \ldots, X_j)$ is an $R$-free resolution of $M$. The fact that $X_1, \ldots, X_j$ is $N$-regular implies that

$$\operatorname{Ext}^i_R(M, N) \cong \begin{cases} 0 & \text{for all} i \neq j = \dim(N) - 1 \\ N & \text{for all} i = j = \dim(N) - 1. \end{cases}$$

In particular, the module $\operatorname{Ext}^i_R(M, N)$ has a compatible $\widehat{R}$-module structure for $i = 0, \ldots, \dim_R(N) - 2$, but $\operatorname{Ext}^{\dim_R(N)-1}_R(M, N)$ does not have a compatible $\widehat{R}$-module structure.

The final result of this paper uses the following definition.

**Definition 11.** Let $R$ be a ring with Jacobson radical $J$. Let $\operatorname{NAK}(R)$ denote the class of all $R$-modules $L$ such that either $L = 0$ or $L/JL \neq 0$.

**Remark 12.** Let $R$ be a ring with Jacobson radical $J$. Nakayama's Lemma implies that every finitely generated $R$-module is in $\operatorname{NAK}(R)$. When $R$ is local, the terminology "$L$ satisfies NAK" from [1] is equivalent to $L \in \operatorname{NAK}(R)$.

**Remark 13.** In [1], it is shown that the conditions of Fact 1 are equivalent to the following condition:

(ix) $\operatorname{Ext}^i_R(S, N)$ is in $\operatorname{NAK}(S)$ (equivalently, in $\operatorname{NAK}(R)$) for $i = 1, \ldots, \dim_R(N)$.

The next result gives a version of this in our situation. (Note that the numbering is chosen to compare with the conditions in Theorem 3.)

**Proposition 14.** *Let* $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ *be a flat local ring homomorphism such that the induced map* $R/\mathfrak{m}\to S/\mathfrak{m}S$ *is an isomorphism, and let* $M$ *and* $N$ *be finitely generated* $R$*-modules. Consider the following conditions:*

(v) $\operatorname{Ext}^i_R(S\otimes_R M,N)$ *is finitely generated over* $R$ *for all* $i\geqslant 1$.

(v') $\operatorname{Ext}^i_R(S\otimes_R M,N)=0$ *for all* $i\geqslant 1$.

(viii) $\operatorname{Ext}^i_R(S\otimes_R M,N)$ *is in* $\operatorname{NAK}(R)$ *(i.e., in* $\operatorname{NAK}(S)$*) for* $i=1,\dots,\dim_R(N)$.

*The implications* (v') $\Longrightarrow$ (v) $\Longrightarrow$ (viii) *always hold, and the three conditions are equivalent when* $\operatorname{Ext}^i_R(M,N)=0$ *for all* $i\geqslant 1$.

*Proof.* Note that, for any $S$-module $L$, one has $L/\mathfrak{m}L=L/\mathfrak{n}L$ since $\mathfrak{n}=\mathfrak{m}S$. Thus, one has $L\in\operatorname{NAK}(R)$ if and only if $L\in\operatorname{NAK}(S)$. Also, the implication (v') $\Longrightarrow$ (v) is trivial.

(v) $\Longrightarrow$ (viii). Assume that $\operatorname{Ext}^i_R(S\otimes_R M,N)$ is finitely generated over $R$. Since $\operatorname{Ext}^i_R(S\otimes_R M,N)$ has a compatible $S$-module structure, the previous paragraph and Remark 12 imply that $\operatorname{Ext}^i_R(S\otimes_R M,N)$ is in $\operatorname{NAK}(R)$ and $\operatorname{NAK}(S)$.

(viii) $\Longrightarrow$ (v'). Assume $\operatorname{Ext}^i_R(M,N)=0$ for all $i\geqslant 1$ and $\operatorname{Ext}^i_R(S\otimes_R M,N)$ is in $\operatorname{NAK}(R)$ for $i=1,\dots,\dim_R(N)$. The vanishing of $\operatorname{Ext}^i_R(M,N)$ and the flatness of $S$ imply that we have the following isomorphisms in $\mathcal{D}(R)$.

$$\begin{aligned}\mathbf{R}\operatorname{Hom}_R(S\otimes_R M,N)&\simeq\mathbf{R}\operatorname{Hom}_R(S\otimes^{\mathbf{L}}_R M,N)\\&\simeq\mathbf{R}\operatorname{Hom}_R(S,\mathbf{R}\operatorname{Hom}_R(M,N))\\&\simeq\mathbf{R}\operatorname{Hom}_R(S,\operatorname{Hom}_R(M,N))\end{aligned}$$

Taking homology, we conclude that

$$\operatorname{Ext}^i_R(S,\operatorname{Hom}_R(M,N))\cong\operatorname{Ext}^i_R(S\otimes_R M,N)$$

for $i=1,\dots,\dim_R(N)$. By assumption, we have $\operatorname{Ext}^i_R(S\otimes_R M,N)\in\operatorname{NAK}(R)$, so Remark 13 implies that we have

$$\operatorname{Ext}^i_R(S\otimes_R M,N)\cong\operatorname{Ext}^i_R(S,\operatorname{Hom}_R(M,N))=0$$

for $i\geqslant 1$, as desired. □

**Acknowledgements.** I am grateful to the anonymous referees for helpful comments.

## References


[1] B. J. Anderson and S. Sather-Wagstaff, NAK for Ext and ascent of module structures, Proc. Amer. Math. Soc. **142** (2014), 1165–1174.

[2] L. W. Christensen, Gorenstein dimensions, Lecture Notes in Mathematics, vol. 1747, Springer-Verlag, Berlin, 2000.

[3] H.-B. Foxby and S. Iyengar, Depth and amplitude for unbounded complexes, Commutative algebra. Interactions with Algebraic Geometry, Contemp. Math., vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 119–137.

[4] A. J. Frankild and S. Sather-Wagstaff, Detecting completeness from Ext-vanishing, Proc. Amer. Math. Soc. **136** (2008), 2303–2312.

[5] A. J. Frankild, S. Sather-Wagstaff, and R. A. Wiegand, Ascent of module structures, vanishing of Ext, and extended modules, Michigan Math. J. **57** (2008), 321–337, Special volume in honor of Melvin Hochster.

[6] R. Hartshorne, Residues and duality, Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966.

[7] H. Holm, L. W. Christensen, and H.-B. Foxby, Derived category methods in commutative algebra, in preparation.

[8] J.-L. Verdier, Catégories dérivées, SGA $4\frac{1}{2}$, Springer-Verlag, Berlin, 1977, Lecture Notes in Mathematics, Vol. 569, pp. 262–311.

[9] J.-L. Verdier, Des catégories dérivées des catégories abéliennes, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

Sean Sather-Wagstaff
Department of Mathematics,
NDSU Department # 2750,
PO Box 6050,
Fargo, ND 58108–6050,
USA
e-mail: Sean.Sather-Wagstaff@ndsu.edu
URL: http://www.ndsu.edu/pubweb/∼ssatherw/

Received: 17 January 2014