

Ascent properties for test modules

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Dedicated to Roger and Sylvia Wiegand on the occasion of their 150th birthday.

ABSTRACT. We investigate modules for which vanishing of Tor-modules implies finiteness of homological dimensions (e.g., projective dimension and G-dimension). In particular, we answer a question of O. Celikbas and Sather-Wagstaff about ascent properties of such modules over residually algebraic flat local ring homomorphisms. To accomplish this, we consider ascent and descent properties over local ring homomorphisms of finite flat dimension, and for flat extensions of finite dimensional differential graded algebras.

1. Introduction

Throughout, $(R, \mathfrak{m}, k) \xrightarrow{\varphi} (S, \mathfrak{n}, l)$ is a local homomorphism between (commutative noetherian) local rings.

A classical result in homological commutative algebra says that for each finitely generated R -module N , if $\mathrm{Tor}_i^R(k, N) = 0$ for all $i \gg 0$, then $\mathrm{pd}_R(N) < \infty$. In the language of [22], this says that the residue field k is a “pd-test R -module”: a finitely generated R -module M is a pd-test module over R provided that for each finitely generated R -module N , if $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, then $\mathrm{pd}_R(N) < \infty$.

A key point of our work with O. Celikbas [22] is to prove that if M is pd-test over R , then the \mathfrak{m} -adic completion \widehat{M} is pd-test over \widehat{R} . (The converse is relatively straightforward to prove.) This is accomplished by considering the more general notion of pd-test complexes. This level of generality allows us to overcome certain technical difficulties involved in considering this ascent problem. However, it did not answer the following.

QUESTION 1.1 ([22, Question 3.7]). If φ is flat with regular closed fibre $S/\mathfrak{m}S$ and M is a pd-test R -module, must $S \otimes_R M$ be pd-test over S ?

It is straightforward to show that the assumption on the closed fibre in this question is necessary: if $R = k$ and $S = k[X]/\langle X^2 \rangle$, then k is pd-test over k , but $S = S \otimes_k k$ is not pd-test over S since S is not regular.

In the current paper, we answer Question 1.1 in the case where the induced residue field extension $k \rightarrow l$ is algebraic; see Theorem 4.8. Moreover, the proof

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shows that the general case reduces to the special case of a purely transcendental field extension with finite transcendence degree.

THEOREM 4.8 (module case). *Assume that φ is flat with regular closed fibre, and let M be a finitely generated R -module. Assume the induced field extension $k \rightarrow l$ is algebraic. Then M is a pd-test module over R if and only if $S \otimes_R M$ is a pd-test module over S .*

Our proof of Theorem 4.8 uses two more significant expansions of our original context. First, we consider the more general setting of local ring homomorphisms of finite flat dimension. This perspective was pioneered by Avramov and Foxby in a sequence of articles [7–9, 11, 13–15] where they used this notion to answer open questions about flat local ring homomorphisms. Section 3 consists of foundational material about test modules and homomorphisms of finite flat dimension.

Second, we expand our perspective from modules and complexes over local rings to the setting of differential graded (DG) modules over commutative DG algebras. This point of view was developed by Avramov and his collaborators; see, e.g., [4, 5, 7, 8, 12, 13, 15–20]. It provides a construction whereby if one can solve a homological commutative algebra problem for finite dimensional algebras over a field, then one can sometimes solve the general problem by passing to an associated finite dimensional DG algebra over a field where one can solve a related problem. We have used this approach in several projects [1, 21, 25, 34–36]. For the current setting, we develop some technology around pd-test DG modules over finite dimensional DG algebras in Section 4 and then use it to prove Theorem 4.8.

To be clear, we consider these generalizations because we do not know how to prove Theorem 4.8 without them. In addition, our techniques allow us to answer similar ascent questions for objects that test for finiteness of G-dimensions. This is the subject of Section 5. In addition, we include some necessary background material in Section 2.

2. Derived categories and semidualizing complexes

Throughout this paper we work in the derived category $\mathcal{D}(R)$ whose objects are the chain complexes of R -modules, i.e., the R -complexes

$$X = \cdots \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \xrightarrow{\partial_{i-1}^X} \cdots$$

References for this include [24, 30, 38, 39]. We denote by $\mathbf{R}\mathrm{Hom}_R(X, Y)$ and $X \otimes_R^{\mathbf{L}} Y$ the derived Hom-complex and derived tensor product of two R -complexes X and Y . Isomorphisms in $\mathcal{D}(R)$ are identified by the symbol \simeq . A complex $X \in \mathcal{D}(R)$ has *finite projective dimension*, signified $\mathrm{pd}_R(X) < \infty$, if it is isomorphic to a bounded complex of projective R -modules. Finiteness of *flat dimension* and *injective dimension* are signified and defined similarly. The ring homomorphism $R \xrightarrow{\varphi} S$ has *finite flat dimension* provided that $\mathrm{fd}_R(S) < \infty$.

The subcategory of $\mathcal{D}(R)$ consisting of homologically bounded R -complexes (i.e., complexes X such that $H_i(X) = 0$ for $|i| \gg 0$) is denoted $\mathcal{D}_b(R)$. The subcategory of $\mathcal{D}(R)$ consisting of homologically finite R -complexes (i.e., complexes X such that $H(X) := \bigoplus_{i \in \mathbb{Z}} H_i(X)$ is finitely generated) is denoted $\mathcal{D}_b^f(R)$.

A complex $C \in \mathcal{D}_b^f(R)$ is *semidualizing* if the natural homothety morphism $R \rightarrow \mathbf{R}\mathrm{Hom}_R(C, C)$ in $\mathcal{D}(R)$ is an isomorphism. Consequently, an R -module is semidualizing if and only if $\mathrm{Hom}_R(C, C) \cong R$ and $\mathrm{Ext}_R^i(C, C) = 0$ for $i \geq 1$. In

particular, R is a semidualizing R -module. A *dualizing R -complex* is a semidualizing R -complex of finite injective dimension. Dualizing complexes were introduced in [30]. The more general semidualizing complexes were defined in [23], based on special cases in [10, 27, 29, 37].

FACT 2.1. If R is a homomorphic image of a local Gorenstein ring Q , then R has a dualizing complex, by [30, §V.10]. (The converse holds by [32, Corollary 6.2].) In particular, the Cohen Structure Theorem shows that the completion \widehat{R} has a dualizing complex. When R has a dualizing complex D , and C is a semidualizing R -complex, the dual $\mathbf{R}\mathrm{Hom}_R(C, D)$ is also semidualizing over R , by [23, (2.12) Corollary].

FACT 2.2. Assume that φ has finite flat dimension, and let C be a semidualizing R -complex. Then the S -complex $S \otimes_R^{\mathbf{L}} C$ is semidualizing, by [23, (5.10) Theorem]. If φ is flat and the closed fibre $S/\mathfrak{m}S$ is Gorenstein and R has a dualizing complex D^R , then $D^S := S \otimes_R^{\mathbf{L}} D^R$ is dualizing for S by [8, (5.1) Theorem].

In order to describe another class of ring homomorphisms where dualizing complexes base change to dualizing complexes, we need the following. A *Cohen factorization* of φ is a decomposition of φ as a composition of local ring homomorphisms $R \xrightarrow{\psi} R' \xrightarrow{\varphi'} S$ such that ψ is flat with regular closed fibre, R' is complete, and φ' is surjective. Such a decomposition exists if and only if S is complete by [14, (1.1) Theorem]. In particular, the *semicompletion* of φ , which is the composition $\widehat{\psi}$ of φ with the natural map from S to its completion \widehat{S} , has a Cohen factorization.

One idea with Cohen factorizations is that it allows one to transfer questions and properties for S as an R -module to related questions and properties for \widehat{S} has an R' -module. This is valuable because \widehat{S} is finitely generated over R' while S often is not finitely generated over R . An example of this transfer is found in [14, (3.2) Lemma] which implies that $\mathrm{fd}(\varphi) < \infty$ if and only if $\mathrm{pd}_{R'}(\widehat{S}) < \infty$.

Recall that an ideal I of R is *Gorenstein* if $g = \mathrm{pd}_R(R/I) < \infty$ satisfies $\mathrm{Ext}_R^i(R/I, R) = 0$ for all $i \neq g$ and $\mathrm{Ext}_R^g(R/I, R) \cong R/I$. Our local ring homomorphism φ is *Gorenstein* provided that in some (equivalently, every) Cohen factorization $R \xrightarrow{\psi} R' \xrightarrow{\varphi'} \widehat{S}$ of $\widehat{\psi}$ the ideal $\mathrm{Ker}(\varphi')$ of R' is Gorenstein; see [7, 8].

FACT 2.3. If φ is flat, then φ is Gorenstein if and only if its closed fibre is Gorenstein. In general, if φ is Gorenstein and R has a dualizing complex D^R , then $D^S := S \otimes_R^{\mathbf{L}} D^R$ is dualizing for S by [8, (5.1) Theorem].

As the name suggests, semidualizing complexes are built for duality. Let C be a semidualizing R -complex and $X \in \mathcal{D}_b^f(R)$. We say that X is *derived C -reflexive* and write $\mathrm{G}_C\text{-dim}_R(X) < \infty$ when $\mathbf{R}\mathrm{Hom}_R(X, C) \in \mathcal{D}_b(R)$ and the natural morphism $X \rightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$ in $\mathcal{D}(R)$ is an isomorphism. In the case $C = R$, we write $\mathrm{G}\text{-dim}_R(X) < \infty$ instead of $\mathrm{G}_R\text{-dim}_R(X) < \infty$. In the special case where C is dualizing, this is from [30], while the case $C = R$ comes from [2, 3, 40]. The general situation is in [23].

FACT 2.4. A semidualizing complex C is dualizing if and only if every R -complex in $\mathcal{D}_b^f(R)$ is derived C -reflexive, by [23, (8.4) Proposition]. In particular, R is Gorenstein if and only if every R -complex $X \in \mathcal{D}_b^f(R)$ has $\mathrm{G}\text{-dim}_R(X) < \infty$.

FACT 2.5. Assume that $\text{fd}(\varphi) < \infty$. If $X \in \mathcal{D}_b^f(R)$, then $\text{G}_C\text{-dim}_R(X) < \infty$ if and only if $\text{G}_{S \otimes_R^L C}\text{-dim}_S(S \otimes_R^L X) < \infty$, by [23, (5.10) Theorem]; Fact 2.2 shows that this is reasonable.

Auslander and Bass classes, defined next, arrived in special cases in [10, 27], again with the general case described in [23]. Let C be a semidualizing R -complex. The Auslander class $\mathcal{A}_C(R)$ consists of the R -complexes $X \in \mathcal{D}_b(R)$ such that $C \otimes_R^L X \in \mathcal{D}_b(R)$ and the natural morphism $\gamma_X^C: X \rightarrow \mathbf{RHom}_R(C, C \otimes_R^L X)$ in $\mathcal{D}(R)$ is an isomorphism. The Bass class $\mathcal{B}_C(R)$ consists of all the R -complexes $X \in \mathcal{D}_b(R)$ such that $\mathbf{RHom}_R(C, X) \in \mathcal{D}_b(R)$ and such that the natural evaluation morphism $\xi_X^C: C \otimes_R^L \mathbf{RHom}_R(C, X) \rightarrow X$ in $\mathcal{D}(R)$ is an isomorphism.

FACT 2.6. When R has a dualizing complex D , given an R -complex $X \in \mathcal{D}_b^f(R)$, one has $\text{G}_C\text{-dim}_R(X) < \infty$ if and only if $X \in \mathcal{A}_{\mathbf{RHom}_R(C,D)}(R)$, by [23, (4.7) Theorem]; this uses Facts 2.1 and 2.4, which imply that $\mathbf{RHom}_R(C, D)$ is semidualizing and $C \simeq \mathbf{RHom}_R(\mathbf{RHom}_R(C, D), D)$.

FACT 2.7. Assume that $\text{fd}(\varphi) < \infty$, and let $X \in \mathcal{D}_b(S)$. Then $X \in \mathcal{A}_C(R)$ if and only if $X \in \mathcal{A}_{S \otimes_R^L C}(S)$, by [23, (5.3.a) Proposition and (5.10) Theorem].

The proof of the following lemma is similar to that of [31, 7.3 Corollary], but it is different enough that we include a proof.

LEMMA 2.8. Assume that φ is Gorenstein, and let $X \in \mathcal{D}_b^f(S)$ be such that each homology module $H_i(X)$ is finitely generated over R .

- (a) One has $\text{G}_C\text{-dim}_R(X) < \infty$ if and only if $\text{G}_{S \otimes_R^L C}\text{-dim}_S(X) < \infty$.
- (b) One has $\text{G-dim}_R(X) < \infty$ if and only if $\text{G-dim}_S(X) < \infty$.

PROOF. (a)

CASE 1. R has a dualizing complex D^R . From Fact 2.3, we have $D^S \simeq S \otimes_R^L D^R$ is dualizing for S . And Facts 2.6–2.7 say that $\text{G}_C\text{-dim}_R(X) < \infty$ if and only if $X \in \mathcal{A}_{\mathbf{RHom}_R(C,D^R)}(R)$ if and only if $X \in \mathcal{A}_{S \otimes_R^L \mathbf{RHom}_R(C,D^R)}(S)$. Standard isomorphisms give

$$S \otimes_R^L \mathbf{RHom}_R(C, D^R) \simeq \mathbf{RHom}_S(S \otimes_R^L C, S \otimes_R^L D^R) \simeq \mathbf{RHom}_S(S \otimes_R^L C, D^S)$$

so $\text{G}_C\text{-dim}_R(X) < \infty$ if and only if $X \in \mathcal{A}_{\mathbf{RHom}_S(S \otimes_R^L C, D^S)}(S)$; Facts 2.4 and 2.6 show that these are equivalent to $\text{G}_{S \otimes_R^L C}\text{-dim}_S(X) < \infty$, as desired.

CASE 2. we have $\mathfrak{m}H(X) = 0$. In this case, let \widehat{R} and \widetilde{S} denote the \mathfrak{m} -adic completions of R and S , respectively, and let $\widetilde{\varphi}: \widehat{R} \rightarrow \widetilde{S}$ be the \mathfrak{m} -adic completion of φ , which is Gorenstein. By Fact 2.5 we have $\text{G}_C\text{-dim}_R(X) < \infty$ if and only if $\text{G}_{\widehat{R} \otimes_R^L C}\text{-dim}_{\widehat{R}}(\widehat{R} \otimes_R^L X) < \infty$. As in the proof of [31, 7.1 Theorem] we have an \widehat{R} -isomorphism $\widehat{R} \otimes_R^L X \simeq \widetilde{S} \otimes_S^L X$ since $\mathfrak{m}H(X) = 0$, so $\text{G}_{\widehat{R} \otimes_R^L C}\text{-dim}_{\widehat{R}}(\widehat{R} \otimes_R^L X) < \infty$ if and only if $\text{G}_{\widetilde{S} \otimes_S^L (\widehat{R} \otimes_R^L C)}\text{-dim}_{\widetilde{S}}(\widetilde{S} \otimes_S^L X) < \infty$ by Case 1, i.e., if and only if $\text{G}_{\widetilde{S} \otimes_S^L C}\text{-dim}_{\widetilde{S}}(\widetilde{S} \otimes_S^L X) < \infty$. Again by Fact 2.5 we have $\text{G}_{S \otimes_R^L C}\text{-dim}_S(X) < \infty$ if and only if $\text{G}_{\widetilde{S} \otimes_S^L (S \otimes_R^L C)}\text{-dim}_{\widetilde{S}}(\widetilde{S} \otimes_S^L X) < \infty$ if and only if $\text{G}_{\widetilde{S} \otimes_S^L C}\text{-dim}_{\widetilde{S}}(\widetilde{S} \otimes_S^L X) < \infty$, as desired.

CASE 3. the general case. Let K^R be the Koszul complex over R on a finite generating set for \mathfrak{m} . Then $\text{G}_C\text{-dim}_R(X) < \infty$ if and only if $\text{G}_C\text{-dim}_R(K^R \otimes_R^L X) < \infty$.

$X) < \infty$ by [28, Theorem 4.4], that is, if and only if $G_C\text{-dim}_R((K^R \otimes_R^L S) \otimes_S^L X) < \infty$, since $(K^R \otimes_R^L S) \otimes_S^L X \simeq K^R \otimes_R^L X$. The homology of this complex is annihilated by \mathfrak{m} , so Case 2 says that $G_C\text{-dim}_R((K^R \otimes_R^L S) \otimes_S^L X) < \infty$ if and only if $G_{S \otimes_R^L C}\text{-dim}_R((K^R \otimes_R^L S) \otimes_S^L X) < \infty$, which is equivalent to $G_{S \otimes_R^L C}\text{-dim}_S(X) < \infty$ by another application of [28, Theorem 4.4].

(b) This is the special case $C = R$ of part (a). □

3. Test complexes and ring homomorphisms of finite flat dimension

In this section, let C be a semidualizing R -complex.

Let $M \in \mathcal{D}_b^f(R)$, and let H-dim denote either pd or $G_C\text{-dim}$. Then M is an $\text{H-dim-test complex}$ over R if the following condition holds for all $N \in \mathcal{D}_b^f(R)$: If $\text{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, i.e., if $M \otimes_R^L N \in \mathcal{D}_b(R)$, then $\text{H-dim}_R(N) < \infty$. See [22, Section 3] for examples and basic properties of these objects.

The following two results are proved like [22, Theorems 3.2 and 3.4], using Lemma 2.8 and Fact 2.5.

PROPOSITION 3.1. *Assume that $\text{fd}(\varphi) < \infty$, and let $M \in \mathcal{D}_b^f(R)$.*

- (a) *If $S \otimes_R^L M$ is $G_{S \otimes_R^L C}\text{-dim-test}$ over S , then M is $G_C\text{-dim-test}$ over R .*
- (b) *If $S \otimes_R^L M$ is $G\text{-dim-test}$ over S , then M is $G\text{-dim-test}$ over R .*
- (c) *If $S \otimes_R^L M$ is pd-test over S , then M is pd-test over R .*

PROPOSITION 3.2. *Assume that φ is Gorenstein, and let $M \in \mathcal{D}_b^f(R)$. Assume that the induced residue field extension $k \rightarrow l$ is finite.*

- (a) *M is $G_C\text{-dim-test}$ over R if and only if $S \otimes_R^L M$ is $G_{S \otimes_R^L C}\text{-dim-test}$ over S .*
- (b) *M is $G\text{-dim-test}$ over R if and only if $S \otimes_R^L M$ is $G\text{-dim-test}$ over S .*

REMARK 3.3. Assume that φ is flat with regular closed fibre. Let $\mathbf{y} = y_1, \dots, y_n \in \mathfrak{n}$ be a sequence that forms a regular system of parameters for the regular local ring $S/\mathfrak{m}S$. Then \mathbf{y} is S -regular and the composition $\tau\varphi: R \rightarrow \overline{S} = S/(\mathbf{y})$ of φ with the quotient map $\tau: S \rightarrow \overline{S}$ is flat by, e.g., [33, Corollary to Theorem 22.5]. By construction, the closed fibre of $\tau\varphi$ is $S/(\mathfrak{m}, \mathbf{y}) = l$.

Here is one of our main results. One point is that one can prove better results about flat local maps by widening the context to finite flat dimension. Note that the case where $\mathfrak{m}S = \mathfrak{n}$ is covered by [22, Theorem 3.5]. As we note in [22, Example 3.6], if $S/\mathfrak{m}S$ is not regular, then the ascent portion of this result fails.

THEOREM 3.4. *Assume that φ is flat with regular closed fibre, and let $M \in \mathcal{D}_b^f(R)$. Assume the induced field extension $k \rightarrow l$ is finite. Then M is a pd-test complex over R if and only if $S \otimes_R^L M$ is a pd-test complex over S .*

PROOF. One implication is covered by Proposition 3.1(c). For the reverse implication, assume that M is a pd-test complex over R . Follow the notation from Remark 3.3. From [22, Theorem 3.5], it follows that $\overline{S} \otimes_R^L M = \overline{S} \otimes_S^L (S \otimes_R^L M)$ is pd-test over \overline{S} , so Proposition 3.1(c) implies that $S \otimes_R^L M$ is pd-test over S . □

4. PD-test results

As we discuss in the introduction, the point of this section is to prove Theorem 4.8, and the proof relies heavily on a version of pd-test objects for finite

dimensional DG algebras. One may consult any of the following for background on DG algebras and derived categories of DG modules [4, 12, 19, 21, 26].

DEFINITION 4.1. We say that a (positively graded commutative) DG algebra A is *weakly local* if $H_0(A)$ is local and noetherian and $H(A)$ is finitely generated as a module over $H_0(A)$. In particular, $H(A)$ is bounded. In this situation, let \mathfrak{m}_A be the augmentation ideal of A corresponding to the maximal ideal $\mathfrak{m}_{H_0(A)} \subset H_0(A)$, and set $k = A/\mathfrak{m}_A$. We sometimes summarize this by writing that (A, \mathfrak{m}_A, k) is a weakly local DG algebra.

REMARK 4.2. If k is a field and A is a finite dimensional DG k -algebra such that $A_0 = k$ and $H_0(A) \neq 0$, then A is weakly local with $\mathfrak{m}_A = A_+$ and $\partial_1^A = 0$ and $A/\mathfrak{m}_A \cong k \cong H_0(A)$.

FACT 4.3. Let (A, \mathfrak{m}_A, k) be a weakly local DG algebra. For a given $N \in \mathcal{D}_b^f(A)$, the following conditions are equivalent.

- (i) N has a semi-free resolution $F \simeq N$ over A with a finite semi-basis.
- (ii) For all $L \in \mathcal{D}_b(A)$, one has $N \otimes_A^L L \in \mathcal{D}_b(A)$.
- (iii) One has $N \otimes_A^L k \in \mathcal{D}_b(A)$.

For the implication (i) \implies (ii), the weakly local assumption implies that A is homologically bounded, so it is quasiisomorphic as a DG algebra to a bounded truncation A' . Then A and A' have equivalent derived categories (e.g., by [36, Lemma A.13(a)]), so we may replace A with A' and use F to compute $N \otimes_A^L L$.

The implication (iii) \implies (i) is in [19, Proposition B.9]. Note that in this situation, the resolution F will be homologically bounded but not necessarily bounded, unless A is bounded. Furthermore, for the implication (i) \implies (ii), it is crucial that A be homologically bounded.

Here is the DG version of pd-test objects we use to prove Theorem 4.8.

DEFINITION 4.4. Let A be a weakly local DG algebra. We say that $N \in \mathcal{D}_b^f(A)$ is *perfect* over A if N satisfies the equivalent conditions of Fact 4.3. Then $M \in \mathcal{D}_b^f(A)$ is *pd-test* over A if the following condition holds for all $N \in \mathcal{D}_b^f(A)$: If $M \otimes_A^L N \in \mathcal{D}_b(A)$, then N is perfect.

EXAMPLE 4.5. If (A, \mathfrak{m}_A, k) is a weakly local DG algebra, then Fact 4.3 implies that k is pd-test over A .

Our proof of Theorem 4.8 follows almost directly from the following result via a construction of Avramov.

THEOREM 4.6. Let A be a finite-dimensional DG k -algebra with $A_0 = k$ and $H_0(A) \neq 0$. Let $k \rightarrow l$ be a field extension, set $B = l \otimes_k A$, and consider the natural morphism of DG algebras $A \rightarrow B$. Let $M \in \mathcal{D}_b^f(A)$ be given.

- (a) If $B \otimes_A^L M$ is pd-test over B , then M is pd-test over A .
- (b) The converse of part (a) holds if the extension $k \rightarrow l$ is algebraic.

PROOF. (a) We argue as in the proof of [22, Theorem 3.2]. Assume that $B \otimes_A^L M$ is pd-test over B . To show that M is pd-test over A , let $N \in \mathcal{D}_b^f(A)$ be such that $M \otimes_A^L N \in \mathcal{D}_b(A)$. As N is homologically finite over A , which is finite dimensional over k , we have

$$B \otimes_A^L N \simeq (l \otimes_k A) \otimes_A^L N \simeq l \otimes_k N \in \mathcal{D}_b^f(l)$$

so $B \otimes_A^{\mathbf{L}} N \in \mathcal{D}_b^f(B)$, and similarly $B \otimes_A^{\mathbf{L}} M, B \otimes_A^{\mathbf{L}} (M \otimes_A^{\mathbf{L}} N) \in \mathcal{D}_b^f(B)$. Moreover, we have the following isomorphisms in $\mathcal{D}(B)$:

$$(B \otimes_A^{\mathbf{L}} M) \otimes_B^{\mathbf{L}} (B \otimes_A^{\mathbf{L}} N) \simeq (B \otimes_A^{\mathbf{L}} M) \otimes_A^{\mathbf{L}} N \simeq B \otimes_A^{\mathbf{L}} (M \otimes_A^{\mathbf{L}} N).$$

As $B \otimes_A^{\mathbf{L}} M$ is pd-test over B , the DG B -module $B \otimes_A^{\mathbf{L}} N$ is perfect. It is straightforward to show (arguing as above or using a minimal semi-free resolution of N over A , with Fact 4.3) that N is perfect over A , as desired.

(b) Assume now that M is pd-test over A .

CASE 1. $k \rightarrow l$ is finite. To show that $B \otimes_A^{\mathbf{L}} M$ is pd-test over B , let $N \in \mathcal{D}_b^f(B)$ be such that $(B \otimes_A^{\mathbf{L}} M) \otimes_B^{\mathbf{L}} N \in \mathcal{D}_b(B)$. Since $k \rightarrow l$ is finite, we have $B \in \mathcal{D}_b^f(A)$, so $N \in \mathcal{D}_b^f(A)$. Moreover, $M \otimes_A^{\mathbf{L}} N \simeq (B \otimes_A^{\mathbf{L}} M) \otimes_B^{\mathbf{L}} N \in \mathcal{D}_b(A)$. As M is pd-test over A , the DG module N is perfect over A . To conclude that N is perfect over B , it suffices to show that $N \otimes_B^{\mathbf{L}} l \in \mathcal{D}_b(B)$, equivalently, that $N \otimes_B^{\mathbf{L}} l \in \mathcal{D}_b(A)$. By assumption, we have $N \otimes_A^{\mathbf{L}} k \in \mathcal{D}_b(A)$. By construction, we have

$$B \otimes_A k \cong (l \otimes_k A) \otimes_A k \cong l \otimes_k k \simeq l$$

so

$$N \otimes_B^{\mathbf{L}} l \simeq N \otimes_B^{\mathbf{L}} (B \otimes_A k) \simeq N \otimes_A^{\mathbf{L}} k \in \mathcal{D}_b(A)$$

as desired.

CASE 2. $k \rightarrow l$ is algebraic. To show that $B \otimes_A^{\mathbf{L}} M$ is pd-test over B , let $N \in \mathcal{D}_b^f(B)$ be such that $(B \otimes_A^{\mathbf{L}} M) \otimes_B^{\mathbf{L}} N \in \mathcal{D}_b(B)$. Truncate a degreewise finite B -semifree resolution of N if necessary to assume without loss of generality that N is finite dimensional over l . It follows that the differential and scalar multiplication on N are represented by matrices consisting of finitely many elements of l . Let k' be the intermediate field extension $k \rightarrow k' \rightarrow l$ generated over k by this finite set of elements. Since l is algebraic over k , the extension $k \rightarrow k'$ is finite. Set $A' = k' \otimes_k A$.

CLAIM. There is a DG A' -module L that is bounded and degreewise finite over k' such that $N \cong l \otimes_{k'} L \cong B \otimes_{A'} L$. Indeed, let E be a semi-basis for A over k , which is also a basis for A' over k' and for B over l . Let F be a semi-basis for N over B , and let L be the graded k' -vector space with basis F . Define a differential on L using the same matrices as those representing ∂^N with respect to F . Similarly, define scalar multiplication of A' on L using the same rule as for scalar multiplication of B on N : for semi-basis elements $e \in E$ and $f \in F$, the element $ef \in N$ is actually in L , by the construction of k' , so this is well-defined. Since N is a DG B -module, it is straightforward to show, from the definition of L , that L is a DG A' -module such that $N \cong l \otimes_{k'} L \cong B \otimes_{A'} L$ as DG B -modules. This establishes the claim.

Using these constructions, we compute:

$$\begin{aligned} (B \otimes_A^{\mathbf{L}} M) \otimes_B^{\mathbf{L}} N &\simeq (B \otimes_{A'}^{\mathbf{L}} (A' \otimes_A^{\mathbf{L}} M)) \otimes_B^{\mathbf{L}} (B \otimes_{A'} L) \\ &\simeq B \otimes_{A'}^{\mathbf{L}} ((A' \otimes_A^{\mathbf{L}} M) \otimes_{A'} L) \\ &\simeq (l \otimes_{k'}^{\mathbf{L}} A') \otimes_{A'}^{\mathbf{L}} ((A' \otimes_A^{\mathbf{L}} M) \otimes_{A'} L) \\ &\simeq l \otimes_{k'}^{\mathbf{L}} ((A' \otimes_A^{\mathbf{L}} M) \otimes_{A'} L) \end{aligned}$$

Thus, we have $l \otimes_{k'}^{\mathbf{L}} ((A' \otimes_A^{\mathbf{L}} M) \otimes_{A'} L) \simeq (B \otimes_A^{\mathbf{L}} M) \otimes_B^{\mathbf{L}} N \in \mathcal{D}_b(B)$. Since $k' \rightarrow l$ is faithfully flat, it follows that $(A' \otimes_A^{\mathbf{L}} M) \otimes_{A'}^{\mathbf{L}} L$ is homologically bounded as well.

Case 1 implies that $A' \otimes_A^{\mathbf{L}} M$ is pd-test over A' , so the homological boundedness of $(A' \otimes_A^{\mathbf{L}} M) \otimes_{A'}^{\mathbf{L}} L$ implies that L is perfect over A' . It follows readily that $N \simeq B \otimes_{A'}^{\mathbf{L}} L$ is perfect over B , as desired. \square

The next result uses the exterior DG algebra structure on the Koszul complex.

LEMMA 4.7. *Let $K = K^R(\mathbf{x})$ be the Koszul complex on $\mathbf{x} = x_1, \dots, x_n \in \mathfrak{m}$, and let $M \in \mathcal{D}_b^f(R)$.*

- (a) *If $K \otimes_R^{\mathbf{L}} M$ is pd-test over K , then M is pd-test over R .*
- (b) *The converse of part (a) holds when \mathbf{x} is part of a minimal generating sequence for \mathfrak{m} .*

PROOF. (a) Argue as in the descent result [22, Theorem 3.2].

(b) We argue as in the proof of Theorem 4.6(b), with a twist. Assume that M is a pd-test complex over R . To show that $K \otimes_R^{\mathbf{L}} M$ is pd-test over K , let $N \in \mathcal{D}_b^f(K)$ be such that $(K \otimes_R^{\mathbf{L}} M) \otimes_K^{\mathbf{L}} N \in \mathcal{D}_b(K)$. Since $K \in \mathcal{D}_b^f(R)$, we have $N \in \mathcal{D}_b^f(R)$. Moreover, $M \otimes_R^{\mathbf{L}} N \simeq (K \otimes_R^{\mathbf{L}} M) \otimes_K^{\mathbf{L}} N \in \mathcal{D}_b(R)$. As M is pd-test over R , the DG module N is perfect over R , so $k \otimes_R^{\mathbf{L}} N \in \mathcal{D}_b(R)$. To conclude that N is perfect over K , we compute:

$$k \otimes_R^{\mathbf{L}} N \simeq (k \otimes_R^{\mathbf{L}} K) \otimes_K^{\mathbf{L}} N \simeq \mathbf{H}(k \otimes_R^{\mathbf{L}} K) \otimes_K^{\mathbf{L}} N \simeq \mathbf{H}(k \otimes_R^{\mathbf{L}} K) \otimes_k^{\mathbf{L}} (k \otimes_K^{\mathbf{L}} N)$$

The second isomorphism here is from [6, Theorem 9.1]; this is where we use the fact that \mathbf{x} is part of a minimal generating sequence for \mathfrak{m} . Since $k \otimes_R^{\mathbf{L}} N$ is homologically bounded, the Künneth formula implies that $k \otimes_K^{\mathbf{L}} N \in \mathcal{D}_b(K)$, so N is perfect. \square

Here is our main result about pd-test complexes.

THEOREM 4.8. *Assume that φ is flat with regular closed fibre, and let $M \in \mathcal{D}_b^f(R)$. Assume the induced field extension $k \rightarrow l$ is algebraic. Then M is a pd-test complex over R if and only if $S \otimes_R^{\mathbf{L}} M$ is a pd-test complex over S .*

PROOF. As with Theorem 3.4, one implication is covered by Proposition 3.1(c). For the reverse implication, assume that M is a pd-test complex over R .

CASE 1. R and S are complete and $\mathfrak{m}S = \mathfrak{n}$. Let $\tau: Q \rightarrow R$ be a minimal Cohen presentation of R , so the map is a surjective ring homomorphism where Q is a complete regular local ring with the same embedding dimension as R . Apply [14, (1.6) Theorem] to find a commutative diagram of local homomorphisms of complete local rings

$$\begin{array}{ccc} Q & \xrightarrow{\alpha} & Q' \\ \tau \downarrow & & \downarrow \tau' \\ R & \xrightarrow{\varphi} & S \end{array}$$

such that α is flat with regular closed fibre, τ' is surjective, and the induced map $R \otimes_Q Q' \rightarrow S$ is an isomorphism. Moreover, the proof of *loc. cit.* shows that $\text{edim}(Q'/\mathfrak{m}_Q Q') = \text{edim}(S/\mathfrak{m}S) = 0$, so the closed fibre of α is $Q'/\mathfrak{m}_Q Q' \cong l$.

Let $F \xrightarrow{\simeq} R$ be a bounded degreewise finite semi-free DG algebra resolution of R over Q . The isomorphism $R \otimes_Q Q' \cong S$ from the previous paragraph implies that $F' = F \otimes_Q Q' \xrightarrow{\simeq} S$ is a bounded degreewise finite semi-free DG algebra resolution of S over Q' since α is flat. Let \mathbf{x} be a minimal generating sequence for \mathfrak{m} . Our assumptions imply that $\varphi(\mathbf{x})$ is a minimal generating sequence for \mathfrak{n} . Set $K^R =$

$K^R(\mathbf{x})$ and $K^S = K^S(\varphi(\mathbf{x}))$ and consider the following standard commutative diagram of morphisms of DG algebras where K^Q is the Koszul complex over Q on a lift \mathbf{y} of the sequence \mathbf{x} , and $K^{Q'} = K^{Q'}(\alpha(\mathbf{y})) \cong Q' \otimes_Q K^Q$.

$$(4.8.1) \quad \begin{array}{ccccccccc} R & \longrightarrow & K^R & \xleftarrow{\cong} & R \otimes_Q K^Q & \xleftarrow{\cong} & F \otimes_Q K^Q & \xrightarrow{\cong} & F \otimes_Q k \\ \varphi \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & K^S & \xleftarrow{\cong} & S \otimes_{Q'} K^{Q'} & \xleftarrow{\cong} & F' \otimes_{Q'} K^{Q'} & \xrightarrow{\cong} & F' \otimes_{Q'} l \end{array}$$

Using the properties catalogued above, it is straightforward to show that each square in this diagram is a pushout square.

Since M is pd-test over R , Lemma 4.7 implies that $K^R \otimes_R^{\mathbf{L}} M$ is pd-test over K^R . Since the other arrows in the top row of the above diagram are equivalences, it is straightforward to show that $K^R \otimes_R^{\mathbf{L}} M$ is pd-test over $F \otimes_Q K^Q$ by restriction of scalars, and $(F \otimes_Q k) \otimes_{F \otimes_Q K^Q}^{\mathbf{L}} (K^R \otimes_R^{\mathbf{L}} M)$ is pd-test over $F \otimes_Q k$ by base-change.

From our assumptions, it is straightforward to show that $F' \otimes_{Q'} l \cong l \otimes_k (F \otimes_Q k)$. Thus, Theorem 4.6 implies that the following is pd-test over $F' \otimes_{Q'} l$.

$$\begin{aligned} (F' \otimes_{Q'} l) \otimes_{F \otimes_Q k}^{\mathbf{L}} ((F \otimes_Q k) \otimes_{F \otimes_Q K^Q}^{\mathbf{L}} (K^R \otimes_R^{\mathbf{L}} M)) \\ \simeq (F' \otimes_{Q'} l) \otimes_{F' \otimes_{Q'} K^{Q'}}^{\mathbf{L}} ((F' \otimes_{Q'} K^{Q'}) \otimes_{F \otimes_Q K^Q}^{\mathbf{L}} (K^R \otimes_R^{\mathbf{L}} M)) \end{aligned}$$

From the bottom-right equivalence in the diagram (4.8.1), we conclude that the DG module $(F' \otimes_{Q'} K^{Q'}) \otimes_{F \otimes_Q K^Q}^{\mathbf{L}} (K^R \otimes_R^{\mathbf{L}} M)$ is pd-test over $F' \otimes_{Q'} K^{Q'}$ by restriction of scalars. From the other equivalences in the bottom row of the diagram, we see that the following DG module is pd-test over K^S ;

$$(F \otimes_Q k) \otimes_{F \otimes_Q K^Q}^{\mathbf{L}} (K^R \otimes_R^{\mathbf{L}} M) \simeq K^S \otimes_{K^R}^{\mathbf{L}} (K^R \otimes_R^{\mathbf{L}} M) \simeq K^S \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} M)$$

the first isomorphism comes from the pushout property of the diagram. Lemma 4.7 implies that $S \otimes_R^{\mathbf{L}} M$ is pd-test over S . This concludes the proof in Case 1.

CASE 2. $mS = n$. In this case, consider the following natural diagram of flat local ring homomorphisms.

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ \widehat{R} & \longrightarrow & \widehat{S} \end{array}$$

Then $\widehat{R} \otimes_R^{\mathbf{L}} M$ is pd-test over \widehat{R} by Theorem 3.4. Case 1 then implies that the following complex is pd-test over \widehat{S} .

$$\widehat{S} \otimes_{\widehat{R}}^{\mathbf{L}} (\widehat{R} \otimes_R^{\mathbf{L}} M) \simeq \widehat{S} \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} M)$$

Another application of Theorem 3.4 implies that $S \otimes_R^{\mathbf{L}} M$ is pd-test over S .

CASE 3. the general case follows from Case 2 by modding out by an S -regular sequence as in the proof of Theorem 3.4. \square

REMARK 4.9. It would be nice to answer Question 1.1 in general. This is equivalent to proving that the conclusion of Theorem 4.8 holds without the assumption that $k \rightarrow l$ is algebraic.

Furthermore, we claim that it suffices to verify the conclusion of Theorem 4.6(b) in the case where $k \rightarrow l$ is purely transcendental of finite transcendence degree.

Indeed, assume that the conclusion of Theorem 4.6(b) in the case where $k \rightarrow l$ is purely transcendental of finite transcendence degree. As in Case 2 of the proof of Theorem 4.6(b) to show that $B \otimes_A^{\mathbf{L}} M$ is pd-test over B , let $N \in \mathcal{D}_b^f(B)$ be such that $(B \otimes_A^{\mathbf{L}} M) \otimes_B^{\mathbf{L}} N \in \mathcal{D}_b(B)$, and assume that N is finite dimensional over l . It follows that the differential and scalar multiplication on N are represented by matrices consisting of finitely many elements of l . Let k' be the intermediate field extension $k \rightarrow k' \rightarrow l$ generated over k by this finite set of elements. Let T be a transcendence basis for k' over k , and set $k'' = k(T)$ and $A'' = k'' \otimes_k A$. Since the extension $k \rightarrow k'$ is finitely generated, the set T is finite, and the extension $k \rightarrow k''$ is purely transcendental of finite transcendence degree, and the extension $k'' \rightarrow k'$ is algebraic. By assumption, the DG module $A'' \otimes_A^{\mathbf{L}} M$ is pd-test over A'' . Thus, Theorem 4.6(b) implies that $A' \otimes_A^{\mathbf{L}} M \simeq A' \otimes_{A''}^{\mathbf{L}} (A'' \otimes_A^{\mathbf{L}} M)$ is pd-test over A'' . Now, argue as in Case 2 of the proof of Theorem 4.6(b) to establish the claim.

Unfortunately, we do not know how to establish the conclusion of Theorem 4.6(b) even when $k \rightarrow l$ is purely transcendental of transcendence degree 1.

5. G-Dim-Test results

Throughout this section, (A, \mathfrak{m}_A, k) is a weakly local DG algebra.

Here, we analyze ascent properties for G_C -dim- and G-dim-test modules using the techniques of the preceding section. Our main result here is Theorem 5.7.

We say that $C \in \mathcal{D}_b^f(A)$ is a *semidualizing* DG A -module if the natural homothety morphism $\chi_C^A: A \rightarrow \mathbf{R}\mathrm{Hom}_A(C, C)$ is an isomorphism in $\mathcal{D}(A)$. For instance, since A is homologically bounded, A itself is a semidualizing DG A -module.

Assume that C is a semidualizing DG A -module. We say that $N \in \mathcal{D}_b^f(A)$ is *derived C -reflexive* over A if $\mathbf{R}\mathrm{Hom}_A(N, C) \in \mathcal{D}_b(A)$ and the natural biduality morphism $\delta_C^N: N \rightarrow \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(N, C), C)$ is an isomorphism in $\mathcal{D}(A)$. Then $M \in \mathcal{D}_b^f(A)$ is G_C -dim-test over A if the following condition holds for all $N \in \mathcal{D}_b^f(A)$: If $M \otimes_A^{\mathbf{L}} N \in \mathcal{D}_b(A)$, then N is derived C -reflexive. In the case $C = A$, we write G-dim-test instead of G_A -dim-test. Fact 4.3 implies that k is G_C -dim-test over A for each semidualizing DG A -module C because every perfect DG A -module is derived C -reflexive. In particular, k is G-dim-test.

We proceed as in the preceding section with some preparatory lemmas.

LEMMA 5.1. *Assume that A is a finite-dimensional DG k -algebra with $A_0 = k$ and $H_0(A) \neq 0$. Let $k \rightarrow l$ be a field extension, set $B = l \otimes_k A$, and consider the natural morphism of DG algebras $A \rightarrow B$. Let $C \in \mathcal{D}_b^f(A)$ be given. Then $B \otimes_A^{\mathbf{L}} C$ is semidualizing over B if and only if C is semidualizing over A .*

PROOF. Note that since l is free over k , we have $B \simeq l \otimes_k^{\mathbf{L}} A$. Furthermore, the A -algebra $B = l \otimes_k A$ is free with semi-basis concentrated in degree 0. These facts justify three of the unspecified isomorphisms in the following diagram, while

the fourth one is tensor-cancellation.

$$\begin{array}{ccc}
 l \otimes_k^{\mathbf{L}} A & \xrightarrow{l \otimes_k^{\mathbf{L}} \chi_C^A} & l \otimes_k^{\mathbf{L}} \mathbf{RHom}_A(C, C) \\
 \cong \downarrow & & \downarrow \cong \\
 B & & (l \otimes_k^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(C, C) \\
 \chi_{B \otimes_A^{\mathbf{L}} C}^B \downarrow & & \downarrow \cong \\
 \mathbf{RHom}_B(B \otimes_A^{\mathbf{L}} C, B \otimes_A^{\mathbf{L}} C) & \xleftarrow{\cong} & B \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(C, C)
 \end{array}$$

It follows that $\chi_{B \otimes_A^{\mathbf{L}} C}^B$ is an isomorphism if and only if $l \otimes_k^{\mathbf{L}} \chi_C^A$ is an isomorphism; since l is free and non-zero over k , the morphism $l \otimes_k^{\mathbf{L}} \chi_C^A$ is an isomorphism is and only if χ_C^A is one. Since $B \otimes_A^{\mathbf{L}} C \in \mathcal{D}_b^f(B)$, the desired conclusion now follows. \square

LEMMA 5.2. *Let A be a finite-dimensional DG k -algebra with $A_0 = k$ and $H_0(A) \neq 0$. Let $k \rightarrow l$ be a field extension, set $B = l \otimes_k A$, and consider the natural morphism of DG algebras $A \rightarrow B$. Let $M \in \mathcal{D}_b^f(A)$ be given, and let C be a semidualizing DG A -module. Then $B \otimes_A^{\mathbf{L}} M$ is derived $(B \otimes_A^{\mathbf{L}} C)$ -reflexive over B if and only if M is derived C -reflexive over A .*

PROOF. As in the proof of Lemma 5.1, in $\mathcal{D}(A)$ we have $\mathbf{RHom}_B(B \otimes_A^{\mathbf{L}} M, B \otimes_A^{\mathbf{L}} C) \simeq l \otimes_k^{\mathbf{L}} \mathbf{RHom}_A(M, C)$ and therefore $\mathbf{RHom}_B(B \otimes_A^{\mathbf{L}} M, B \otimes_A^{\mathbf{L}} C) \in \mathcal{D}_b(B)$ if and only if $\mathbf{RHom}_A(M, C) \in \mathcal{D}_b(A)$. Assume therefore that $\mathbf{RHom}_A(M, C) \in \mathcal{D}_b(A)$, and use the next diagram

$$\begin{array}{ccc}
 l \otimes_k^{\mathbf{L}} M & \xrightarrow{l \otimes_k^{\mathbf{L}} \delta_C^M} & l \otimes_k^{\mathbf{L}} \mathbf{RHom}_A(\mathbf{RHom}_A(M, C), C) \\
 \cong \downarrow & & \downarrow \cong \\
 (l \otimes_k^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} M & & (l \otimes_k^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(\mathbf{RHom}_A(M, C), C) \\
 \cong \downarrow & & \downarrow \cong \\
 B \otimes_A^{\mathbf{L}} M & \xrightarrow{\delta_{B \otimes_A^{\mathbf{L}} C}^{B \otimes_A^{\mathbf{L}} M}} & \mathbf{RHom}_B(B \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(M, C), B \otimes_A^{\mathbf{L}} C) \\
 & & \uparrow \cong \\
 & & \mathbf{RHom}_B(B \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(M, C), B \otimes_A^{\mathbf{L}} C) \\
 & & \downarrow \cong \\
 & & B \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(\mathbf{RHom}_A(M, C), C) \\
 & & \downarrow \cong \\
 & & (l \otimes_k^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(\mathbf{RHom}_A(M, C), C) \\
 & & \downarrow \cong \\
 & & l \otimes_k^{\mathbf{L}} \mathbf{RHom}_A(\mathbf{RHom}_A(M, C), C)
 \end{array}$$

as in the proof of Lemma 5.1 to complete the argument. \square

Here are the versions of Theorem 4.6 and Lemma 4.7 for this context.

THEOREM 5.3. *Let A be a finite-dimensional DG k -algebra with $A_0 = k$ and $H_0(A) \neq 0$. Let $k \rightarrow l$ be a field extension, set $B = l \otimes_k A$, and consider the natural morphism of DG algebras $A \rightarrow B$. Let $M \in \mathcal{D}_b^f(A)$ be given, and let C be a semidualizing DG A -module.*

- (a) *If $B \otimes_A^{\mathbf{L}} M$ is $G_{B \otimes_A^{\mathbf{L}} C}$ -dim-test over B , then M is G_C -dim-test over A .*
- (b) *The converse of part (a) holds if the extension $k \rightarrow l$ is algebraic.*
- (c) *If $B \otimes_A^{\mathbf{L}} M$ is G -dim-test over B , then M is G -dim-test over A .*
- (d) *The converse of part (a) holds if the extension $k \rightarrow l$ is algebraic.*

PROOF. (a) Argue as in the proof of Theorem 4.6(a), using Lemma 5.2.
 (b) Assume now that M is a G_C -dim-test complex over A .

CASE 1. $k \rightarrow l$ is finite. To show that $B \otimes_A^{\mathbf{L}} M$ is $G_{B \otimes_A^{\mathbf{L}} C}$ -dim-test over B , let $N \in \mathcal{D}_b^f(B)$ be such that $(B \otimes_A^{\mathbf{L}} M) \otimes_B^{\mathbf{L}} N \in \mathcal{D}_b(B)$. Since $k \rightarrow l$ is finite, we have $B \in \mathcal{D}_b^f(A)$, so $N \in \mathcal{D}_b^f(A)$. Moreover, $M \otimes_A^{\mathbf{L}} N \simeq (B \otimes_A^{\mathbf{L}} M) \otimes_B^{\mathbf{L}} N \in \mathcal{D}_b(A)$. As M is G_C -dim-test over A , the DG module N is derived C -reflexive over A . To conclude that N is derived $(B \otimes_A^{\mathbf{L}} C)$ -reflexive over B , we use the Auslander class.

Since A is finite dimensional over k , it has a dualizing DG module D^A . From the definition of B as $l \otimes_k A$, it is straightforward to show that $D^B = B \otimes_A^{\mathbf{L}} D^A$ is a dualizing DG B -module. As in [23, (4.7) Theorem] or [10, (4.1.7)] one shows that N is derived reflexive over A if and only if N is in the Auslander class $\mathcal{A}_{\mathbf{R}\mathrm{Hom}_A(C, D^A)}(A)$, and similarly over B . Moreover, arguing as in [23, (5.3) Proposition], one sees that N is in $\mathcal{A}_{\mathbf{R}\mathrm{Hom}_A(C, D^A)}(A)$ if and only if it is in $\mathcal{A}_{B \otimes_A^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_A(C, D^A)}(B) = \mathcal{A}_{\mathbf{R}\mathrm{Hom}_B(B \otimes_A^{\mathbf{L}} C, D^B)}(B)$, that is, N is derived C -reflexive over A if and only if it is derived $(B \otimes_A^{\mathbf{L}} C)$ -reflexive over B . This concludes the proof in Case 1.

CASE 2. $k \rightarrow l$ is algebraic. Argue as in Case 2 of the proof of Theorem 4.6(b).
 (c)–(d) These are the special case $C = A$ of parts (a)–(b). \square

As in Section 4, we use the exterior DG algebra structure on the Koszul complex to prove our ascent result for G -dim-test complexes.

FACT 5.4. Let $K = K^R(\mathbf{x})$ be the Koszul complex on $\mathbf{x} = x_1, \dots, x_n \in \mathfrak{m}$, and let $C \in \mathcal{D}_b^f(R)$. Then $K \otimes_R^{\mathbf{L}} C$ is semidualizing over K if and only if C is semidualizing over R by [25, Lemma A.3(a)].

LEMMA 5.5. Let $K = K^R(\mathbf{x})$ be the Koszul complex on $\mathbf{x} = x_1, \dots, x_n \in \mathfrak{m}$. Let $M \in \mathcal{D}_b^f(R)$ be given, and let C be a semidualizing R -complex.

- (a) $K \otimes_R^{\mathbf{L}} M$ is $G_{K \otimes_R^{\mathbf{L}} C}$ -dim-test over K if and only if M is G_C -dim-test over R .
- (b) $K \otimes_R^{\mathbf{L}} M$ is G -dim-test over K if and only if M is G -dim-test over R .

PROOF. Again, we focus on part (a). For one implication, argue as in the descent result [22, Theorem 3.2]. For the converse, assume that M is G_C -dim-test over R .

CASE 1. R has a dualizing complex D^R . To show that $K \otimes_R^{\mathbf{L}} M$ is $G_{K \otimes_R^{\mathbf{L}} C}$ -dim-test over K , let $N \in \mathcal{D}_b^f(K)$ be such that $(K \otimes_R^{\mathbf{L}} M) \otimes_K^{\mathbf{L}} N \in \mathcal{D}_b(K)$. Since $K \in \mathcal{D}_b^f(R)$, we have $N \in \mathcal{D}_b^f(R)$. Moreover, $M \otimes_R^{\mathbf{L}} N \simeq (K \otimes_R^{\mathbf{L}} M) \otimes_K^{\mathbf{L}} N \in \mathcal{D}_b(R)$. As M is G_C -dim-test over R , the DG module N is derived C -reflexive over R . Now argue as in the proof of Theorem 5.3 using D^R and $D^K = K \otimes_R^{\mathbf{L}} D^R$ to prove that N is derived $K \otimes_R^{\mathbf{L}} C$ -reflexive over K .

CASE 2. $(\mathbf{x})R$ is \mathfrak{m} -primary. Proposition 3.2 implies that $\widehat{R} \otimes_R^{\mathbf{L}} M$ is $G_{\widehat{R} \otimes_R^{\mathbf{L}} C}$ -dim-test for \widehat{R} . Set $\widehat{K} = \widehat{R} \otimes_R^{\mathbf{L}} K$. Recall that \widehat{R} has a dualizing complex, so Case 1 implies that the DG \widehat{K} -module

$$\widehat{K} \otimes_{\widehat{R}}^{\mathbf{L}} (\widehat{R} \otimes_R^{\mathbf{L}} M) \simeq \widehat{K} \otimes_R^{\mathbf{L}} M \simeq K \otimes_R^{\mathbf{L}} M$$

is $G_{\widehat{K} \otimes_R^{\mathbf{L}} C}$ -dim-test for \widehat{K} ; the second isomorphism here uses the assumption on $(\mathbf{x})R$, which implies that the natural morphism $K \rightarrow \widehat{K}$ is a quasiisomorphism of

DG algebras. This quasiisomorphism furthermore implies that this DG module is $G_{K \otimes_R^{\mathbf{L}} C}$ -dim-test for K , by restriction of scalars.

CASE 3. the general case. Let $\mathbf{y} = y_1, \dots, y_m \in \mathfrak{m}$ be a generating sequence for \mathfrak{m} . By Case 2, the DG $K^R(\mathbf{x}, \mathbf{y})$ -module $K^R(\mathbf{x}, \mathbf{y}) \otimes_R^{\mathbf{L}} M$ is $G_{K^R(\mathbf{x}, \mathbf{y}) \otimes_R^{\mathbf{L}} C}$ -dim-test over $K^R(\mathbf{x}, \mathbf{y})$. Argue as in the descent implication, from the DG algebra morphism $K \rightarrow K^R(\mathbf{x}, \mathbf{y})$ to conclude that $K \otimes_R^{\mathbf{L}} M$ is $G_{K \otimes_R^{\mathbf{L}} C}$ -dim test over K , as desired. \square

THEOREM 5.6. *Assume that φ is flat with regular closed fibre such that the induced field extension $k \rightarrow l$ is algebraic. Let $M \in \mathcal{D}_b^f(R)$ be given, and let C be a semidualizing R -complex.*

- (a) M is G_C -dim-test over R if and only if $S \otimes_R^{\mathbf{L}} M$ is $G_{S \otimes_R^{\mathbf{L}} C}$ -dim-test over S .
- (b) M is G -dim-test over R if and only if $S \otimes_R^{\mathbf{L}} M$ is G -dim-test over S .

PROOF. Argue as in the proof of Theorem 4.8. \square

We conclude with the main result of this section.

THEOREM 5.7. *Assume that φ is Gorenstein such that the induced field extension $k \rightarrow l$ is algebraic. Let $M \in \mathcal{D}_b^f(R)$, and let C be a semidualizing R -complex.*

- (a) M is G_C -dim-test over R if and only if $S \otimes_R^{\mathbf{L}} M$ is $G_{S \otimes_R^{\mathbf{L}} C}$ -dim-test over S .
- (b) M is G -dim-test over R if and only if $S \otimes_R^{\mathbf{L}} M$ is G -dim-test over S .

PROOF. Again we focus on part (a). One implication is from Proposition 3.1(a). For the converse, assume that M is G_C -dim-test over R . Let $R \rightarrow R' \rightarrow \widehat{S}$ be a Cohen factorization of the semicompletion $\widehat{\varphi}: R \rightarrow \widehat{S}$. This provides isomorphisms

$$\begin{aligned} \widehat{S} \otimes_{R'}^{\mathbf{L}} (R' \otimes_R^{\mathbf{L}} C) &\simeq \widehat{S} \otimes_R^{\mathbf{L}} C \simeq \widehat{S} \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} C) \\ \widehat{S} \otimes_{R'}^{\mathbf{L}} (R' \otimes_R^{\mathbf{L}} M) &\simeq \widehat{S} \otimes_R^{\mathbf{L}} M \simeq \widehat{S} \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} M). \end{aligned}$$

Theorem 5.6 implies that $R' \otimes_R^{\mathbf{L}} M$ is $G_{R' \otimes_R^{\mathbf{L}} C}$ -dim-test over R' . Because of the displayed isomorphisms, Proposition 3.2 implies that $\widehat{S} \otimes_R^{\mathbf{L}} M$ is $G_{\widehat{S} \otimes_R^{\mathbf{L}} C}$ -dim-test over \widehat{S} . Thus, $S \otimes_R^{\mathbf{L}} M$ is $G_{S \otimes_R^{\mathbf{L}} C}$ -dim-test over S by Proposition 3.1(a). \square

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