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Adically finite chain complexes

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We investigate the similarities between adic finiteness and homological finiteness for chain complexes over a commutative Noetherian ring. In particular, we extend the isomorphism properties of certain natural morphisms from homologically finite complexes to adically finite complexes. We do the same for characterizations of certain homological dimensions. In addition, we study transfer of adic finiteness along ring homomorphisms, all with a view toward subsequent applications.

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1. Introduction

Throughout this paper, let R and S be commutative Noetherian rings, let $\mathfrak{a} \subsetneq R$ be a proper ideal of R, and let $\widehat{R}^{\mathfrak{a}}$ be the $\mathfrak{a}$-adic completion of R. Let $\underline{x} = x_1, \ldots, x_n \in R$ be a generating sequence for $\mathfrak{a}$, and consider the Koszul complex $K := K^R(\underline{x})$. We work in the derived category $\mathcal{D}(R)$ with objects the R-complexes indexed homologically $X = \cdots \to X_i \to X_{i-1} \to \cdots$. See, e.g. [10, 28, 29] and Sec. 2 for background/foundational material. Isomorphisms in $\mathcal{D}(R)$ are identified by the symbol $\simeq$.

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This work is part 2 of a series of papers about support and finiteness conditions for complexes, with a view toward derived local cohomology and homology. It builds on our previous paper [25], and it is used in the papers [22–24, 26].

In [25], we introduce the notion of "$\mathfrak{a}$-adic finiteness" for complexes; see Definition 2.6. For example, an R-module M is $\mathfrak{a}$-adically finite if it is $\mathfrak{a}$-torsion and has $\operatorname{Tor}_i^R(R/\mathfrak{a}, M)$ finitely generated for all i. In particular, this recovers two standard finiteness conditions as special cases: first, M is finitely generated (i.e. Noetherian) if and only if it is 0-adically finite; second, over a local ring, M is Artinian if and only if it is adically finite with respect to the ring's maximal ideal.

One goal of this paper is to extend standard results for finitely generated modules (and homologically finite complexes) to the $\mathfrak{a}$-adically finite realm. For instance, given a finitely generated R-module M, a classical result shows that $\operatorname{Tor}_i^R(M, -)$ commutes not only with arbitrary direct sums, but also with arbitrary direct products. One of our main results extends this to the case where M is $\mathfrak{a}$-adically finite.

Theorem 1.1. *Let M be an $\mathfrak{a}$-adically finite R-complex, and let $V \in \mathcal{D}(R)$ such that $\operatorname{supp}_R(V) \subseteq \operatorname{V}(\mathfrak{a})$. Consider a set $\{N_\lambda\}_{\lambda\in\Lambda} \in \mathcal{D}_+(R)$ such that $\operatorname{H}_i(N_\lambda) = 0$ for all $i < s$ and for all $\lambda \in \Lambda$. Consider the natural morphism*

$$M \otimes^{\mathbf{L}}_R \prod_\lambda N_\lambda \xrightarrow{p} \prod_\lambda (M \otimes^{\mathbf{L}}_R N_\lambda)$$

in $\mathcal{D}(R)$. Then the induced morphism

$$V \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R \prod_\lambda N_\lambda \to V \otimes^{\mathbf{L}}_R \prod_\lambda (M \otimes^{\mathbf{L}}_R N_\lambda)$$

is an isomorphism.

This is contained in Theorem 4.7 from the body of the paper. Note the trade-off in this result, as compared to the classical one. We have relaxed the assumptions on M, but we are not claiming that the morphism p is an isomorphism, only that a certain induced morphism is so. This is the theme of the results of Sec. 4. While these results may seem quite specialized, we exhibit applications in [23].

Theorem 1.1 uses Foxby's "small support", as do most of the results of this paper; see Definition 2.3. This is an extremely useful substitute for the standard notion of support (our "large support") for finitely generated modules. For instance, large support allows us to give conditions on two finitely generated modules to decide when their tensor product is nonzero. This does not work in general for nonfinitely generated modules, but one can use small support to detect when at least one of their Tor-modules is nonzero, which ends up being enough for many applications. Thus, small support provides another substitute for finite generation.

The paper continues with Sec. 5 which tracks some transfer behavior of these notions, specifically, support, cosupport, and adic finiteness through restriction and extension of scalars. These are used heavily in the paper [24].

Section 6 contains other results showing how similar adic finiteness is to homological finiteness with respect to homological dimensions. For instance, the next result, contained in Theorem 6.1 below is well known when X is homologically finite; it is somewhat surprising to us that it holds in this generality.

Theorem 1.2. *Let $X \in \mathcal{D}_b(R)$ be $\mathfrak{a}$-adically finite. If X is locally of finite flat dimension, then $\mathrm{pd}_R(X) < \infty$. Moreover, one has $\mathrm{pd}_R(X) = \mathrm{fd}_R(X)$.*

2. Background

Derived Categories. The ith shift (or suspension) of an R-complex X is denoted $\Sigma^i X$. We consider the following full triangulated subcategories of $\mathcal{D}(R)$.

$\mathcal{D}_+(R)$: objects are the complexes X with $\mathrm{H}_i(X) = 0$ for $i \ll 0$.
$\mathcal{D}_-(R)$: objects are the complexes X with $\mathrm{H}_i(X) = 0$ for $i \gg 0$.
$\mathcal{D}_\mathrm{b}(R)$: objects are the complexes X with $\mathrm{H}_i(X) = 0$ for $|i| \gg 0$.
$\mathcal{D}^\mathrm{f}(R)$: objects are the complexes X with $\mathrm{H}_i(X)$ finitely generated for all i.

Doubly ornamented subcategories are intersections, e.g. $\mathcal{D}^\mathrm{f}_\mathrm{b}(R) := \mathcal{D}^\mathrm{f}(R) \bigcap \mathcal{D}_\mathrm{b}(R)$.

Resolutions. An R-complex P is *semi-projective*[a] if it respects surjective quasi-isomorphisms, that is, if it consists of projective R-modules and respects quasi-isomorphisms; see [2, 1.2.P]. A *semi-projective resolution* of an R-complex X is a quasiisomorphism $P \xrightarrow{\simeq} X$ such that P is semi-projective. The *projective dimension* of X is finite, written $\mathrm{pd}_R(X) < \infty$, if it has a bounded semi-projective resolution. The corresponding flat and injective versions of these notions (homotopically flat, etc.) are defined similarly.

For the following items, consult [2, Sec. 1] or [3, Chaps. 3 and 5]. Bounded below complexes of projective modules are semi-projective, bounded below complexes of flat modules are semi-flat, and bounded above complexes of injective modules are semi-injective, semi-projective R-complexes are semi-flat. Every R-complex admits a semi-projective resolution (hence, a semi-flat one) and a semi-injective resolution.

Derived Functors. The right derived functor of Hom is $\mathbf{R}\mathrm{Hom}_R(-,-)$, which is computed via a semi-projective resolution in the first slot or a semi-injective resolution in the second slot. The left derived functor of tensor product is $- \otimes^{\mathbf{L}}_R -$, which is computed via semi-flat resolutions in either slot.

Let $\Lambda^\mathfrak{a}$ denote the $\mathfrak{a}$-adic completion functor, and $\Gamma_\mathfrak{a}$ is the $\mathfrak{a}$-torsion functor, i.e. for an R-module M, we have

$$\Lambda^\mathfrak{a}(M) = \widehat{M}^\mathfrak{a} \qquad\qquad \Gamma_\mathfrak{a}(M) = \{x \in M \mid \mathfrak{a}^n x = 0 \text{ for } n \gg 0\}.$$

A module M is $\mathfrak{a}$-*torsion* if $\Gamma_\mathfrak{a}(M) = M$.

The associated left and right derived functors (i.e. *derived local homology and cohomology* functors) are $\mathbf{L}\Lambda^\mathfrak{a}(-)$ and $\mathbf{R}\Gamma_\mathfrak{a}(-)$. Specifically, given an R-complex $X \in \mathcal{D}(R)$ and a semi-flat resolution $F \xrightarrow{\simeq} X$ and a semi-injective resolution

[a]In the literature, semi-projective complexes are sometimes called "K-projective" or "DG-projective".

$X \xrightarrow{\simeq} I$, then we have $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(F)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(I)$. Note that these definitions yield natural transformations $\mathbf{R}\Gamma_{\mathfrak{a}} \xrightarrow{\varepsilon_{\mathfrak{a}}} \mathrm{id} \xrightarrow{\vartheta^{\mathfrak{a}}} \mathbf{L}\Lambda^{\mathfrak{a}}$, induced by the natural morphisms $\Gamma_{\mathfrak{a}}(I) \xrightarrow{\iota^I_{\mathfrak{a}}} I$ and $F \xrightarrow{\nu^{\mathfrak{a}}_F} \Lambda^{\mathfrak{a}}(F)$. These notions go back to Grothendieck [11], and Matlis [17, 18], respectively; see also [1, 16].

Fact 2.1. By [1, Theorem (0.3) and Corollary (3.2.5.i)], there are natural isomorphisms of functors

$$\mathbf{R}\Gamma_{\mathfrak{a}}(-) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R - \qquad\qquad \mathbf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), -).$$

Koszul Complexes We refer to the following as the "self-dual nature" of K.

Remark 2.2. Recall the isomorphism

$$K \cong \Sigma^n \mathrm{Hom}_R(K, R).$$

Given an R-complex X, there are natural isomorphisms

$$K \otimes_R X \cong \Sigma^n \mathrm{Hom}_R(K, R) \otimes_R X \cong \Sigma^n \mathrm{Hom}_R(K, X)$$

in the category of R-complexes; these are verified by induction on n, using the definitions $K(x_i) \cong \mathrm{Cone}(R \xrightarrow{x_i} R)$ and $K := K^R(x_1) \otimes_R \cdots \otimes_R K(x_n)$, in terms of mapping cones and tensor products. From these, we have isomorphisms in $\mathcal{D}(R)$

$$K \simeq \Sigma^n \mathbf{R}\mathrm{Hom}_R(K, R)$$
$$K \otimes^{\mathbf{L}}_R X \simeq \Sigma^n \mathbf{R}\mathrm{Hom}_R(K, R) \otimes^{\mathbf{L}}_R X \simeq \Sigma^n \mathbf{R}\mathrm{Hom}_R(K, X).$$

Similarly, one verifies the next natural evaluation isomorphisms for all $X, Y \in \mathcal{D}(R)$:

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K, X), Y) \simeq K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, Y) \simeq \mathbf{R}\mathrm{Hom}_R(X, K \otimes^{\mathbf{L}}_R Y).$$

Support and Co-support. The following notions are due to Foxby [8] and Benson, Iyengar and Krause [5].

Definition 2.3. Let $X \in \mathcal{D}(R)$. The *small and large support* and *small co-support* of X are

$$\mathrm{supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \not\simeq 0\}$$
$$\mathrm{Supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_{\mathfrak{p}} \otimes^{\mathbf{L}}_R X \not\simeq 0\}$$
$$\text{co-supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), X) \not\simeq 0\},$$

where $\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. We have a notion of $\text{Co-supp}_R(X)$, as well, but do not need it in the current paper.

Much of the following is from [8] when X and Y are appropriately bounded and from [4, 5] in general. We refer to [25] as a matter of convenience.

Fact 2.4. Let $X, Y \in \mathcal{D}(R)$. Then we have $\mathrm{supp}_R(X) = \emptyset$ if and only if $X \simeq 0$ if and only if $\text{co-supp}_R(X) = \emptyset$, because of [25, Fact 3.4 and Proposition 4.7(a)].

Also, by [25, Propositions 3.12 and 4.10], we have

$$\operatorname{supp}_R(X \otimes^{\mathbf{L}}_R Y) = \operatorname{supp}_R(X) \bigcap \operatorname{supp}_R(Y)$$

$$\operatorname{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \operatorname{supp}_R(X) \bigcap \operatorname{co-supp}_R(Y).$$

In addition, we know that $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ if and only if the natural morphism $\varepsilon^X_{\mathfrak{a}} \colon \mathbf{R}\Gamma_{\mathfrak{a}}(X) \to X$ is an isomorphism, that is, if and only if each homology module $\mathrm{H}_i(X)$ is $\mathfrak{a}$-torsion, by [25, Proposition 5.4] and [20, Corollary 4.32]. Similarly, we have $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ if and only if the natural morphism $\vartheta^{\mathfrak{a}}_X \colon X \to \mathbf{L}\Lambda^{\mathfrak{a}}(X)$ is an isomorphism, by [25, Propositions 5.9].

Adic Finiteness. The next fact and definition from [25] take their cues from work of Hartshorne [12], Kawasaki [13, 14], and Melkersson [19].

Fact 2.5 ([25, Theorem 1.3]). For $X \in \mathcal{D}_{\mathrm{b}}(R)$, the next conditions are equivalent.

(i) One has $K^R(\underline{y}) \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ for some (equivalently, for every) generating sequence $\underline{y}$ of $\mathfrak{a}$.
(ii) One has $X \otimes^{\mathbf{L}}_R R/\mathfrak{a} \in \mathcal{D}^{\mathrm{f}}(R)$.
(iii) One has $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X) \in \mathcal{D}^{\mathrm{f}}(R)$.

Definition 2.6. An R-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ is $\mathfrak{a}$-*adically finite* if it satisfies the equivalent conditions of Fact 2.5 and $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

Example 2.7. Let $X \in \mathcal{D}_{\mathrm{b}}(R)$ be given.

(a) If $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then we have $\operatorname{supp}_R(X) = \mathrm{V}(\mathfrak{b})$ for some ideal $\mathfrak{b}$, and it follows that X is $\mathfrak{a}$-adically finite whenever $\mathfrak{a} \subseteq \mathfrak{b}$. (The case $\mathfrak{a} = 0$ is from [25, Proposition 7.8(a)], and the general case follows readily.)
(b) K and $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ are $\mathfrak{a}$-adically finite, by [25, Fact 3.4 and Theorem 7.10].
(c) If $(R, \mathfrak{m})$ is local, then the homology modules of X are Artinian if and only if X is $\mathfrak{m}$-adically finite, by [25, Proposition 7.8(b)]. See Proposition 5.11 for an extension of this.

Bookkeeping. We use some convenient accounting tools due to Foxby [7].

Definition 2.8. The *supremum*, *infimum*, *amplitude*, $\mathfrak{a}$-*depth*, and $\mathfrak{a}$-*width* of an R-complex Z are

$$\sup(Z) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(Z) \neq 0\}$$

$$\inf(Z) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(Z) \neq 0\}$$

$$\operatorname{amp}(Z) = \sup(Z) - \inf(Z)$$

$$\operatorname{depth}_{\mathfrak{a}}(Z) = -\sup(\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, Z))$$

$$\operatorname{width}_{\mathfrak{a}}(Z) = \inf((R/\mathfrak{a}) \otimes^{\mathbf{L}}_R Z)$$

with the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

Fact 2.9. Let $Y, Z \in \mathcal{D}(R)$.

(a) By definition, one has $\sup(Z) < \infty$ if and only if $Z \in \mathcal{D}_-(R)$. Also, one has $\inf(Z) > -\infty$ if and only if $Z \in \mathcal{D}_+(R)$, and one has $\operatorname{amp}(Z) < \infty$ if and only if $Z \in \mathcal{D}_{\mathrm{b}}(R)$.
(b) By [7, Lemma 2.1], there are inequalities

$$\inf(Y) + \inf(Z) \leqslant \inf(Y \otimes^{\mathbf{L}}_R Z)$$
$$\sup(\mathbf{R}\mathrm{Hom}_R(Y, Z)) \leqslant \sup(Z) - \inf(Y).$$

3. Koszul Homology

We begin this section by showing how, with appropriate support conditions, bounded Koszul homology implies bounded homology. Note that the self-dual nature 2.2 of the Koszul complex implies that these also give results for Koszul cohomology; see, e.g. the proof of Lemma 3.2.

Lemma 3.1. *Let* $Z \in \mathcal{D}(R)$, *let* $\mathbf{y} = y_1, \ldots, y_m \in \mathfrak{a}$, *and set* $L := K^R(\mathbf{y})$ *and* $\mathfrak{b} = (\mathbf{y})R$. *Assume that* $\operatorname{supp}_R(Z) \subseteq \mathrm{V}(\mathfrak{a})$, e.g. *that each homology module* $\mathrm{H}_i(Z)$ *is annihilated by a power of* $\mathfrak{a}$.

(a) *There are (in)equalities*

$$\inf(L \otimes^{\mathbf{L}}_R Z) \leqslant m + \inf(Z) \qquad \sup(L \otimes^{\mathbf{L}}_R Z) = m + \sup(Z)$$
$$\operatorname{amp}(Z) \leqslant \operatorname{amp}(L \otimes^{\mathbf{L}}_R Z) \qquad \operatorname{depth}_{\mathfrak{b}}(Z) = -\sup(Z).$$

(b) *For each* $* \in \{+, -, b\}$, *one has* $L \otimes^{\mathbf{L}}_R Z \in \mathcal{D}_*(R)$ *if and only if* $Z \in \mathcal{D}_*(R)$.

Proof. Claim: For any $i \in \mathbb{Z}$, if $\mathrm{H}_i(K^R(y_1) \otimes^{\mathbf{L}}_R Z) = 0$, then $\mathrm{H}_{i-1}(Z) = 0$. To prove the claim, assume that $\mathrm{H}_i(K^R(y_1) \otimes^{\mathbf{L}}_R Z) = 0$. The standard long exact sequence for Koszul homology contains the following:

$$\mathrm{H}_i(K^R(y_1) \otimes^{\mathbf{L}}_R Z) \to \mathrm{H}_{i-1}(Z) \xrightarrow{y_1} \mathrm{H}_{i-1}(Z).$$

It follows that the map $\mathrm{H}_{i-1}(Z) \xrightarrow{y_1} \mathrm{H}_{i-1}(Z)$ is injective. However, the condition $\operatorname{supp}_R(Z) \subseteq \mathrm{V}(\mathfrak{a})$ implies that $\mathrm{H}_{i-1}(Z)$ is $\mathfrak{a}$-torsion by Fact 2.4. Since y_1 is in $\mathfrak{a}$, the fact that the map $\mathrm{H}_{i-1}(Z) \xrightarrow{y_1} \mathrm{H}_{i-1}(Z)$ is injective therefore implies that $\mathrm{H}_{i-1}(Z) = 0$, as claimed.

(a) We now show that $\inf(L \otimes^{\mathbf{L}}_R Z) \leqslant m + \inf(Z)$. For this, it suffices to show that for any $i \in \mathbb{Z}$, if $\mathrm{H}_i(L \otimes^{\mathbf{L}}_R Z) = 0$, then $\mathrm{H}_{i-m}(Z) = 0$. We verify this by induction on m, the base case $m = 1$ being covered by the above claim. For the induction step, it suffices to note that Fact 2.4 implies that $\operatorname{supp}_R(K^R(y_1) \otimes^{\mathbf{L}}_R Z) \subseteq \operatorname{supp}_R(Z) \subseteq \mathrm{V}(\mathfrak{a})$.

For the equality $\sup(L \otimes^{\mathbf{L}}_R Z) = m + \sup(Z)$, note that the condition $\operatorname{supp}_R(Z) \subseteq \mathrm{V}(\mathfrak{a}) \subseteq \mathrm{V}(\mathfrak{b})$ implies that $\mathbf{R}\Gamma_{\mathfrak{b}}(Z) \simeq Z$, by Fact 2.4. Thus, we have the following equalities by [9, Theorem 2.1]

$$\sup(L \otimes^{\mathbf{L}}_R Z) = \sup(\mathbf{R}\Gamma_{\mathfrak{b}}(Z)) + m = \sup(Z) + m.$$

The inequality for amp follows directly. For the equality $\operatorname{depth}_{\mathfrak{b}}(Z) = \sup(Z)$, note that [9, Theorem 2.1] shows that we have

$$\operatorname{depth}_{\mathfrak{b}}(Z) = -\sup(L \otimes^{\mathbf{L}}_R Z) + m = -\sup(Z)$$

by what we have already shown.

(b) This follows from part (a), via Fact 2.9. □

Note that some items in the next result use L, while others use K.

Lemma 3.2. *Let $Z \in \mathcal{D}(R)$, let $\mathbf{y} = y_1, \ldots, y_m \in \mathfrak{a}$, and set $L := K^R(\mathbf{y})$ and $\mathfrak{b} := (\mathbf{y})R \subseteq \mathfrak{a}$. Assume that* $\operatorname{co-supp}_R(Z) \subseteq \operatorname{V}(\mathfrak{a})$, e.g. *that each homology module $\operatorname{H}_i(Z)$ is annihilated by a power of $\mathfrak{a}$.*

(a) *There are (in)equalities*

$$\operatorname{width}_{\mathfrak{b}}(Z) = \inf(Z) = \inf(L \otimes^{\mathbf{L}}_R Z)$$
$$\sup(Z) - n \leqslant \sup(K \otimes^{\mathbf{L}}_R Z)$$
$$\operatorname{amp}(Z) - n \leqslant \operatorname{amp}(K \otimes^{\mathbf{L}}_R Z).$$

(b) *For each $* \in \{+, -, b\}$, one has $K \otimes^{\mathbf{L}}_R Z \in \mathcal{D}_*(R)$ if and only if $Z \in \mathcal{D}_*(R)$.*

Proof. (a) The assumption $\operatorname{co-supp}_R(Z) \subseteq \operatorname{V}(\mathfrak{a}) \subseteq \operatorname{V}(\mathfrak{b})$ implies that $Z \simeq \mathbf{L}\Lambda^{\mathfrak{b}}(Z)$, by Fact 2.4. This explains the first equality in the next sequence

$$\inf(Z) = \inf(\mathbf{L}\Lambda^{\mathfrak{b}}(Z)) = \inf(L \otimes^{\mathbf{L}}_R Z) = \inf((R/\mathfrak{b}) \otimes^{\mathbf{L}}_R Z) = \operatorname{width}_{\mathfrak{b}}(Z)$$

while the remaining equalities are from [9, Theorem 4.1] and by definition. This also explains the first isomorphism in the next sequence

$$Z \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(Y) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(Z)) \simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\Gamma_{\mathfrak{a}}(Z))$$

while the other isomorphisms are from [1, Theorem (0.3)* and Corollary]. This explains the first step in the next sequence.

$$\begin{aligned}\sup(Z) &= \sup(\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\Gamma_{\mathfrak{a}}(Z))) \\ &\leqslant \sup(\mathbf{R}\Gamma_{\mathfrak{a}}(Z)) - \inf(\mathbf{R}\Gamma_{\mathfrak{a}}(R)) \\ &\leqslant \sup(K \otimes^{\mathbf{L}}_R Z) + n.\end{aligned}$$

The second step is from Fact 2.9(b). The third step follows from the equality $\sup(\mathbf{R}\Gamma_{\mathfrak{a}}(Z)) = \sup(K \otimes^{\mathbf{L}}_R Y)$ of [9, Theorem 2.1], and the inequality $\inf(\mathbf{R}\Gamma_{\mathfrak{a}}(R)) \geqslant -n$ which is via the Čech complex. This explains the first two rows of (in)equalities from part (a), and the third row follows by definition.

(b) This follows from part (a) and Fact 2.9. □

Remark 3.3. Since $Z \simeq 0$ if and only if $\inf(Z) = \infty$, the previous two results also have the following conclusions: one has $Z \simeq 0$ if and only if $K \otimes^{\mathbf{L}}_R Z \simeq 0$. However, we already know this because of Remark 2.2 and Fact 2.4.

We continue with some useful computations of homological dimensions. Note that the next result shows that the quantities $\operatorname{fd}_R(L \otimes^{\mathbf{L}}_R Z)$ and $\operatorname{fd}_R(Z)$ are simultaneously finite, and similarly for pd.

Lemma 3.4. *Let $Z \in \mathcal{D}(R)$, let $\mathbf{y} = y_1, \ldots, y_m \in \mathfrak{a}$, and set $L := K^R(\mathbf{y})$. Assume that* $\operatorname{supp}_R(Z) \subseteq \operatorname{V}(\mathfrak{a})$. *Then we have*

$$\operatorname{fd}_R(L \otimes^{\mathbf{L}}_R Z) = m + \operatorname{fd}_R(Z)$$

$$\operatorname{pd}_R(L \otimes^{\mathbf{L}}_R Z) = m + \operatorname{pd}_R(Z).$$

Proof. Let N be an R-module, and note that $\operatorname{supp}_R(N \otimes^{\mathbf{L}}_R Z) \subseteq \operatorname{supp}_R(Z) \subseteq \operatorname{V}(\mathfrak{a})$. Using this with the associativity isomorphism $N \otimes^{\mathbf{L}}_R (L \otimes^{\mathbf{L}}_R Z) \simeq L \otimes^{\mathbf{L}}_R (N \otimes^{\mathbf{L}}_R Z)$, we conclude from Lemma 3.1(a) that

$$\sup(N \otimes^{\mathbf{L}}_R (L \otimes^{\mathbf{L}}_R Z)) = \sup(L \otimes^{\mathbf{L}}_R (N \otimes^{\mathbf{L}}_R Z)) = m + \sup(N \otimes^{\mathbf{L}}_R Z).$$

Thus, it follows from [2, Proposition 2.4.F] that we have

$$\begin{aligned} \operatorname{fd}_R(L \otimes^{\mathbf{L}}_R Z) &= \sup\{\sup(N \otimes^{\mathbf{L}}_R (L \otimes^{\mathbf{L}}_R Z)) \mid N \text{ is an } R\text{-module}\} \\ &= m + \sup\{\sup(N \otimes^{\mathbf{L}}_R Z) \mid N \text{ is an } R\text{-module}\} \\ &= m + \operatorname{fd}_R(Z). \end{aligned}$$

One verifies the equality $\operatorname{pd}_R(L \otimes^{\mathbf{L}}_R Z) = m + \operatorname{pd}_R(Z)$ similarly. □

The next result is verified like the previous one.

Lemma 3.5. *Let $Z \in \mathcal{D}(R)$, let $\mathbf{y} = y_1, \ldots, y_m \in \mathfrak{a}$, and set $L := K^R(\mathbf{y})$. Assume that* $\operatorname{co-supp}_R(Z) \subseteq \operatorname{V}(\mathfrak{a})$. *Then we have* $\operatorname{id}_R(\mathbf{R}\operatorname{Hom}_R(L, Z)) = m + \operatorname{id}_R(Z)$.

4. Induced Isomorphisms

This section contains Theorem 1.1 from the introduction, with several other results of the same ilk. The main idea is to replace the homologically finite assumption with $\mathfrak{a}$-adically finiteness in some well-known isomorphism theorems. We begin with tensor-evaluation.

Theorem 4.1. *Let M be an $\mathfrak{a}$-adically finite R-complex, and let $Y \in \mathcal{D}_-(R)$ and $U, V, Z \in \mathcal{D}(R)$. Assume that* $\operatorname{supp}_R(V), \operatorname{co-supp}_R(U) \subseteq \operatorname{V}(\mathfrak{a})$ *and that either* $\operatorname{fd}_R(M)$ *or* $\operatorname{fd}_R(Z)$ *is finite. Consider an exact triangle*

$$\mathbf{R}\operatorname{Hom}_R(M, Y) \otimes^{\mathbf{L}}_R Z \xrightarrow{\omega_{MYZ}} \mathbf{R}\operatorname{Hom}_R(M, Y \otimes^{\mathbf{L}}_R Z) \to C \to$$

in $\mathcal{D}(R)$, where ω_{MYZ} is the natural tensor evaluation morphism.

(a) *One has* $\operatorname{supp}_R(C)\bigcap \mathrm{V}(\mathfrak{a}) = \emptyset = \text{co-supp}_R(C)\bigcap \mathrm{V}(\mathfrak{a})$.
(b) *The morphisms* $V \otimes^{\mathbf{L}}_R \omega_{MYZ}$ *and* $\mathbf{R}\mathrm{Hom}_R(V, \omega_{MYZ})$ *and* $\mathbf{R}\mathrm{Hom}_R(\omega_{MYZ}, U)$ *are isomorphisms.*

Proof. The fact that M is $\mathfrak{a}$-adically finite implies that $\mathbf{R}\mathrm{Hom}_R(K, M) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. If $\mathrm{fd}_R(M) < \infty$, then $\mathrm{fd}_R(\mathbf{R}\mathrm{Hom}_R(K, M)) < \infty$, that is, $\mathrm{pd}_R(\mathbf{R}\mathrm{Hom}_R(K, M)) < \infty$ because of the homological finiteness. Thus, when either $\mathrm{fd}_R(M)$ or $\mathrm{fd}_R(Z)$ is finite, the tensor evaluation morphism $\omega_{\mathbf{R}\mathrm{Hom}_R(K,M)YZ}$ in the next commutative diagram

$$\begin{array}{ccc}
K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, Y) \otimes^{\mathbf{L}}_R Z & \xrightarrow[\simeq]{\theta_{KMY\otimes^{\mathbf{L}}_R Z}} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K, M), Y) \otimes^{\mathbf{L}}_R Z \\
\Big\downarrow {\scriptstyle K\otimes^{\mathbf{L}}_R \omega_{MYZ}} & & \simeq \Big\downarrow {\scriptstyle \omega_{\mathbf{R}\mathrm{Hom}_R(K,M)YZ}} \\
K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, Y \otimes^{\mathbf{L}}_R Z) & \xrightarrow[\simeq]{\theta_{KMY\otimes^{\mathbf{L}}_R Z}} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K, M), Y \otimes^{\mathbf{L}}_R Z)
\end{array}$$

is an isomorphism in $\mathcal{D}(R)$, by [2, Lemma 4.4(F)]; this result also explains the horizontal isomorphisms. It follows that $K \otimes^{\mathbf{L}}_R \omega_{MYZ}$ is an isomorphism as well. We conclude that $K \otimes^{\mathbf{L}}_R C \simeq 0$, so by Fact 2.4, we have

$$\emptyset = \operatorname{supp}_R(K \otimes^{\mathbf{L}}_R C) = \operatorname{supp}_R(K)\bigcap \operatorname{supp}_R(C) = \mathrm{V}(\mathfrak{a})\bigcap \operatorname{supp}_R(C).$$

This implies that $\emptyset = \operatorname{supp}_R(V)\bigcap \operatorname{supp}_R(C)$, so $V \otimes^{\mathbf{L}}_R C \simeq 0$. The induced triangle

$$V \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, Y) \otimes^{\mathbf{L}}_R Z \xrightarrow{V\otimes^{\mathbf{L}}_R \omega_{MYZ}} V \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, Y \otimes^{\mathbf{L}}_R Z) \to V \otimes^{\mathbf{L}}_R C \to$$

shows that $V \otimes^{\mathbf{L}}_R \omega_{MYZ}$ is an isomorphism. The isomorphism $\mathbf{R}\mathrm{Hom}_R(\omega_{MYZ}, U)$ is verified similarly.

On the other hand, the self-dual nature 2.2 of K implies that $\mathbf{R}\mathrm{Hom}_R(K, -) \simeq \Sigma^{-n} K \otimes^{\mathbf{L}}_R -$, so the induced morphism $\mathbf{R}\mathrm{Hom}_R(K, \omega_{MYZ})$ is also an isomorphism. We conclude as above that $\emptyset = \mathrm{V}(\mathfrak{a})\bigcap \text{co-supp}_R(C)$[b] and that the morphism $\mathbf{R}\mathrm{Hom}_R(V, \omega_{MYZ})$ is an isomorphism, as desired. □

Next, we consider Hom-evaluation.

Theorem 4.2. *Let* M *be an* $\mathfrak{a}$*-adically finite* R*-complex, and let* $Y \in \mathcal{D}_b(R)$ *and* $U, V, Z \in \mathcal{D}(R)$. *Assume that* $\operatorname{supp}_R(V), \text{co-supp}_R(U) \subseteq \mathrm{V}(\mathfrak{a})$ *and that either* $\mathrm{fd}_R(M)$ *or* $\mathrm{id}_R(Z)$ *is finite. Consider an exact triangle*

$$M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(Y, Z) \xrightarrow{\theta_{MYZ}} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, Y), Z) \to C \to$$

in $\mathcal{D}(R)$ *where* θ_{MYZ} *is the natural Hom-evaluation morphism.*

(a) *One has* $\operatorname{supp}_R(C)\bigcap \mathrm{V}(\mathfrak{a}) = \emptyset = \text{co-supp}_R(C)\bigcap \mathrm{V}(\mathfrak{a})$.
(b) *The morphisms* $V \otimes^{\mathbf{L}}_R \theta_{MYZ}$ *and* $\mathbf{R}\mathrm{Hom}_R(V, \theta_{MYZ})$ *and* $\mathbf{R}\mathrm{Hom}_R(\theta_{MYZ}, U)$ *are isomorphisms.*

[b] See also [5, Corollary 4.9].

Proof. The fact that M is $\mathfrak{a}$-adically finite implies that $K \otimes^{\mathbf{L}}_R M \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. If $\mathrm{fd}_R(M) < \infty$, then $\mathrm{fd}_R(K \otimes^{\mathbf{L}}_R M) < \infty$, that is, $\mathrm{pd}_R(K \otimes^{\mathbf{L}}_R M) < \infty$ because of the homological finiteness. Thus, when either $\mathrm{fd}_R(M)$ or $\mathrm{id}_R(Z)$ is finite, the morphism $\theta_{K\otimes^{\mathbf{L}}_R MYZ}$ is an isomorphism in $\mathcal{D}(R)$, by [2, Lemma 4.4(I)]. This result also explains the second vertical isomorphism in the following commutative diagram

$$\begin{array}{ccc} K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(Y,Z) & \xrightarrow{K\otimes^{\mathbf{L}}_R \theta_{MYZ}} & K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,Y),Z) \\ {\scriptstyle \theta_{K\otimes^{\mathbf{L}}_R MYZ}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \theta_{K\mathbf{R}\mathrm{Hom}_R(M,Y)Z}} \\ \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K \otimes^{\mathbf{L}}_R M, Y), Z) & \xrightarrow{\simeq} & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\mathrm{Hom}_R(M,Y)), Z) \end{array}$$

in $\mathcal{D}(R)$. The unspecified isomorphism is induced by Hom-tensor adjointness. It follows that $K \otimes^{\mathbf{L}}_R \theta_{MYZ}$ is an isomorphism, and the rest of the desired conclusions follow as in the proof of Theorem 4.1. ∎

Similarly, we have the next result, via [6, Proposition 2.2].

Theorem 4.3. *Let M be an $\mathfrak{a}$-adically finite R-complex, and let $U, V, Y \in \mathcal{D}(R)$ and $Z \in \mathcal{D}_-(R)$ be such that $\mathrm{supp}_R(V), \text{co-supp}_R(U) \subseteq \mathrm{V}(\mathfrak{a})$. Assume that at least one of the following conditions holds:*

(1) $\mathrm{pd}_R(Y) < \infty$, *or*
(2) $Z \in \mathcal{D}_b(R)$ *and* $\mathrm{fd}_R(M) < \infty$.

Consider an exact triangle

$$\mathbf{R}\mathrm{Hom}_R(Y,Z) \otimes^{\mathbf{L}}_R M \xrightarrow{\omega_{YZM}} \mathbf{R}\mathrm{Hom}_R(Y, Z \otimes^{\mathbf{L}}_R M) \to C \to$$

in $\mathcal{D}(R)$, where ω_{MYZ} is the natural tensor evaluation morphism.

(a) *One has* $\mathrm{supp}_R(C) \bigcap \mathrm{V}(\mathfrak{a}) = \emptyset = \text{co-supp}_R(C) \bigcap \mathrm{V}(\mathfrak{a})$.
(b) *The morphisms $V \otimes^{\mathbf{L}}_R \omega_{YZM}$ and $\mathbf{R}\mathrm{Hom}_R(V, \omega_{YZM})$ and $\mathbf{R}\mathrm{Hom}_R(\omega_{YZM}, U)$ are isomorphisms.*

Next, we document some special cases of the previous results.

Corollary 4.4. *Let M be an $\mathfrak{a}$-adically finite R-complex, and let $Y \in \mathcal{D}_-(R)$ and $U, V \in \mathcal{D}(R)$ such that $\mathrm{supp}_R(V), \text{co-supp}_R(U) \subseteq \mathrm{V}(\mathfrak{a})$. Let $R \to S$ be a ring homomorphism. Assume that either $\mathrm{fd}_R(M) < \infty$ or $\mathrm{fd}_R(S) < \infty$. Consider an exact triangle*

$$\mathbf{R}\mathrm{Hom}_R(M,Y) \otimes^{\mathbf{L}}_R S \xrightarrow{\alpha_{MYS}} \mathbf{R}\mathrm{Hom}_S(M \otimes^{\mathbf{L}}_R S, Y \otimes^{\mathbf{L}}_R S) \to C \to$$

in $\mathcal{D}(R)$, where α_{MYS} is the natural morphism.

(a) *One has* $\mathrm{supp}_R(C) \bigcap \mathrm{V}(\mathfrak{a}) = \emptyset = \text{co-supp}_R(C) \bigcap \mathrm{V}(\mathfrak{a})$.
(b) *The morphisms $V \otimes^{\mathbf{L}}_R \alpha_{MYZ}$ and $\mathbf{R}\mathrm{Hom}_R(V, \alpha_{MYZ})$ and $\mathbf{R}\mathrm{Hom}_R(\alpha_{MYZ}, U)$ are isomorphisms.*

Proof. Consider the following commutative diagram in $\mathcal{D}(R)$

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_R(M,Y)\otimes^{\mathbf{L}}_R S & \xrightarrow{\omega_{MYS}} & \mathbf{R}\mathrm{Hom}_R(M,Y\otimes^{\mathbf{L}}_R S) \\
\Big\downarrow{\scriptstyle \alpha_{MYS}} & & \Big\uparrow{\scriptstyle \simeq} \\
\mathbf{R}\mathrm{Hom}_S(M\otimes^{\mathbf{L}}_R S, Y\otimes^{\mathbf{L}}_R S) & \xrightarrow{\simeq} & \mathbf{R}\mathrm{Hom}_R(M,\mathbf{R}\mathrm{Hom}_S(S,Y\otimes^{\mathbf{L}}_R S)).
\end{array}$$

The unspecified isomorphisms are Hom-cancellation and Hom-tensor adjointness. Apply the functors $V\otimes^{\mathbf{L}}_R -$ and $\mathbf{R}\mathrm{Hom}_R(V,-)$ and $\mathbf{R}\mathrm{Hom}_R(-,U)$ to this diagram, and use Theorem 4.1 to prove part (4.4). Then part (4.4) follows from Fact 2.4. $\square$

The next two lemmas are probably well known. In the absence of suitable references, we include some proof.

Lemma 4.5. *Let $X\in\mathcal{D}^f_+(R)$, and consider a set $\{N_\lambda\}_{\lambda\in\Lambda}\in\mathcal{D}_+(R)$ such that $\inf(N_\lambda)\geqslant s$ for all $\lambda\in\Lambda$. Then the natural morphism*

$$X\otimes^{\mathbf{L}}_R \prod_\lambda N_\lambda \to \prod_\lambda (X\otimes^{\mathbf{L}}_R N_\lambda)$$

is an isomorphism in $\mathcal{D}(R)$.

Proof. Let $F\xrightarrow{\simeq} X$ be a degree-wise finite semi-free resolution, that is, a quasi-isomorphism, where F is a bounded below complex of finitely generated free R-modules. Truncate each N_λ if necessary to assume that $(N_\lambda)_q=0$ for all $q<s$. Since each F_p is a finite-rank free module, the natural map

$$F_p\otimes_R \prod_\lambda (N_\lambda)_q \to \prod_\lambda \left(F_p\otimes_R (N_\lambda)_q\right)$$

is an isomorphism for each q. Thus, the product map

$$\prod_{p+q=i}\left(F_p\otimes_R \prod_\lambda (N_\lambda)_q\right) \to \prod_{p+q=i}\left(\prod_\lambda \left(F_p\otimes_R (N_\lambda)_q\right)\right)$$

is an isomorphism as well, for each i. Using the natural isomorphism

$$\prod_{p+q=i}\left(\prod_\lambda F_p\otimes_R (N_\lambda)_q\right) \cong \prod_\lambda\left(\prod_{p+q=i} F_p\otimes_R (N_\lambda)_q\right)$$

we conclude that the natural map

$$\prod_{p+q=i}\left(F_p\otimes_R \prod_\lambda (N_\lambda)_q\right) \to \prod_\lambda\left(\prod_{p+q=i} F_p\otimes_R (N_\lambda)_q\right) \tag{1}$$

is an isomorphism a well.

Our boundedness assumptions on F and N_λ imply that, for each i, we have

$$\bigoplus_{p+q=i}\left(F_p \otimes_R \prod_\lambda (N_\lambda)_q\right) = \prod_{p+q=i}\left(F_p \otimes_R \prod_\lambda (N_\lambda)_q\right)$$

$$\prod_\lambda\left(\bigoplus_{p+q=i} F_p \otimes_R (N_\lambda)_q\right) = \prod_\lambda\left(\prod_{p+q=i} F_p \otimes_R (N_\lambda)_q\right)$$

so the isomorphism (1) with these equalities shows that the natural map

$$\bigoplus_{p+q=i}\left(F_p \otimes_R \prod_\lambda (N_\lambda)_q\right) \to \prod_\lambda\left(\bigoplus_{p+q=i} F_p \otimes_R (N_\lambda)_q\right)$$

is an isomorphism for each i. Thus, the chain map

$$F \otimes_R \prod_\lambda N_\lambda \to \prod_\lambda (F \otimes_R N_\lambda)$$

is an isomorphism. By design, this represents the natural morphism

$$X \otimes^{\mathbf{L}}_R \prod_\lambda N_\lambda \to \prod_\lambda \left(X \otimes^{\mathbf{L}}_R N_\lambda\right)$$

in $\mathcal{D}(R)$, so this morphism is an isomorphism, as desired. □

The next result is proved like the previous one.

Lemma 4.6. *Let $X \in \mathcal{D}^f_+(R)$, and consider a set $\{N_\lambda\}_{\lambda\in\Lambda} \in \mathcal{D}_-(R)$ such that $\sup(N_\lambda) \leqslant s$ for all $\lambda \in \Lambda$. Then the natural morphism*

$$\bigoplus_\lambda (\mathbf{R}\mathrm{Hom}_R(X, N_\lambda)) \to \mathbf{R}\mathrm{Hom}_R\left(X, \bigoplus_\lambda N_\lambda\right)$$

is an isomorphism in $\mathcal{D}(R)$.

Next, we soup up the previous two results, first by proving Theorem 1.1 from the introduction.

Theorem 4.7. *Let M be an $\mathfrak{a}$-adically finite R-complex, and let $U, V \in \mathcal{D}(R)$ be such that $\mathrm{supp}_R(V), \text{co-supp}_R(U) \subseteq \mathrm{V}(\mathfrak{a})$. Consider a set $\{N_\lambda\}_{\lambda\in\Lambda} \in \mathcal{D}_+(R)$ such that $\inf(N_\lambda) \geqslant s$ for all $\lambda \in \Lambda$. Consider an exact triangle*

$$M \otimes^{\mathbf{L}}_R \prod_\lambda N_\lambda \xrightarrow{p} \prod_\lambda \left(M \otimes^{\mathbf{L}}_R N_\lambda\right) \to A \to$$

where p is the natural morphism.

(a) *One has $\mathrm{supp}_R(A) \bigcap \mathrm{V}(\mathfrak{a}) = \emptyset = \text{co-supp}_R(A) \bigcap \mathrm{V}(\mathfrak{a})$.*
(b) *The morphisms $V \otimes^{\mathbf{L}}_R p$ and $\mathbf{R}\mathrm{Hom}_R(V, p)$ and $\mathbf{R}\mathrm{Hom}_R(p, U)$ are isomorphisms.*

Proof. As in our previous results, it suffices to show that $K\otimes^{\mathbf{L}}_R p$ is an isomorphism in $\mathcal{D}(R)$. As M is $\mathfrak{a}$-adically finite, we have $K, K\otimes^{\mathbf{L}}_R M \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $\inf(M\otimes^{\mathbf{L}}_R N_\lambda) \geqslant s+\inf(M)$ by Fact 2.9(b). Thus, Lemma 4.5 explains the unspecified isomorphisms in the next commutative diagram.

$$\begin{array}{ccc} K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R \prod_\lambda N_\lambda & & \\ \downarrow{\scriptstyle K\otimes^{\mathbf{L}}_R p} & \searrow^{\simeq} & \\ K \otimes^{\mathbf{L}}_R \prod_\lambda (M \otimes^{\mathbf{L}}_R N_\lambda) & \xrightarrow{\simeq} & \prod_\lambda (K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R N_\lambda). \end{array}$$

We conclude that $K \otimes^{\mathbf{L}}_R p$ is also an isomorphism, as desired. □

The next result follows similarly from Lemma 4.6.

Theorem 4.8. *Let M be an $\mathfrak{a}$-adically finite R-complex, and let $U, V \in \mathcal{D}(R)$ such that* $\operatorname{supp}_R(V), \operatorname{co-supp}_R(U) \subseteq \operatorname{V}(\mathfrak{a})$. *Consider a set $\{N_\lambda\}_{\lambda\in\Lambda} \in \mathcal{D}_-(R)$ such that $\sup(N_\lambda) \leqslant s$ for all $\lambda \in \Lambda$. Consider an exact triangle*

$$\bigoplus_\lambda (\mathbf{R}\mathrm{Hom}_R(M, N_\lambda)) \xrightarrow{q} \mathbf{R}\mathrm{Hom}_R\left(M, \bigoplus_\lambda N_\lambda\right) \to B \to$$

where q is the natural morphism.

(a) *One has* $\operatorname{supp}_R(B) \bigcap \operatorname{V}(\mathfrak{a}) = \emptyset = \operatorname{co-supp}_R(B) \bigcap \operatorname{V}(\mathfrak{a})$.
(b) *The morphisms $V\otimes^{\mathbf{L}}_R p$ and $\mathbf{R}\mathrm{Hom}_R(V, p)$ and $\mathbf{R}\mathrm{Hom}_R(p, U)$ are isomorphisms.*

The case $M = \mathbf{R}\Gamma_{\mathfrak{a}}(R)$ is useful for subsequent results, so we document it next.

Corollary 4.9. *Let $U, V, Z \in \mathcal{D}(R)$ such that* $\operatorname{co-supp}_R(U), \operatorname{supp}_R(V) \subseteq \operatorname{V}(\mathfrak{a})$.

(a) *If $Y \in \mathcal{D}_-(R)$, then there are natural isomorphisms in $\mathcal{D}(R)$*

$$V \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(Y) \otimes^{\mathbf{L}}_R Z \xrightarrow{\simeq} V \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(Y \otimes^{\mathbf{L}}_R Z)$$

$$\mathbf{R}\mathrm{Hom}_R(V, \mathbf{L}\Lambda^{\mathfrak{a}}(Y) \otimes^{\mathbf{L}}_R Z) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(V, \mathbf{L}\Lambda^{\mathfrak{a}}(Y \otimes^{\mathbf{L}}_R Z))$$

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(Y \otimes^{\mathbf{L}}_R Z), U) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(Y) \otimes^{\mathbf{L}}_R Z, U).$$

(b) *If $Y \in \mathcal{D}_b(R)$, then there are natural isomorphisms in $\mathcal{D}(R)$*

$$V \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(Y, Z)) \xrightarrow{\simeq} V \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(Y), Z)$$

$$\mathbf{R}\mathrm{Hom}_R(V, \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(Y, Z))) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(V, \mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(Y), Z))$$

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(Y), Z), U) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(Y, Z)), U).$$

(c) *If $Z \in \mathcal{D}_b(R)$ or both $Z \in \mathcal{D}_-(R)$ and $\mathrm{pd}_Y(R) < \infty$, then there are natural isomorphisms in $\mathcal{D}(R)$*

$$V \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(Y, Z)) \xrightarrow{\simeq} V \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(Z))$$

$$\mathbf{R}\mathrm{Hom}_R(V, \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(Y, Z))) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(V, \mathbf{R}\mathrm{Hom}_R(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(Z)))$$

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(Y, \mathbf{R}\Gamma_{\mathfrak{a}}(Z)), U) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(Y, Z)), U).$$

(d) *Let* $R \to S$ *be a ring homomorphism. If* $Y \in \mathcal{D}_-(R)$, *then there are natural isomorphisms in* $\mathcal{D}(S)$

$$V \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(Y) \otimes^{\mathbf{L}}_R S \xrightarrow{\simeq} V \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}S}(Y \otimes^{\mathbf{L}}_R S)$$
$$\mathbf{R}\mathrm{Hom}_R(V, \mathbf{L}\Lambda^{\mathfrak{a}}(Y) \otimes^{\mathbf{L}}_R S) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(V, \mathbf{L}\Lambda^{\mathfrak{a}S}(Y \otimes^{\mathbf{L}}_R S))$$
$$\mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}S}(Y \otimes^{\mathbf{L}}_R S), U) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(Y) \otimes^{\mathbf{L}}_R S, U).$$

(e) *Given a set* $\{N_\lambda\}_{\lambda\in\Lambda} \in \mathcal{D}_+(R)$ *such that* $\inf(N_\lambda) \geqslant s$ *for all* $\lambda \in \Lambda$, *there are natural isomorphisms in* $\mathcal{D}(R)$

$$V \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}\left(\prod_\lambda N_\lambda\right) \xrightarrow{\simeq} V \otimes^{\mathbf{L}}_R \prod_\lambda \mathbf{R}\Gamma_{\mathfrak{a}}(N_\lambda)$$
$$\mathbf{R}\mathrm{Hom}_R\left(V, \mathbf{R}\Gamma_{\mathfrak{a}}\left(\prod_\lambda N_\lambda\right)\right) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R\left(V, \prod_\lambda \mathbf{R}\Gamma_{\mathfrak{a}}(N_\lambda)\right)$$
$$\mathbf{R}\mathrm{Hom}_R\left(\prod_\lambda \mathbf{R}\Gamma_{\mathfrak{a}}(N_\lambda), U\right) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}\left(\prod_\lambda N_\lambda\right), U\right).$$

(f) *Given a set* $\{N_\lambda\}_{\lambda\in\Lambda} \in \mathcal{D}_+(R)$ *such that* $\inf(N_\lambda) \geqslant s$ *for all* $\lambda \in \Lambda$, *there are natural isomorphisms in* $\mathcal{D}(R)$

$$V \otimes^{\mathbf{L}}_R \bigoplus_\lambda \mathbf{L}\Lambda^{\mathfrak{a}}(N_\lambda) \xrightarrow{\simeq} V \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}\left(\bigoplus_\lambda N_\lambda\right)$$
$$\mathbf{R}\mathrm{Hom}_R\left(V, \bigoplus_\lambda \mathbf{L}\Lambda^{\mathfrak{a}}(N_\lambda)\right) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R\left(V, \mathbf{L}\Lambda^{\mathfrak{a}}\left(\bigoplus_\lambda N_\lambda\right)\right)$$
$$\mathbf{R}\mathrm{Hom}_R\left(\mathbf{L}\Lambda^{\mathfrak{a}}\left(\bigoplus_\lambda N_\lambda\right), U\right) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R\left(\bigoplus_\lambda \mathbf{L}\Lambda^{\mathfrak{a}}(N_\lambda), U\right).$$

Proof. The complex $M = \mathbf{R}\Gamma_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-adically finite by Example 2.7(b) and has finite flat dimension via the Čech complex. Thus, the result follows from Fact 2.1 and the preceding results. □

5. Transfer of Support, Co-support and Finiteness

In this section, we consider various transfer relations along a ring homomorphism.

Notation 5.1. Throughout this section, let $\varphi\colon R \to S$ be a ring homomorphism such that $\mathfrak{a}S \neq S$, and consider the forgetful functor $Q\colon \mathcal{D}(S) \to \mathcal{D}(R)$.

Restriction of Scalars

We begin this subsection with two useful isomorphisms.

Lemma 5.2. *Given an S-complex $Y \in \mathcal{D}(S)$, there are natural isomorphisms*

$$Q(\mathbf{L}\Lambda^{\mathfrak{a}S}(Y)) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(Q(Y)) \qquad\qquad Q(\mathbf{R}\Gamma_{\mathfrak{a}S}(Y)) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Q(Y)).$$

Proof. In the following computation, the first and last isomorphisms are from Fact 2.1, and the second one comes from Čech complexes.

$$\begin{aligned} Q(\mathbf{L}\Lambda^{\mathfrak{a}S}(Y)) &\simeq Q(\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\Gamma_{\mathfrak{a}S}(S), Y)) \\ &\simeq Q(\mathbf{R}\mathrm{Hom}_S(S \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R), Y)) \\ &\simeq Q(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\mathrm{Hom}_S(S, Y))) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), Q(Y)) \\ &\simeq \mathbf{L}\Lambda^{\mathfrak{a}}(Q(Y)) \end{aligned}$$

The other isomorphisms are from adjointness and cancellation. The explains the first isomorphism, and the second one is verified similarly. □

Lemma 5.3. *Given an S-complex $Y \in \mathcal{D}(S)$, one has* $\mathrm{supp}_S(Y) \subseteq \mathrm{V}(\mathfrak{a}S)$ *if and only if* $\mathrm{supp}_R(Q(Y)) \subseteq \mathrm{V}(\mathfrak{a})$.

Proof. From Lemma 5.2, the natural morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(Q(Y)) \to Q(Y)$ is equivalent to the induced morphism $Q(\mathbf{R}\Gamma_{\mathfrak{a}S}(Y)) \to Q(Y)$. In particular, this says that the morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(Q(Y)) \to Q(Y)$ is an isomorphism in $\mathcal{D}(R)$ if and only if $Q(\mathbf{R}\Gamma_{\mathfrak{a}S}(Y)) \to Q(Y)$ is an isomorphism in $\mathcal{D}(R)$, that is, if and only if the morphism $\mathbf{R}\Gamma_{\mathfrak{a}S}(Y) \to Y$ is an isomorphism in $\mathcal{D}(S)$. Two applications of Fact 2.4, then give the desired result.

Alternately, we know that $\mathrm{supp}_S(Y) \subseteq \mathrm{V}(\mathfrak{a}S)$ if and only if each homology module $\mathrm{H}_i(Y)$ is $\mathfrak{a}S$-torsion, and similarly for $Q(Y)$; see Fact 2.4. Since we have $\mathrm{H}_i(Q(Y)) \cong \mathrm{H}_i(Y)$, these are $\mathfrak{a}$-torsion if and only if they are $\mathfrak{a}S$-torsion, so the result follows. □

Lemma 5.4. *Given an S-complex $Y \in \mathcal{D}(S)$, one has* $\text{co-supp}_S(Y) \subseteq \mathrm{V}(\mathfrak{a}S)$ *if and only if* $\text{co-supp}_R(Q(Y)) \subseteq \mathrm{V}(\mathfrak{a})$.

Proof. Argue as in the first paragraph of the proof of the previous result. □

Theorem 5.5. *Let $Y \in \mathcal{D}(S)$ be given. If $Q(Y)$ is $\mathfrak{a}$-adically finite over R, then Y is $\mathfrak{a}S$-adically finite over S; the converse holds when the induced map $\overline{\varphi}\colon R/\mathfrak{a} \to S/\mathfrak{a}S$ is module finite, e.g. when φ is module finite or $S = \widehat{R}^{\mathfrak{a}}$.*

Proof. It is clear that we have $Q(Y) \in \mathcal{D}_{\mathrm{b}}(R)$ if and only if $Y \in \mathcal{D}_{\mathrm{b}}(S)$, and we have $\mathrm{supp}_S(Y) \subseteq \mathrm{V}(\mathfrak{a}S)$ if and only if $\mathrm{supp}_R(Q(Y)) \subseteq \mathrm{V}(\mathfrak{a})$ by Lemma 5.3. Note that $K' := S \otimes^{\mathbf{L}}_R K$ is the Koszul complex over S on a finite generating sequence for $\mathfrak{a}S$. Consider the following isomorphisms:

$$K' \otimes^{\mathbf{L}}_S Y \simeq (S \otimes^{\mathbf{L}}_R K) \otimes^{\mathbf{L}}_S Y \simeq K \otimes^{\mathbf{L}}_R Q(Y).$$

If each homology module $\mathrm{H}_i(K \otimes^{\mathbf{L}}_R Q(Y))$ is finitely generated over R, then the above isomorphisms imply that $\mathrm{H}_i(K' \otimes^{\mathbf{L}}_S Y)$ is finitely generated over R, hence over S. Thus, if $Q(Y)$ is $\mathfrak{a}$-adically finite over R, then Y is $\mathfrak{a}S$-adically finite over S, by definition.

For the converse, assume that Y is $\mathfrak{a}S$-adically finite over S and that the induced map $\overline{\varphi}$ is module finite. Since each module $\mathrm{H}_i(K' \otimes^{\mathbf{L}}_S Y)$ is finitely generated over S and has a natural $S/\mathfrak{a}S$-module structure, it is finitely generated over $S/\mathfrak{a}S$. Our finiteness assumption on $S/\mathfrak{a}S$ conspires with the above isomorphisms to imply that each module $\mathrm{H}_i(K \otimes^{\mathbf{L}}_R Q(Y))$ is finitely generated over $R/\mathfrak{a}$, hence over R, so $Q(Y)$ is $\mathfrak{a}$-adically finite over R. □

The next result explains the relation between the condition $\varphi^*(\mathrm{supp}_S(F)) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$ from [22, Theorem 4.1] and the seemingly more natural condition $\mathrm{supp}_R(F) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$. Here, φ^* is the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. These ideas will be explored further in [21].

Proposition 5.6. *Let $Y \in \mathcal{D}(S)$.*

(a) *One has* $\varphi^*(\mathrm{Spec}(S)) = \mathrm{supp}_R(S)$.
(b) *One has* $\mathrm{supp}_R(Y) \subseteq \varphi^*(\mathrm{Spec}(S))$.
(c) *If* $\mathrm{supp}_R(Y) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, *then* $\varphi^*(\mathrm{supp}_S(Y)) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$.

Proof. Let $\mathfrak{p} \in \mathrm{Spec}(R)$, and set $S_\mathfrak{p} := R_\mathfrak{p} \otimes_R S$.

(a) For one implication, assume that $\mathfrak{p} \in \varphi^*(\mathrm{Spec}(S))$, and let $P \in \mathrm{Spec}(S)$ be such that $\mathfrak{p} = \varphi^{-1}(P)$. It follows that P represents a prime ideal in the ring $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \cong \kappa(\mathfrak{p}) \otimes_R S$. In particular, we have $0 \neq \kappa(\mathfrak{p}) \otimes_R S \cong \mathrm{H}_0(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R S)$. We conclude that $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R S \not\simeq 0$, so $\mathfrak{p} \in \mathrm{supp}_R(S)$.

For the converse, assume that $\mathfrak{p} \in \mathrm{supp}_R(S)$. By definition, this implies that $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R S \not\simeq 0$, so there is an integer i such that $\mathrm{H}_i(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R S) \neq 0$. From the isomorphism $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R S \simeq (R/\mathfrak{p}) \otimes^{\mathbf{L}}_R S_\mathfrak{p}$, we conclude that $\mathrm{H}_i(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R S)$ is a nonzero $S_\mathfrak{p}$-module that is annihilated by $\mathfrak{p}$, i.e., it is a nonzero module over $(R/\mathfrak{p}) \otimes_R S_\mathfrak{p} \cong \kappa(\mathfrak{p}) \otimes_R S$. It follows that the ring $\kappa(\mathfrak{p}) \otimes_R S$ is nonzero, so it has a prime ideal P, which necessarily satisfies $\mathfrak{p} = \varphi^{-1}(P) \in \varphi^*(\mathrm{Spec}(S))$, as desired.

(b) Assume that $\mathfrak{p} \in \mathrm{supp}_R(Y)$. It follows that we have

$$0 \not\simeq \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R Y \simeq (\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R S) \otimes^{\mathbf{L}}_S Y$$

so we conclude that $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R S \not\simeq 0$. That is, by part (a), we have $\mathfrak{p} \in \mathrm{supp}_R(S) = \varphi^*(\mathrm{Spec}(S))$, as desired.

(c) Assume that $\mathrm{supp}_R(Y) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, and let $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$. Let $\mathbf{y}$ be a finite generating sequence for $\mathfrak{m}$, and set $L := K^R(\mathbf{y})$

and $L' := S \otimes^{\mathbf{L}}_R L \cong K^S(\mathbf{y})$. By assumption, we have $\mathfrak{m} \in \operatorname{supp}_R(Y) \bigcap \operatorname{V}(\mathfrak{m}) = \operatorname{supp}_R(Y \otimes^{\mathbf{L}}_R L)$, so we conclude that $0 \not\simeq Y \otimes^{\mathbf{L}}_R L \simeq Y \otimes^{\mathbf{L}}_S L'$. Thus, there is a prime $P \in \operatorname{supp}_S(Y \otimes^{\mathbf{L}}_S L') = \operatorname{supp}_S(Y) \bigcap \operatorname{V}(\mathfrak{m}S)$, so $\varphi^{-1}(P) \supseteq \varphi^{-1}(\mathfrak{m}S) \supseteq \mathfrak{m}$. As $\mathfrak{m}$ is maximal and $\varphi^{-1}(P)$ is prime, we conclude that $\mathfrak{m} = \varphi^{-1}(P) \in \varphi^*(\operatorname{supp}_S(Y))$, as desired. □

Base Change and Co-base Change. We now switch to extension of scalars. For perspective in the results of this subsection, recall the characterization of $\operatorname{supp}_R(S)$ from Proposition 5.6(a).

Lemma 5.7. *Let $X \in \mathcal{D}(R)$ be given. If* $\operatorname{supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$, *then* $\operatorname{supp}_S(S \otimes^{\mathbf{L}}_R X) \subseteq \operatorname{V}(\mathfrak{a}S)$; *the converse holds when* $\operatorname{supp}_R(S) \supseteq \operatorname{supp}_R(X)$, *e.g. when the map φ is faithfully flat or when it is injective and integral.*

Proof. As in the second paragraph of the proof of Lemma 5.3, one has the containment $\operatorname{supp}_S(S \otimes^{\mathbf{L}}_R X) \subseteq \operatorname{V}(\mathfrak{a}S)$ if and only if each module $\operatorname{H}_i(S \otimes^{\mathbf{L}}_R X)$ is $\mathfrak{a}S$-torsion, i.e. if and only if each homology module $\operatorname{H}_i(S \otimes^{\mathbf{L}}_R X)$ is $\mathfrak{a}$-torsion, that is, if and only if $\operatorname{supp}_R(S \otimes^{\mathbf{L}}_R X) \subseteq \operatorname{V}(\mathfrak{a})$. Since we have $\operatorname{supp}_R(S \otimes^{\mathbf{L}}_R X) = \operatorname{supp}_R(S) \bigcap \operatorname{supp}_R(X)$ by Fact 2.4, the desired implications follow readily. (When φ is faithfully flat or when it is injective and integral, we have $\operatorname{supp}_R(S) = \operatorname{Spec}(R)$.) □

Lemma 5.8. *Let $X \in \mathcal{D}(R)$ be given. If* $\operatorname{V}(\mathfrak{a}) \supseteq \operatorname{co-supp}_R(X)$, *then* $\operatorname{V}(\mathfrak{a}S) \supseteq \operatorname{co-supp}_S(\mathbf{R}\operatorname{Hom}_R(S, X))$; *the converse holds when* $\operatorname{supp}_R(S) \supseteq \operatorname{co-supp}_R(X)$, e.g. *when the map φ is faithfully flat or when it is injective and integral.*

Proof. Given a prime ideal $P \in \operatorname{Spec}(S)$ with contraction $\mathfrak{p} \in \operatorname{Spec}(R)$, we have the following commutative diagram of natural/induced ring homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ \kappa(\mathfrak{p}) & \longrightarrow & \kappa(P). \end{array}$$

From this, we conclude that there are isomorphisms

$$\begin{aligned} \mathbf{R}\operatorname{Hom}_S(\kappa(P), \mathbf{R}\operatorname{Hom}_R(S, X)) &\simeq \mathbf{R}\operatorname{Hom}_R(\kappa(P), X) \\ &\simeq \mathbf{R}\operatorname{Hom}_{\kappa(\mathfrak{p})}(\kappa(P), \mathbf{R}\operatorname{Hom}_R(\kappa(\mathfrak{p}), X)). \end{aligned}$$

Since $\kappa(P)$ is a nonzero $\kappa(\mathfrak{p})$-vector space, we conclude that $\mathfrak{p} \in \operatorname{co-supp}_R(X)$ if and only if $P \in \operatorname{co-supp}_S(\mathbf{R}\operatorname{Hom}_R(S, X))$.

Now, for one implication in the result, assume that $\operatorname{co-supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$ and let $P \in \operatorname{co-supp}_S(\mathbf{R}\operatorname{Hom}_R(S, X))$. With $\mathfrak{p}$ as above, we then have $\mathfrak{p} \in \operatorname{co-supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$, so we conclude that $\mathfrak{a} \subseteq \mathfrak{p}$, which implies that $\mathfrak{a}S \subseteq \mathfrak{p}S \subseteq P$. Thus, we have $P \in \operatorname{V}(\mathfrak{a}S)$, as desired.

Next, assume that we have $\operatorname{co-supp}_S(\mathbf{R}\operatorname{Hom}_R(S, X)) \subseteq \operatorname{V}(\mathfrak{a})$ and $\operatorname{supp}_R(S) \supseteq \operatorname{co-supp}_R(X)$, and let $\mathfrak{p} \in \operatorname{co-supp}_R(X)$ be given. It follows that $\mathfrak{p} \in \operatorname{supp}_R(S) =$

$\varphi^*(\operatorname{Spec}(S))$, by Proposition 5.6(a), i.e. S has a prime ideal P lying over $\mathfrak{p}$. We conclude as above that $P \in \mathrm{V}(\mathfrak{a}S)$, so $\mathfrak{a} \subseteq \varphi^{-1}(\mathfrak{a}S) \subseteq \varphi^{-1}(P) = \mathfrak{p}$, as desired. $\square$

Remark 5.9. The proof of Lemma 5.8 can be used to give another proof of Lemma 5.7, but not vice versa. This is due in part to the differences between [20, Corollary 4.32] and [30, Theorem 3].

The next result explains base-change behavior for adic finiteness.

Theorem 5.10. *Let $X \in \mathcal{D}(R)$ be given. If X is $\mathfrak{a}$-adically finite over R and $S \otimes^{\mathbf{L}}_R X \in \mathcal{D}_b(S)$, then $S \otimes^{\mathbf{L}}_R X$ is $\mathfrak{a}S$-adically finite over S; the converse holds when the map φ is flat with* $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcup \operatorname{supp}_R(X)$, e.g. *when φ is faithfully flat.*

Proof. Note that $K' := S \otimes^{\mathbf{L}}_R K$ is the Koszul complex over S on a finite generating sequence for $\mathfrak{a}S$. Consider the following isomorphisms in $\mathcal{D}(S)$:

$$K' \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R X) \simeq K' \otimes^{\mathbf{L}}_R X \simeq (S \otimes^{\mathbf{L}}_R K) \otimes^{\mathbf{L}}_R X \simeq S \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R X).$$

For one implication, assume that X is $\mathfrak{a}$-adically finite over R and $S \otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(S)$. Lemma 5.7 implies that $\operatorname{supp}_S(S \otimes^{\mathbf{L}}_R X) \subseteq \mathrm{V}(\mathfrak{a}S)$. Our finiteness assumption implies that $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$; using a degree-wise finite semi-free resolution of $K \otimes^{\mathbf{L}}_R X$ over R, we deduce that $K' \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R X) \simeq S \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R X) \in \mathcal{D}^{\mathrm{f}}_{+}(S)$. (We actually have $K' \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R X) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S)$, but we do not need that here.) Thus, the S-complex $S \otimes^{\mathbf{L}}_R X$ is $\mathfrak{a}S$-adically finite.

Conversely, assume that φ is flat with $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a})$. Assume further that the S-complex $S \otimes^{\mathbf{L}}_R X$ is $\mathfrak{a}S$-adically finite. Lemma 5.7 implies that $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

Claim 1. The induced map $\overline{\varphi}$ is faithfully flat. Since φ is flat, so is $\overline{\varphi}$. To show that it is faithfully flat, let $\mathfrak{m}/\mathfrak{a} \in \text{m-Spec}(R/\mathfrak{a})$. We need to show that $(R/\mathfrak{a})/(\mathfrak{m}/\mathfrak{a}) \otimes_{R/\mathfrak{a}} S/\mathfrak{a} \neq 0$, i.e. that $(R/\mathfrak{m}) \otimes_R S \neq 0$. By assumption, we have $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \subseteq \operatorname{supp}_R(S)$, so flatness gives us $(R/\mathfrak{m}) \otimes_R S \simeq \kappa(\mathfrak{m}) \otimes^{\mathbf{L}}_R S \not\simeq 0$, as desired.

Claim 2. Each module $\mathrm{H}_i(K \otimes^{\mathbf{L}}_R X)$ is finitely generated over R. By assumption, we have $K' \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R X) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S)$. By flatness, we have the following

$$\mathrm{H}_i(K' \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R X)) \cong \mathrm{H}_i(S \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R X)) \cong S \otimes_R \mathrm{H}_i(K \otimes^{\mathbf{L}}_R X)$$

so $S \otimes_R \mathrm{H}_i(K \otimes^{\mathbf{L}}_R X)$ is finitely generated over S. It follows that the next module is finitely generated over $S/\mathfrak{a}S$.

$$(S/\mathfrak{a}S) \otimes_S (S \otimes_R \mathrm{H}_i(K \otimes^{\mathbf{L}}_R X)) \cong (S/\mathfrak{a}S) \otimes_{R/\mathfrak{a}} ((R/\mathfrak{a}) \otimes_R \mathrm{H}_i(K \otimes^{\mathbf{L}}_R X))$$

Faithful flatness implies that $(R/\mathfrak{a}) \otimes_R \mathrm{H}_i(K \otimes^{\mathbf{L}}_R X) \cong \mathrm{H}_i(K \otimes^{\mathbf{L}}_R X)$ is finitely generated over $R/\mathfrak{a}$, hence over R.

Now we complete the proof. By assumption, we have

$$S \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R X) \simeq K' \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R X) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S).$$

Claim 2 implies that $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}(R)$. Also, as in Claim 2, one verifies that if $\mathrm{H}_i(K' \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R X)) = 0$, then $\mathrm{H}_i(K \otimes^{\mathbf{L}}_R X) = 0$. Thus, we have $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(R)$.

From Lemma 3.1(b), we have $X \in \mathcal{D}_b(R)$, so we conclude that X is $\mathfrak{a}$-adically finite, as desired. □

As an application of the previous result, we characterize complexes with Artinian total homology. Recall that an ideal $\mathfrak{b}$ has *finite colength* if the ring $R/\mathfrak{a}$ has finite length, i.e. is Artinian.

Proposition 5.11. *Let $X \in \mathcal{D}_b(R)$ be given. Then each module $\mathrm{H}_i(X)$ is Artinian over R if and only if there is an ideal $\mathfrak{b}$ of R with finite colength such that X is $\mathfrak{b}$-adically finite.*

Proof. Assume first that each R-module $\mathrm{H}_i(X)$ is Artinian. Since we have $X \in \mathcal{D}_b(R)$, a result of Sharp [27, Proposition 1.4] implies that there is a finite list $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of maximal ideals of R such that $\mathrm{H}_i(X) \cong \oplus_{j=1}^n \Gamma_{\mathfrak{m}_j}(\mathrm{H}_i(X))$ for all i. Set $\mathfrak{b} = \bigcap_{j=1}^n \mathfrak{m}_j$, which has finite colength. The above isomorphism implies that each module $\mathrm{H}_i(X)$ is $\mathfrak{b}$-torsion, so we have $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{b})$ by Fact 2.4. To show that X is $\mathfrak{b}$-adically finite, it remains to show that $K \otimes_R^{\mathbf{L}} X \in \mathcal{D}^{\mathrm{f}}(R)$. Using a standard cone/truncation argument, it suffices to show this in the case, where $X \simeq \mathrm{H}_0(X)$ is a module. Using the above direct sum decomposition, we reduce to the case where the module X is $\mathfrak{m}_j$-torsion. In this case, the module $X_{\mathfrak{m}_j}$ has a natural $R_{\mathfrak{m}_j}$-module structure and is Artinian over both R and $R_{\mathfrak{m}_j}$. By construction, we have $\mathfrak{b}R_{\mathfrak{m}_j} = \mathfrak{m}_j R_{\mathfrak{m}_j}$ and $K_{\mathfrak{m}_j} \simeq R_{\mathfrak{m}_j} \otimes_R^{\mathbf{L}} K$ is the Koszul complex over $R_{\mathfrak{m}_j}$ on a finite generating sequence for $\mathfrak{m}_j R_{\mathfrak{m}_j}$. Thus, from the local case in Example 2.7(c), we conclude that $K \otimes_R^{\mathbf{L}} X \simeq (R_{\mathfrak{m}_j} \otimes_R^{\mathbf{L}} K) \otimes_{R_{\mathfrak{m}_j}}^{\mathbf{L}} X \in \mathcal{D}^{\mathrm{f}}(R_{\mathfrak{m}_j})$. It follows that $K \otimes_R^{\mathbf{L}} X$ has homology of finite length over $R_{\mathfrak{m}_j}$ and over R, so we have $K \otimes_R^{\mathbf{L}} X \in \mathcal{D}^{\mathrm{f}}(R)$, as desired.

Conversely, assume that R has an ideal $\mathfrak{b}$ of finite colength such that X is $\mathfrak{b}$-adically finite. Using [25, Proposition 7.1], we may replace $\mathfrak{b}$ with $\mathrm{rad}(\mathfrak{b})$ to assume that $\mathfrak{b}$ is an intersection of finitely many maximal ideals of R, say $\mathfrak{b} = \bigcap_{j=1}^n \mathfrak{m}_j$. Theorem 5.10 implies that $R_{\mathfrak{m}_j} \otimes_R^{\mathbf{L}} X$ is $\mathfrak{b}R_{\mathfrak{m}_j}$-adically finite (i.e. $\mathfrak{m}_j R_{\mathfrak{m}_j}$-adically finite) over $R_{\mathfrak{m}_j}$ for each j. Example 2.7(c) implies that each of the modules $\mathrm{H}_i(R_{\mathfrak{m}_j} \otimes_R^{\mathbf{L}} X) \cong \mathrm{H}_i(X)_{\mathfrak{m}_j}$ is Artinian over $R_{\mathfrak{m}_j}$. Since each module $\mathrm{H}_i(X)$ is $\mathfrak{b}$-torsion, we have $\mathrm{Supp}_R(\mathrm{H}_i(X)) \subseteq \mathrm{V}(\mathfrak{b}) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. So, we conclude that each $\mathrm{H}_i(X)$ is Artinian over R by [15, Lemma 3.2]. □

6. Adic Finiteness and Homological Dimensions

The point of this section is to show that certain homological dimension computations extend from the setting of homologically finite complexes to the realm of $\mathfrak{a}$-adically finite complexes. To begin, we prove Theorem 1.2 from the introduction. Note that the special case $Z = \mathbf{R}\Gamma_{\mathfrak{a}}(R)$ of this result is also well-known, using the "telescope complex". However, the general result showcases the power (or, if one prefers, the restrictiveness) of adic finiteness in general.

Theorem 6.1. *Let $X \in \mathcal{D}_b(R)$ be $\mathfrak{a}$-adically finite. The following are equivalent.*

(i) *For all* $\mathfrak{m} \in \text{m-Spec}(R) \bigcap \mathrm{V}(\mathfrak{a})$, *one has* $\mathrm{fd}_{R_\mathfrak{m}}(X_\mathfrak{m}) < \infty$.
(ii) *For all* $\mathfrak{m} \in \text{m-Spec}(R) \bigcap \mathrm{V}(\mathfrak{a})$, *one has* $\mathrm{pd}_{R_\mathfrak{m}}(X_\mathfrak{m}) < \infty$.
(iii) *One has* $\mathrm{fd}_R(X) < \infty$.
(iv) *One has* $\mathrm{pd}_R(X) < \infty$.

Moreover, one has $\mathrm{pd}_R(X) = \mathrm{fd}_R(X)$.

Proof. (iii) $\Leftrightarrow$ (iv) By Lemma 3.4, we have the first and last equalities next.

$$n + \mathrm{fd}_R(X) = \mathrm{fd}_R(K \otimes^{\mathbf{L}}_R X) = \mathrm{pd}_R(K \otimes^{\mathbf{L}}_R X) = n + \mathrm{pd}_R(X).$$

The second equality is from the assumption $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

From Theorem 5.10, we know that for all $\mathfrak{m} \in \text{m-Spec}(R) \bigcap \mathrm{V}(\mathfrak{a})$, the $R_\mathfrak{m}$-complex $X_\mathfrak{m} \simeq R_\mathfrak{m} \otimes^{\mathbf{L}}_R X$ is $\mathfrak{a}R_\mathfrak{m}$-adically finite. Thus, the equivalence of conditions (i) and (ii) follows from what we have just shown. The implication (iv) $\Longrightarrow$ (ii) is from [2, Proposition 5.1(P)], so it remains to prove the implication (ii) $\Longrightarrow$ (iv).

Assume that for all $\mathfrak{m} \in \text{m-Spec}(R) \bigcap \mathrm{V}(\mathfrak{a})$, one has $\mathrm{pd}_{R_\mathfrak{m}}(X_\mathfrak{m}) < \infty$. It follows that for all $\mathfrak{m} \in \text{m-Spec}(R) \bigcap \mathrm{V}(\mathfrak{a})$, we have

$$\mathrm{pd}_{R_\mathfrak{m}}((K \otimes^{\mathbf{L}}_R X)_\mathfrak{m}) = \mathrm{pd}_{R_\mathfrak{m}}(K_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} X_\mathfrak{m}) < \infty.$$

From the condition $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, we have the finiteness in the next display

$$\mathrm{pd}_R(X) = \mathrm{pd}_R(K \otimes^{\mathbf{L}}_R X) - n < \infty$$

and the equality is from Lemma 3.4. □

Remark 6.2. It is worth noting that in conditions (i) and (ii) of Proposition 6.3, one can replace $\text{m-Spec}(R) \bigcap \mathrm{V}(\mathfrak{a})$ with other sets, e.g. $\text{m-Spec}(R) \bigcap \mathrm{supp}_R(X)$ or $\mathrm{supp}_R(X)$ or $\mathrm{V}(\mathfrak{a})$ or $\text{m-Spec}(R)$. Indeed, the fact that X is $\mathfrak{a}$-adically finite implies that $\mathrm{supp}_R(X) = \mathrm{Supp}_R(X)$, by [25, Theorem 7.11]. Since $X_\mathfrak{p} \simeq 0$ if and only if $\mathfrak{p} \notin \mathrm{Supp}_R(X) = \mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, it is straightforward to show that $\text{m-Spec}(R) \bigcap \mathrm{V}(\mathfrak{a})$ can be replaced by any of the sets listed.

Similarly, in the next result, one can replace $\mathrm{Spec}(R)$ with $\mathrm{supp}_R(X)$ or $\mathrm{V}(\mathfrak{a})$; in the expression $\sup\{\mathrm{pd}_{R_\mathfrak{p}}(X_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}$, one can even replace $\mathrm{Spec}(R)$ with $\text{m-Spec}(R)$ or $\text{m-Spec}(R) \bigcap \mathrm{V}(\mathfrak{a})$ or $\text{m-Spec}(R) \bigcap \mathrm{supp}_R(X)$. This result extends [2, Proposition 5.3.P] to our setting.

Proposition 6.3. *Let $X \in \mathcal{D}_b(R)$ be $\mathfrak{a}$-adically finite. Then we have*

$$\begin{aligned}\mathrm{pd}_R(X) &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X, R/\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}\\ &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, \kappa(\mathfrak{p}))) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}\\ &= \sup\{\mathrm{pd}_{R_\mathfrak{p}}(X_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}.\end{aligned}$$

Proof. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be given. As in the proof of Lemma 3.4, we have

$$\text{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(X, R/\mathfrak{p})) \subseteq \mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$$

so we conclude that

$$\begin{aligned}\inf(\mathbf{R}\mathrm{Hom}_R(K \otimes^{\mathbf{L}}_R X, R/\mathfrak{p})) &= \inf(\mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\mathrm{Hom}_R(X, R/\mathfrak{p}))) \\ &= \inf(\Sigma^{-n} K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, R/\mathfrak{p})) \\ &= -n + \inf(K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, R/\mathfrak{p})) \\ &= -n + \inf(\mathbf{R}\mathrm{Hom}_R(X, R/\mathfrak{p})).\end{aligned}$$

This explains the third step in the next sequence

$$\begin{aligned}n + \mathrm{pd}_R(X) &= \mathrm{pd}_R(K \otimes^{\mathbf{L}}_R X) \\ &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(K \otimes^{\mathbf{L}}_R X, R/\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} \\ &= n + \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X, R/\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} \\ &\leqslant n + \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X, N)) \mid N \text{ is an } R\text{-module}\} \\ &= n + \mathrm{pd}_R(X).\end{aligned}$$

The first step is from Lemma 3.4, and the second step is from [2, Proposition 5.3.P], using the assumption $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. The fourth step is routine, and the last one is from [2, Corollary 2.5.P]. This explains the first equality in our result.

For the other equalities in the statement of our result, we use the condition $\mathrm{supp}_{R_\mathfrak{p}}(X_\mathfrak{p}) \subseteq \mathrm{V}(\mathfrak{a}R_\mathfrak{p})$ from Lemma 5.7 to compute similarly

$$\begin{aligned}-\inf(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}((K \otimes^{\mathbf{L}}_R X)_\mathfrak{p}, \kappa(\mathfrak{p}))) &= -\inf(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(K_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} X_\mathfrak{p}, \kappa(\mathfrak{p}))) \\ &= -\inf(\Sigma^{-n} K_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, \kappa(\mathfrak{p}))) \\ &= n - \inf(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, \kappa(\mathfrak{p})))\end{aligned}$$

and from this, we have

$$\begin{aligned}n + \mathrm{pd}_R(X) &= \mathrm{pd}_R(K \otimes^{\mathbf{L}}_R X) \\ &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}((K \otimes^{\mathbf{L}}_R X)_\mathfrak{p}, \kappa(\mathfrak{p}))) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} \\ &= n + \sup\{-\inf(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, \kappa(\mathfrak{p}))) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} \\ &\leqslant n + \sup\{-\inf(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, N)) \mid \\ &\qquad \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } N \text{ is an } R_\mathfrak{p}\text{-module}\} \\ &= n + \sup\{\mathrm{pd}_{R_\mathfrak{p}}(X_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} \\ &\leqslant n + \mathrm{pd}_R(X)\end{aligned}$$

and hence the desired result. □

In the next result, we tackle [2, Proposition 5.5]. Note that the local assumption on φ implies that $\mathfrak{a}S \neq S$.

Proposition 6.4. *Let* $\varphi\colon (R,\mathfrak{m},k) \to S$ *be a local homomorphism, and let* $Y \in \mathcal{D}_b(S)$ *be* $\mathfrak{a}S$*-adically finite.*

(a) *There are equalities*

$$\mathrm{fd}_R(Y) = \sup(k \otimes^{\mathbf{L}}_R Y) \qquad\qquad \mathrm{id}_R(Y) = -\inf(\mathbf{R}\mathrm{Hom}_R(k,Y)).$$

(b) *If* φ *is the identity map, then*

$$\mathrm{pd}_R(Y) = -\inf(\mathbf{R}\mathrm{Hom}_R(Y,k)).$$

Proof. Argue as in the first paragraph of the proof of Theorem 6.1, using Lemmas 3.4–3.5 to reduce to the homologically finite case of [2, Proposition 5.5]. □

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