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Abstract

We continue our work on adic semidualizing complexes over a commutative noetherian ring R by investigating the associated Auslander and Bass classes (collectively known as Foxby classes), following Foxby and Christensen. Fundamental properties of these classes include Foxby Equivalence, which provides an equivalence between the Auslander and Bass classes associated to a given adic semidualizing complex. We prove a variety of stability results for these classes, for instance, with respect to $F\otimes^{\mathbf{L}}_R-$ where F is an R-complex finite flat dimension, including special converses of these results. We also investigate change of rings and local-global properties of these classes.

Keywords Adic finiteness · Adic semidualizing complexes · Auslander classes · Bass classes · Quasi-dualizing modules · Support

Mathematics Subject Classification (2010) 13B35 · 13C12 · 13D09 · 13D45

1 Introduction

Throughout this paper let R be a commutative noetherian ring, let $\mathfrak{a} \subsetneq R$ be a proper ideal of R, and let $\widehat{R}^{\mathfrak{a}}$ be the $\mathfrak{a}$-adic completion of R. Let $K = K^R(\underline{x})$ denote the Koszul complex over R on a generating sequence $\underline{x} = x_1, \ldots, x_n$ for $\mathfrak{a}$. We work in the derived category $\mathcal{D}(R)$ with objects the R-complexes indexed homologically $X = \cdots \to X_i \to X_{i-1} \to \cdots$; see, e.g., [20, 41, 42] for foundations of this construction and Section 2 for

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background and notation. Isomorphisms in $\mathcal{D}(R)$ are identified by the symbol $\simeq$, e.g., by writing $X \xrightarrow{\simeq} Y$ if there is a chain map $X \to Y$ that represents an isomorphism in $\mathcal{D}(R)$. We also consider the full triangulated subcategory $\mathcal{D}_b(R)$, with objects the R-complexes X such that $H_i(X) = 0$ for $|i| \gg 0$. The appropriate derived functors of Hom and $\otimes$ are denoted $\mathbf{R}$Hom and $\otimes^{\mathbf{L}}$.

This paper is part 6 of a series of papers on homological constructions over R; see also [34–38]. The genesis of the current paper goes back at least to Auslander and Bridger's monograph [2] on G-dimension of finitely generated modules, defined in terms of resolutions by modules of G-dimension 0, i.e., totally reflexive modules. This was extended to the non-finite arena by Enochs, Jenda, and Torrecillas [12, 13] yielding the G-projective, G-flat, and G-injective dimensions.

One difficulty with these dimensions is found in their definitions in terms of resolutions. As opposed to the standard characterization of projective dimension in terms of Ext-vanishing, for instance, a functorial characterization of the modules of finite G-projective dimension took significantly more work. This goal was achieved, first for Cohen-Macaulay rings admitting a dualizing (i.e., canonical) module, by Enochs, Jenda, and Xu [14], then for rings admitting a dualizing complex, by Christensen, Frankild, and Holm [9], then in general by Esmkhani and Tousi [15] and Christensen and Sather-Wagstaff [11]. The first of these uses Foxby's Auslander and Bass classes [16] with respect to a dualizing module, and the later ones use Avramov and Foxby's [4] more general Auslander and Bass classes with respect to a dualizing complex.

Motivated in part by Avramov and Foxby's [4] use of *relative* dualizing complexes to study ring homomorphisms of finite G-dimension, Christensen [8] introduced and studied Auslander and Bass classes with respect to semidualizing complexes. A complex $C \in \mathcal{D}_b(R)$ with finitely generated homology is *semidualizing* if the natural homothety morphism $R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathcal{D}(R)$. Examples of these include dualizing complexes (in particular, dualizing modules), relative dualizing complexes, and the free R-module of rank 1, each of which has important duality properties. When C is an *R-module*, it is semidualizing over R if it is finitely generated with $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$ such that the natural homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism

The *Bass class* $\mathcal{B}_C(R)$ with respect to C is the class of R-complexes $X \in \mathcal{D}_b(R)$ with $\mathbf{R}\mathrm{Hom}_R(C, X) \in \mathcal{D}_b(R)$ where the natural morphism $C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, X) \to X$ is an isomorphism in $\mathcal{D}(R)$. The Auslander class $\mathcal{A}_C(R)$ is defined similarly; see 3.1. (Fact 3.3 describes these classes in the case of modules.) By work of Holm and Jørgensen [23], these classes give functorial characterizations of certain generalized Gorenstein homological dimensions. In summary, these classes are powerful tools for studying homological dimensions of modules and complexes.

Note that the definitions of $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ do not a priori require C to be semidualizing. However, when C is not semidualizing, these classes tend to lose their nice properties, especially when C has non-finitely generated homology. On the other hand, the class of semidualizing R-modules misses some modules and complexes with important duality properties, e.g., the injective hull $E_R(R/\mathfrak{m})$ over a local ring $(R, \mathfrak{m})$, used for Matlis duality and Grothendieck's local duality. To fill this gap, in [34] we introduce and study the $\mathfrak{a}$-adic semidualizing complexes; see Definition 2.16 below. In the case $\mathfrak{a} = 0$, these are exactly Christensen's semidualizing complexes. When $(R, \mathfrak{m})$ is local and $\mathfrak{a} = \mathfrak{m}$, these recover Kubik's [26] quasi-dualizing modules (e.g., $E_R(R/\mathfrak{m})$) as a special case.

The point of the current paper is to investigate the Auslander and Bass classes with respect to an $\mathfrak{a}$-adic semidualizing complex M. Even though M does not generally have finitely generated homology, the definition allows us to retain many of the nice properties from Christensen's setting. For instance, we have the following version of Foxby Equivalence (originally from [4, 8, 16]); it is Theorem 3.6 below, one of several foundational properties documented in Section 3.

Theorem 1.1 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex.*

(a) *The functors* $\mathbf{R}\mathrm{Hom}_R(M,-)\colon \mathcal{B}_M(R) \to \mathcal{A}_M(R)$ *and* $M \otimes^{\mathbf{L}}_R -\colon \mathcal{A}_M(R) \to \mathcal{B}_M(R)$ *are quasi-inverse equivalences.*
(b) *An R-complex $Y \in \mathcal{D}(R)$ is in $\mathcal{B}_M(R)$ if and only if* $\mathbf{R}\mathrm{Hom}_R(M,Y) \in \mathcal{A}_M(R)$ *and* $\mathrm{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$.
(c) *An R-complex $X \in \mathcal{D}(R)$ is in $\mathcal{A}_M(R)$ if and only if one has* $M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ *and* $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

A difference between this result and its predecessors is the presence of support/co-support conditions; see Definition 2.4. On the other hand, these conditions are present in these earlier results, but they are invisible. Indeed, we have $\mathfrak{a} = 0$ in those cases, so the condition $\mathrm{supp}_R(Y) \subseteq \mathrm{V}(0) = \mathrm{Spec}(R)$ is satisfied trivially, and similarly for $\text{co-supp}_R(X)$. This is a necessary feature of our constructions. Another difference worth noticing is the lack of boundedness assumptions in parts (b) – (c).

Section 4 is devoted to stability properties of these classes, i.e., their behavior with respect to direct sum and product, in addition to well-behaved derived functors. For example, the next result is Theorem 4.3 from the body of the paper.

Theorem 1.2 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex. Let $F \in \mathcal{D}_b(R)$ be such that* $\mathrm{fd}_R(F) < \infty$*, and let $X \in \mathcal{D}(R)$. If $X \in \mathcal{B}_M(R)$, then $X \otimes^{\mathbf{L}}_R F \in \mathcal{B}_M(R)$ and* $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. *The converse of this statement holds when at least one of the following conditions is satisfied.*

(1) *F is $\mathfrak{a}$-adically finite such that* $\mathrm{supp}_R(F) = \mathrm{V}(\mathfrak{a})$.
(2) *There is a homomorphism $\varphi\colon R \to S$ of commutative noetherian rings with $\mathfrak{a}S \neq S$ such that $F \in \mathcal{D}_b(S)$ is $\mathfrak{a}S$-adically finite over S with* $\varphi^*(\mathrm{supp}_S(F)) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$*, and we have $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$. Here* $\varphi^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is the induced map and K is the Koszul complex over R on a generating sequence for $\mathfrak{a}$.*
(3) *F is a flat R-module with* $\mathrm{supp}_R(F) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$*, e.g., F is faithfully flat, e.g., free.*

In Section 5 we focus on transfer properties for these classes with respect to a ring homomorphism $\varphi\colon R \to S$. As a sample, here is Theorem 5.5 from this section.

Theorem 1.3 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex. Let $\varphi\colon R \to S$ be a homomorphism of commutative noetherian rings, and assume that* $\mathrm{fd}_R(S) < \infty$. *Let $X \in \mathcal{D}(R)$ be given, and consider the following conditions.*

(i) *$X \in \mathcal{B}_M(R)$.*
(ii) *$S \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ and* $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.
(iii) *$S \otimes^{\mathbf{L}}_R X \in \mathcal{B}_{S \otimes^{\mathbf{L}}_R M}(S)$ and* $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

Then we have (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii). *The conditions* (i) – (iii) *are equivalent when at least one of the following conditions is satisfied.*

(1) *S is $\mathfrak{a}$-adically finite over R such that* $\operatorname{supp}_R(S) = \mathrm{V}(\mathfrak{a})$.
(2) $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \operatorname{m-Spec}(R)$, *and* $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$, *where K is the Koszul complex over R on a generating sequence for* $\mathfrak{a}$.
(3) *S is flat over R with* $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \operatorname{m-Spec}(R)$, *e.g., S is faithfully flat.*

This result yields local-global behavior for Bass classes; see Theorem 5.13. This section also contains versions of these results for Auslander classes.

A reader familiar with the paper of Christensen [8] will undoubtedly see numerous similarities between that paper and this one. However, the fact that $\mathfrak{a}$-adic semidualizing complexes do not usually have finitely generated homology makes for some technical and subtle differences. On the other hand, some of our results, including parts of Theorem 1.3, are new even for Christensen's setting.

2 Background

2.1 Derived Categories

We consider the following full subcategories of $\mathcal{D}(R)$.

$\mathcal{D}_+(R)$: objects are the complexes X with $\mathrm{H}_i(X) = 0$ for $i \ll 0$.
$\mathcal{D}_-(R)$: objects are the complexes X with $\mathrm{H}_i(X) = 0$ for $i \gg 0$.
$\mathcal{D}_\mathrm{b}(R) = \mathcal{D}_+(R) \cap \mathcal{D}_-(R)$
$\mathcal{D}^\mathrm{f}(R)$: objects are the complexes X with $\mathrm{H}_i(X)$ finitely generated for all i.

Intersections of these categories are designated with two ornaments, e.g., $\mathcal{D}^\mathrm{f}_\mathrm{b}(R) = \mathcal{D}_\mathrm{b}(R) \bigcap \mathcal{D}^\mathrm{f}(R)$. The ith shift (or suspension) of an R-complex X is denoted $\Sigma^i X$, and the *supremum* and *infimum* of X are

$$\begin{aligned} \sup(X) &= \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\} \\ \inf(X) &= \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\} \end{aligned}$$

with the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

Fact 2.1 Let $Y, Z \in \mathcal{D}(R)$.

(a) By definition, one has $\sup(Z) < \infty$ if and only if $Z \in \mathcal{D}_-(R)$, and one has $\inf(Z) > -\infty$ if and only if $Z \in \mathcal{D}_+(R)$.
(b) By [17, Lemma 2.1(1)], there is an inequality

$$\sup(\mathbf{R}\mathrm{Hom}_R(Y, Z)) \leqslant \sup(Z) - \inf(Y).$$

The next lemma is routine, but we include a proof for the sake of completeness.

Lemma 2.2 *Let $\mathfrak{a}$ be a proper ideal of R and K the Koszul complex over R on a generating sequence for $\mathfrak{a}$. If $X, Y \in \mathcal{D}_+(R)$ and $Z \in \mathcal{D}_-(R)$ are given such that the complexes $K \otimes^{\mathbf{L}}_R X$, $K \otimes^{\mathbf{L}}_R Y$, and $K \otimes^{\mathbf{L}}_R Z$ are in $\mathcal{D}^f(R)$, then one has $K \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R Y) \in \mathcal{D}^f_+(R)$ and $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, Z) \in \mathcal{D}^f_-(R)$.*

Proof First, we observe that our assumptions on X and Y yield the following.

$$K \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R Y)) \simeq (K \otimes^{\mathbf{L}}_R X) \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R Y) \in \mathcal{D}^{\mathrm{f}}_{+}(R)$$

Let n be the length of our given generating sequence for $\mathfrak{a}$. Since each module $\mathrm{H}_i(K \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R Y))$ is annihilated by $\mathfrak{a}$, we have

$$\mathrm{H}_i(K \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R Y))) \cong \bigoplus_{j=0}^{n} \mathrm{H}_{i+j}(K \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R Y))^{\binom{n}{j}}$$

In particular, $\mathrm{H}_i(K \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R Y))$ is a summand of the finitely generated module $\mathrm{H}_i(K \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R Y)))$, so it is finitely generated as well. This shows that we have $K \otimes^{\mathbf{L}}_R (X \otimes^{\mathbf{L}}_R Y) \in \mathcal{D}^{\mathrm{f}}(R)$. The conclusion $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X, Z) \in \mathcal{D}^{\mathrm{f}}(R)$ is obtained similarly, using the isomorphisms

$$K \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R (\mathbf{R}\mathrm{Hom}_R(X, Z))) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K, X), K \otimes^{\mathbf{L}}_R Z)$$

and $K \otimes^{\mathbf{L}}_R X \simeq \Sigma^n \mathbf{R}\mathrm{Hom}_R(K, X)$. □

2.2 Homological Dimensions

An R-complex F is *semi-flat*[1] if it consists of flat R-modules and the functor $F \otimes_R -$ respects quasiisomorphisms, that is, if it respects injective quasiisomorphisms (see [3, 1.2.F]). A *semi-flat resolution* of an R-complex X is a quasiisomorphism $F \xrightarrow{\simeq} X$ such that F is semi-flat. An R-complex X has *finite flat dimension* if it has a bounded semi-flat resolution; specifically, we have

$$\mathrm{fd}_R(X) = \inf\{\sup\{i \mid F_i \neq 0\} \mid F \xrightarrow{\simeq} X \text{ is a semi-flat resolution}\}.$$

The projective and injective versions of these notions are defined similarly.

For the following items, consult [3, Section 1] or [5, Chapters 3 and 5]. Bounded below complexes of flat modules are semi-flat, bounded below complexes of projective modules are semi-projective, and bounded above complexes of injective modules are semi-injective. Semi-projective R-complexes are semi-flat. Every R-complex admits a semi-projective (hence, semi-flat) resolution and a semi-injective resolution.

2.3 Derived Local (Co)homology

The next notions go back to Grothendieck [21] and Matlis [29, 30]; see also [1, 19, 28, 32]. Let $\Lambda^{\mathfrak{a}}$ denote the $\mathfrak{a}$-adic completion functor with respect to the proper ideal $\mathfrak{a}$, and let $\Gamma_{\mathfrak{a}}$ be the $\mathfrak{a}$-torsion functor, i.e., for an R-module M we have

$$\Lambda^{\mathfrak{a}}(M) = \widehat{M}^{\mathfrak{a}} \qquad\qquad \Gamma_{\mathfrak{a}}(M) = \{x \in M \mid \mathfrak{a}^n x = 0 \text{ for } n \gg 0\}.$$

A module M is $\mathfrak{a}$-*torsion* if $\Gamma_{\mathfrak{a}}(M) = M$.

The associated left and right derived functors (i.e., *derived local homology and cohomology* functors) are $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. Specifically, given an R-complex $X \in \mathcal{D}(R)$ and a semi-flat resolution $F \xrightarrow{\simeq} X$ and a semi-injective resolution $X \xrightarrow{\simeq} I$, then we have $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(F)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(I)$.

[1] In the literature, semi-flat complexes are sometimes called "DG-flat".

Fact 2.3 By [1, Theorem (0.3) and Corollary (3.2.5.i)], with $\mathfrak{a}$ and other notation as above, there are natural isomorphisms of functors

$$\mathbf{R}\Gamma_{\mathfrak{a}}(-) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R - \quad \mathbf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), -).$$

Note that we have $\mathrm{pd}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R)) < \infty$, via the telescope complex of [19]. Thus, if $F \in \mathcal{D}_{\mathrm{b}}(R)$ has finite flat dimension, then so has $\mathbf{L}\Lambda^{\mathfrak{a}}(F) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), F)$.

2.4 Support and Co-support

The following notions of support and co-support, also crucial for our work, are due to Foxby [18] and Benson, Iyengar, and Krause [7].

Definition 2.4 Let $X \in \mathcal{D}(R)$. The *small support* and *small co-support* of X are

$$\begin{aligned} \mathrm{supp}_R(X) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \not\simeq 0\} \\ \text{co-supp}_R(X) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), X) \not\simeq 0\} \end{aligned}$$

where $\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

Much of the following is from [18] when X and Y are appropriately bounded and from [6, 7] in general. We refer to [38] as a matter of convenience.

Fact 2.5 Let $X, Y \in \mathcal{D}(R)$, and continue with the proper ideal $\mathfrak{a}$.

2.5.1 We have $\mathrm{supp}_R(X) = \emptyset$ if and only if $X \simeq 0$ if and only if $\text{co-supp}_R(X) = \emptyset$, because of [38, Fact 3.4 and Proposition 4.7(a)].

2.5.2 By [38, Propositions 3.12, 3.13, 4.10, and 4.11] we have

$$\begin{aligned} \mathrm{supp}_R(X \otimes^{\mathbf{L}}_R Y) &= \mathrm{supp}_R(X) \bigcap \mathrm{supp}_R(Y) \\ \text{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(X, Y)) &= \mathrm{supp}_R(X) \bigcap \text{co-supp}_R(Y) \\ \mathrm{supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(Y)) &= \mathrm{V}(\mathfrak{a}) \bigcap \mathrm{supp}_R(Y) \\ \text{co-supp}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(Y)) &= \mathrm{V}(\mathfrak{a}) \bigcap \text{co-supp}_R(Y). \end{aligned}$$

2.5.3 We know that $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ if and only if the natural morphism $\varepsilon^X_{\mathfrak{a}} \colon \mathbf{R}\Gamma_{\mathfrak{a}}(X) \to X$ is an isomorphism, that is, if and only if each homology module $\mathrm{H}_i(X)$ is $\mathfrak{a}$-torsion, by [38, Proposition 5.4] and [32, Corollary 4.32]. Dually, we have $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ if and only if the natural morphism $X \to \mathbf{L}\Lambda^{\mathfrak{a}}(X)$ in $\mathcal{D}(R)$ is an isomorphism, by [38, Proposition 5.9]. Also, each homology module $\mathrm{H}_i(X)$ is $\mathfrak{a}$-adically complete if and only if each $\mathrm{H}_i(X)$ is $\mathfrak{a}$-adically separated and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, by [43, Theorem 3]. Since $\mathfrak{a}$ annihilates the homology of $K \otimes^{\mathbf{L}}_R X$, it follows from this that $\mathrm{supp}(K \otimes^{\mathbf{L}}_R X), \text{co-supp}(K \otimes^{\mathbf{L}}_R X) \subseteq \mathrm{V}(\mathfrak{a})$.

The next three facts demonstrate some of the flexibility afforded by support and co-support conditions.

Fact 2.6 ([37, Lemmas 3.1(b) and 3.2(b)]) Let $Z \in \mathcal{D}(R)$. If $\mathrm{supp}_R(Z) \subseteq \mathrm{V}(\mathfrak{a})$ or $\text{co-supp}_R(Z) \subseteq \mathrm{V}(\mathfrak{a})$, then one has $K \otimes^{\mathbf{L}}_R Z \in \mathcal{D}_{\mathrm{b}}(R)$ if and only if $Z \in \mathcal{D}_{\mathrm{b}}(R)$.

Fact 2.7 ([35, Lemma 2.8]) Let $X \in \mathcal{D}(R)$ be such that $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Then the following natural transformations from Fact 2.5.3 are isomorphisms.

$$X \otimes^{\mathbf{L}}_R - \xrightarrow[\simeq]{X \otimes^{\mathbf{L}}_R \vartheta^{\mathfrak{a}}} X \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(-)$$

$$\mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\Gamma_{\mathfrak{a}}(-)) \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}_R(X, \varepsilon_{\mathfrak{a}})} \mathbf{R}\mathrm{Hom}_R(X, -)$$

Fact 2.8 ([38, Corollaries 5.3 and 5.8]) Let $f\colon Y \to Z$ be morphism in $\mathcal{D}(R)$, and continue with the proper ideal $\mathfrak{a}$. Assume that either $\operatorname{supp}_R(Y), \operatorname{supp}_R(Z) \subseteq \mathrm{V}(\mathfrak{a})$ or $\text{co-supp}_R(Y), \text{co-supp}_R(Z) \subseteq \mathrm{V}(\mathfrak{a})$. Then f is an isomorphism in $\mathcal{D}(R)$ if and only if $K \otimes^{\mathbf{L}}_R f$ is an isomorphism in $\mathcal{D}(R)$. Here is the main idea of the proof, for instance, in the case $\operatorname{supp}_R(Y), \operatorname{supp}_R(Z) \subseteq \mathrm{V}(\mathfrak{a})$. Consider an exact triangle $Y \xrightarrow{f} Z \to A \to$ in $\mathcal{D}(R)$. The support assumptions on Y and Z imply that we have $\operatorname{supp}_R(A) \subseteq \mathrm{V}(\mathfrak{a})$. Since $K \otimes^{\mathbf{L}}_R f$ is an isomorphism, when we tensor this triangle with K, we find that $K \otimes^{\mathbf{L}}_R A \simeq 0$. On the other hand, we have $\operatorname{supp}_R(K) = \mathrm{V}(\mathfrak{a}) \supseteq \operatorname{supp}_R(A)$, so Fact 2.5.2 implies that $\operatorname{supp}_R(A) = \emptyset$ and so $A \simeq 0$ by Fact 2.5.1. Thus, our triangle shows that f is an isomorphism.

The point of discussing this proof explicitly is as follows. We have many results in [36, Sections 3–5] of the following form: given a functor F and an R-complex A such that $F(A) \simeq 0$, nice assumptions on F and A imply that we have $A \simeq 0$. Using the logic of the previous paragraph, we conclude that if f is a morphism between nice complexes such that $F(f)$ is an isomorphism, then f is an isomorphism. To keep things reasonable, we do not state every possible variation on this theme, though we use this idea several times below.

2.5 Adic Finiteness

The following fact and definition take their cues from work of Hartshorne [22], Kawasaki [24, 25], and Melkersson [31].

Fact 2.9 ([38, Theorem 1.3]) Continue with the proper ideal $\mathfrak{a}$. For $X \in \mathcal{D}_{\mathrm{b}}(R)$, the following conditions are equivalent.

(i) One has $K(\underline{y}) \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ for some (equivalently for every) generating sequence $\underline{y}$ of $\mathfrak{a}$.
(ii) One has $X \otimes^{\mathbf{L}}_R R/\mathfrak{a} \in \mathcal{D}^{\mathrm{f}}(R)$.
(iii) One has $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X) \in \mathcal{D}^{\mathrm{f}}(R)$.

Definition 2.10 For a proper ideal $\mathfrak{a}$, an R-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ is $\mathfrak{a}$-*adically finite* if it satisfies the equivalent conditions of Fact 2.9 and $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$; see Fact 2.5.3.

Example 2.11 Let $X \in \mathcal{D}_{\mathrm{b}}(R)$ be given, and continue with the proper ideal $\mathfrak{a}$.

(a) If $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then we have $\operatorname{supp}_R(X) = \mathrm{V}(\mathfrak{b})$ for some ideal $\mathfrak{b}$, and it follows that X is $\mathfrak{a}$-adically finite whenever $\mathfrak{a} \subseteq \mathfrak{b}$. (The case $\mathfrak{a} = 0$ is from [38, Proposition 7.8(a)], and the general case follows readily.)
(b) K and $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ are $\mathfrak{a}$-adically finite, by [38, Fact 3.4 and Theorem 7.10].
(c) The homology modules of X are artinian if and only if there is an ideal $\mathfrak{a}$ of finite colength (i.e., such that $R/\mathfrak{a}$ is artinian) such that X is $\mathfrak{a}$-adically finite, by [37, Proposition 5.11].

As the name suggests, adically finite complexes can behave like the complexes in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. The next fact gives a taste of this. See [36, 37] for other such results.

Fact 2.12 ([37, Theorems 4.1(b) and 4.2(b)]) Continue with the proper ideal $\mathfrak{a}$. Let M be an $\mathfrak{a}$-adically finite R-complex, and let $Y, V, Z \in \mathcal{D}(R)$ be such that $\operatorname{supp}_R(V) \subseteq \mathrm{V}(\mathfrak{a})$, e.g., $V = M$ or $V = K$. Consider the natural tensor evaluation and Hom-evaluation morphisms

$$\mathbf{R}\mathrm{Hom}_R(M, Y) \otimes^{\mathbf{L}}_R Z \xrightarrow{\omega_{MYZ}} \mathbf{R}\mathrm{Hom}_R(M, Y \otimes^{\mathbf{L}}_R Z)$$

$$M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(Y, Z) \xrightarrow{\theta_{MYZ}} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, Y), Z).$$

(a) If $Y \in \mathcal{D}_-(R)$ and either $\operatorname{fd}_R(M)$ or $\operatorname{fd}_R(Z)$ is finite, then the induced morphisms $V \otimes^{\mathbf{L}}_R \omega_{MYZ}$ and $\mathbf{R}\mathrm{Hom}_R(V, \omega_{MYZ})$ are isomorphisms.
(b) If $Y \in \mathcal{D}_{\mathrm{b}}(R)$ and either $\operatorname{fd}_R(M)$ or $\operatorname{id}_R(Z)$ is finite, then the induced morphisms $V \otimes^{\mathbf{L}}_R \theta_{MYZ}$ and $\mathbf{R}\mathrm{Hom}_R(V, \theta_{MYZ})$ are isomorphisms.

The next results augment some computations from [38]. Note the somewhat strange switching of supp and co-supp, as compared to Fact 2.5.2.

Lemma 2.13 *Let $X \in \mathcal{D}(R)$ and $P \in \mathcal{D}^f_b(R)$ be such that $\operatorname{pd}_R(P) < \infty$. Then there are equalities*

$$\operatorname{supp}_R(\mathbf{R}\mathrm{Hom}_R(P, X)) = \operatorname{supp}_R(P) \bigcap \operatorname{supp}_R(X)$$

$$\operatorname{co-supp}_R(P \otimes^{\mathbf{L}}_R X) = \operatorname{supp}_R(P) \bigcap \operatorname{co-supp}_R(X).$$

Proof Set $P^* := \mathbf{R}\mathrm{Hom}_R(P, R)$. The assumptions $P \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $\operatorname{pd}_R(P) < \infty$ imply that $P \simeq \mathbf{R}\mathrm{Hom}_R(P^*, R)$. In particular, we have $P \simeq 0$ if and only if $P^* \simeq 0$. For any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, it follows that $P_{\mathfrak{p}} \simeq 0$ if and only if $(P^*)_{\mathfrak{p}} \simeq 0$, so we have $\operatorname{Supp}_R(P) = \operatorname{Supp}_R(P^*)$, that is, $\operatorname{supp}_R(P) = \operatorname{supp}_R(P^*)$ since $P, P^* \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

In the next sequence of isomorphisms, the second step is Hom-evaluation [3, Lemma 4.4(I)]

$$\mathbf{R}\mathrm{Hom}_R(P, X) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(P^*, R), X) \simeq P^* \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R, X) \simeq P^* \otimes^{\mathbf{L}}_R X$$

and the other steps are routine. From this, we have

$$\begin{aligned}\operatorname{supp}_R(\mathbf{R}\mathrm{Hom}_R(P, X)) &= \operatorname{supp}_R(P^* \otimes^{\mathbf{L}}_R X)\\ &= \operatorname{supp}_R(P^*) \bigcap \operatorname{supp}_R(X)\\ &= \operatorname{supp}_R(P) \bigcap \operatorname{supp}_R(X)\end{aligned}$$

by Fact 2.5, and hence the first equality from the statement of the result. For the second equality from the statement of the result, argue similarly via the isomorphism $P \otimes^{\mathbf{L}}_R X \simeq \mathbf{R}\mathrm{Hom}_R(P^*, X)$. $\square$

Lemma 2.14 *Continue with the proper ideal $\mathfrak{a}$. Let $X \in \mathcal{D}(R)$, and let $F \in \mathcal{D}_b(R)$ be $\mathfrak{a}$-adically finite such that $\operatorname{fd}_R(F) < \infty$. Then there are equalities*

$$\operatorname{supp}_R(\mathbf{R}\mathrm{Hom}_R(F, X)) \bigcap \mathrm{V}(\mathfrak{a}) = \operatorname{supp}_R(F) \bigcap \operatorname{supp}_R(X) \bigcap \mathrm{V}(\mathfrak{a})$$

$$\operatorname{co-supp}_R(F \otimes^{\mathbf{L}}_R X) \bigcap \mathrm{V}(\mathfrak{a}) = \operatorname{supp}_R(F) \bigcap \operatorname{co-supp}_R(X) \bigcap \mathrm{V}(\mathfrak{a}).$$

Proof The complex $P := K \otimes^{\mathbf{L}}_R F$ is in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, since F is $\mathfrak{a}$-adically finite. Furthermore, since K and F both have finite flat dimension, the same is true for P. Thus, the condition $P \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ implies that P has finite projective dimension. Combining these facts, we see that P satisfies the hypotheses of Lemma 2.13. Using this as in the proof of [38, Theorem 7.12], we obtain the desired conclusions. □

2.6 Adic Semidualizing Complexes

The complexes defined next are introduced and studied in this generality in [34].

Remark 2.15 Let $M \in \mathcal{D}_{\mathrm{b}}(R)$ with $\operatorname{supp}_R(M) \subseteq V(\mathfrak{a})$, where $\mathfrak{a}$ is a proper ideal of R. From [34, Lemma 3.1] we know that M has a bounded above semi-injective resolution $M \xrightarrow{\simeq} J$ over R consisting of injective $\widehat{R}^{\mathfrak{a}}$-modules and $\widehat{R}^{\mathfrak{a}}$-module homomorphisms. This yields a well-defined chain map $\chi^{\widehat{R}^{\mathfrak{a}}}_J : \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(J, J)$ given by $\chi^{\widehat{R}^{\mathfrak{a}}}_J(r)(j) = rj$ and, in turn, a well-defined homothety morphism $\chi^{\widehat{R}^{\mathfrak{a}}}_M : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\operatorname{Hom}_R(M, M)$ in $\mathcal{D}(R)$.

Definition 2.16 Given a proper ideal $\mathfrak{a}$ of R, an *$\mathfrak{a}$-adic semidualizing R-complex* is an $\mathfrak{a}$-adically finite R-complex M (see Definition 2.10) such that the homothety morphism $\chi^{\widehat{R}^{\mathfrak{a}}}_M : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\operatorname{Hom}_R(M, M)$ from Remark 2.15 is an isomorphism.

We end this section with some examples, for perspective in the sequel.

Example 2.17 Let $M \in \mathcal{D}_{\mathrm{b}}(R)$, and continue with the proper ideal $\mathfrak{a}$.

2.17.1 If M is an R-module, then it is $\mathfrak{a}$-adically semidualizing as an R-complex if it is $\mathfrak{a}$-adically finite, the natural homothety map $\widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(M, M)$, defined as in Remark 2.15, is an isomorphism, and $\operatorname{Ext}^i_R(M, M) = 0$ for all $i \geqslant 1$.

2.17.2 The complex M is semidualizing if and only if it is 0-adically semidualizing, by [34, Proposition 4.4].

2.17.3 Assume that $(R, \mathfrak{m})$ is local. Then M is $\mathfrak{m}$-adically semidualizing if and only if each homology module $\mathrm{H}_i(M)$ is artinian and the homothety morphism $\chi^{\widehat{R}^{\mathfrak{m}}}_M : \widehat{R}^{\mathfrak{m}} \to \mathbf{R}\operatorname{Hom}_R(M, M)$ is an isomorphism in $\mathcal{D}(R)$. Hence, an R-module T is quasi-dualizing if and only if it is $\mathfrak{m}$-adically semidualizing; see [26]. Hence, the injective hull $E_R(R/\mathfrak{m})$ is $\mathfrak{m}$-adically semidualizing. See [34, Proposition 4.5].

2.17.4 If C is a semidualizing R-complex, e.g., $C = R$, then the complex $\mathbf{R}\Gamma_{\mathfrak{a}}(C)$ is $\mathfrak{a}$-adically semidualizing, by [34, Corollary 4.8].

3 Foxby Classes

This section develops the foundations of Auslander and Bass classes in the adic context. It contains our version of the ubiquitous “Foxby Equivalence”, which is Theorem 1.1 from the introduction, among other results.

Definition 3.1 Let $M, X, Y \in \mathcal{D}_{\mathrm{b}}(R)$.

(a) The complex X is in the *Auslander class* $\mathcal{A}_M(R)$ if $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_b(R)$ and the natural morphism $\gamma^M_X : X \to \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)$ is an isomorphism in $\mathcal{D}(R)$.

(b) The complex Y is in the *Bass class* $\mathcal{B}_M(R)$ if one has $\mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathcal{D}_b(R)$ and the natural evaluation morphism $\xi^M_Y : M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, Y) \to Y$ is an isomorphism in $\mathcal{D}(R)$.

The next result gives some examples of objects in Foxby classes to keep in mind. See Propositions 3.10 and 3.13(a) for improvements on the conclusion $\widehat{R}^{\mathfrak{a}} \in \mathcal{A}_M(R)$.

Proposition 3.2 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R. Then one has $\widehat{R}^{\mathfrak{a}} \in \mathcal{A}_M(R)$ and $M \in \mathcal{B}_M(R)$.*

Proof First, we show that $\widehat{R}^{\mathfrak{a}} \in \mathcal{A}_M(R)$. Since $\widehat{R}^{\mathfrak{a}}$ is flat over R and $M \in \mathcal{D}_b(R)$, we have $M \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}} \in \mathcal{D}_b(R)$. By [38, Theorem 5.10], the natural morphism $\alpha : M \to M \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}$ is an isomorphism in $\mathcal{D}(R)$. From the next commutative diagram in $\mathcal{D}(R)$

$$\begin{array}{ccc} \widehat{R}^{\mathfrak{a}} & \xrightarrow[\simeq]{\chi^{\widehat{R}^{\mathfrak{a}}}_M} & \mathbf{R}\mathrm{Hom}_R(M, M) \\ & \searrow^{\gamma^M_{\widehat{R}^{\mathfrak{a}}}}_{\simeq} & \downarrow{\scriptstyle \simeq\ \mathbf{R}\mathrm{Hom}_R(M,\alpha)} \\ & & \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}) \end{array}$$

we conclude that $\gamma^M_{\widehat{R}^{\mathfrak{a}}}$ is an isomorphism, so $\widehat{R}^{\mathfrak{a}} \in \mathcal{A}_M(R)$.

Next, we show that $M \in \mathcal{B}_M(R)$. By assumption, the homothety morphism $\chi^{\widehat{R}^{\mathfrak{a}}}_M : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_R(M, M)$ is an isomorphism in $\mathcal{D}(R)$. In particular, we have $\mathbf{R}\mathrm{Hom}_R(M, M) \in \mathcal{D}_b(R)$. The composition of the following morphisms

$$M \xrightarrow[\simeq]{\alpha} M \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}} \xrightarrow[\simeq]{M \otimes^{\mathbf{L}}_R \chi^{\widehat{R}^{\mathfrak{a}}}_M} M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, M) \xrightarrow{\xi^M_M} M$$

is id_M, so we conclude that ξ^M_M is an isomorphism, so $M \in \mathcal{B}_M(R)$. □

Much of this work highlights the similarities between Christensen's setting [8] where $\mathfrak{a} = 0$ and the general case. However, the next two items document some important differences to keep in mind.

Fact 3.3 ([8, Observation 4.10]) Let C be a semidualizing R-module.

3.3.1 An R-module A is in $\mathcal{A}_C(R)$ if and only if the natural map $\gamma^M_A : A \to \mathrm{Hom}_R(M, M \otimes_R A)$ is an isomorphism and for all $i \geqslant 1$ we have $\mathrm{Tor}^R_i(M, A) = 0 = \mathrm{Ext}^i_R(M, M \otimes_R A)$.

3.3.2 An R-module B is in $\mathcal{B}_C(R)$ if and only if the natural evaluation homomorphism $\xi^M_B : M \otimes_R \mathrm{Hom}_R(M, B) \to B$ is an isomorphism and for all $i \geqslant 1$ we have $\mathrm{Ext}^i_R(M, B) = 0 = \mathrm{Tor}^R_i(M, \mathrm{Hom}_R(M, B))$.

Example 3.4 Let k be a field, and let $R = k[\![X]\!]$ be a power series ring in one variable. Let E be the injective hull $E_R(k)$. Example 2.17.3 implies that E is $\mathfrak{m}$-adically semidualizing over R.

For this example, we define $\mathcal{A}^0_E(R)$ to be the class of all R-modules A such that the natural map $\gamma^E_A : A \to \mathrm{Hom}_R(E, E \otimes_R A)$ is an isomorphism and for all $i \geqslant 1$ we have $\mathrm{Tor}^R_i(E, A) = 0 = \mathrm{Ext}^i_R(E, E \otimes_R A)$. Define $\mathcal{B}^0_E(R)$ similarly.

It is straightforward to show that R is in $\mathcal{A}^0_E(R)$ in this example. Based on work in the semidualizing case, one may expect $k = R/XR$ to be in $\mathcal{A}^0_E(R)$, as it is a module with finite flat dimension. However, this module fails the definition of $\mathcal{A}^0_E(R)$ in two ways.

First, we have $\mathrm{Hom}_R(E, E \otimes_R k) \cong \mathrm{Hom}_R(E, 0) = 0$; so it is not possible for the natural map $\gamma^E_k : k \to \mathrm{Hom}_R(E, E \otimes_R k)$ to be an isomorphism. Second, we have $\mathrm{Tor}^R_1(E, k) \cong k$ and $\mathrm{Tor}^R_i(E, k) = 0$ for all $i \neq 1$; this is straightforward to show using the Koszul complex $K^R(X)$ as a free resolution of k.

This example is even more troubling because it shows that $\mathcal{A}^0_E(R)$ does not satisfy the 2-of-3 condition. Indeed, the following is an exact sequence

$$0 \to R \xrightarrow{X} R \to k \to 0$$

and the first two modules are in $\mathcal{A}_E(R)$, but the third is not.

Similarly, by dualizing the above exact sequence with respect to E, one obtains an augmented injective resolution of k

$$0 \to k \to E \xrightarrow{X} E \to 0.$$

From this, we see that $\mathrm{Ext}^1_R(E, k) \cong k$ and $\mathrm{Ext}^i_R(E, k) = 0$ for all $i \neq 1$. As above, this shows that $k \notin \mathcal{B}^0_E(R)$ and that $\mathcal{B}^0_E(R)$ does not satisfy the 2-of-3 condition.

The next result shows that the classes $\mathcal{A}_M(R)$ and $\mathcal{B}_M(R)$ do not have the same flaws as the classes from the previous example.

Proposition 3.5 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R. Then the classes $\mathcal{A}_M(R)$ and $\mathcal{B}_M(R)$ are triangulated and thick.*

Proof By definition, this follows from the next straightforward facts:

1. For each $i \in \mathbb{Z}$, the classes $\mathcal{A}_M(R)$ and $\mathcal{B}_M(R)$ are closed under Σ^i.
2. Given an exact triangle $X \to Y \to Z \to$ in $\mathcal{D}(R)$, if two of the three complexes X, Y, Z are in $\mathcal{A}_M(R)$ (repectively, in $\mathcal{B}_M(R)$), then so is the third.
3. For all $X, Y \in \mathcal{D}_b(R)$, the direct sum $X \oplus Y$ is in $\mathcal{A}_M(R)$ if and only if X and Y are both in $\mathcal{A}_M(R)$, and similarly for $\mathcal{B}_M(R)$. □

Next, we prove the adic version of Foxby Equivalence, which is Theorem 1.1 in the introduction. Note the support and co-support conditions in parts (b) and (c), which are automatic in the semidualizing situation [8, Theorem 4.6]. Example 3.7 below shows that they are crucial in our more general setup. Note also the lack of any a priori boundedness condition in parts (b) and (c).

Theorem 3.6 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R.*

(a) *The functors $\mathbf{R}\mathrm{Hom}_R(M, -) : \mathcal{B}_M(R) \to \mathcal{A}_M(R)$ and $M \otimes^{\mathbf{L}}_R - : \mathcal{A}_M(R) \to \mathcal{B}_M(R)$ are quasi-inverse equivalences.*

(b) *An R-complex $Y \in \mathcal{D}(R)$ is in $\mathcal{B}_M(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathcal{A}_M(R)$ and $\mathrm{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$.*

(c) *An R-complex* $X \in \mathcal{D}(R)$ *is in* $\mathcal{A}_M(R)$ *if and only if one has* $M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ *and* $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

Proof Let $X \in \mathcal{D}_{\mathrm{b}}(R)$, and set $Z := M \otimes^{\mathbf{L}}_R X$. Consider the defining morphisms $\xi^M_Z \colon M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, Z) \to Z$ and $\gamma^M_X \colon X \to \mathbf{R}\mathrm{Hom}_R(M, Z)$. Then the induced morphism in $\mathcal{D}(R)$

$$Z = M \otimes^{\mathbf{L}}_R X M \otimes^{\mathbf{L}}_R \gamma^M_X \xrightarrow{\ -\ } M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, Z)$$

satisfies $\xi^M_Z \circ (M \otimes^{\mathbf{L}}_R \gamma^M_X) = \mathrm{id}_Z$. It follows that $M \otimes^{\mathbf{L}}_R \gamma^M_X$ is an isomorphism if and only if ξ^M_Z is one.

We verify the forward implication of part (c). To this end, assume for this paragraph that $X \in \mathcal{A}_M(R)$. Then we have $Z \in \mathcal{D}_{\mathrm{b}}(R)$, and γ^M_X being an isomorphism implies that $\mathbf{R}\mathrm{Hom}_R(M, Z) = \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X) \simeq X \in \mathcal{D}_{\mathrm{b}}(R)$. Thus, Fact 2.5.2 implies that

$$\operatorname{co-supp}_R(X) = \operatorname{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)) \subseteq \operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a}).$$

The morphism $M \otimes^{\mathbf{L}}_R \gamma^M_X$ is also an isomorphism, since γ^M_X is one. From the previous paragraph, we know that ξ^M_Z is also an isomorphism, and therefore, we have $Z \in \mathcal{B}_M(R)$, as desired.

We now prove the converse of part (c). Assume that $Z = M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ and $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Then we have $Z, \mathbf{R}\mathrm{Hom}_R(M, Z) \in \mathcal{D}_{\mathrm{b}}(R)$, and ξ^M_Z is an isomorphism. By the first paragraph of this proof, the morphism $M \otimes^{\mathbf{L}}_R \gamma^M_X$ is therefore an isomorphism. Fact 2.5.2 implies $\operatorname{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(M, Z)) \subseteq \mathrm{V}(\mathfrak{a})$. Since we also have $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a}) = \operatorname{supp}_R(M)$, by [34, Remark 4.3], we conclude from [38, Theorem 5.7] that γ^M_X is an isomorphism in $\mathcal{D}(R)$. Hence, we have $X \simeq \mathbf{R}\mathrm{Hom}_R(M, Z) \in \mathcal{D}_{\mathrm{b}}(R)$. As we also have $M \otimes^{\mathbf{L}}_R X = Z \in \mathcal{D}_{\mathrm{b}}(R)$, we conclude that $X \in \mathcal{A}_M(R)$, as desired.

Part (b) is verified similarly, and part (a) follows from (b) and (c). □

Next, we show the necessity of the support conditions in Foxby Equivalence 3.6.

Example 3.7 Let k be a field, and set $R := k[\![Y]\!]$ with $E := E_R(k)$ and $\mathfrak{a} := (Y)R$. We show that $R \notin \mathcal{B}_E(R)$ and $\mathbf{R}\mathrm{Hom}_R(E, R) \in \mathcal{A}_E(R)$. We also show that $E \notin \mathcal{A}_E(R)$ and $E \otimes^{\mathbf{L}}_R E \in \mathcal{B}_E(R)$. Notice that these facts do not contradict Foxby Equivalence 3.6, because the support condition is not satisfied: by faithful flatness and faithful injectivity we have $\operatorname{supp}_R(R) = \operatorname{co-supp}_R(E) = \operatorname{Spec}(R) \not\subseteq \mathrm{V}(\mathfrak{a})$.

We first show $\mathbf{R}\mathrm{Hom}_R(E, R) \in \mathcal{A}_E(R)$. The augmented and truncated minimal semi-injective resolutions of R are, respectively,

$$\begin{aligned} {}^{+}J &= \quad 0 \longrightarrow R \longrightarrow Q(R) \longrightarrow E \longrightarrow 0 \\ J &= \qquad\qquad\quad 0 \longrightarrow Q(R) \longrightarrow E \longrightarrow 0 \end{aligned}$$

where $Q(R) = k(\!(Y)\!)$ is the field of fractions of R. An application of $\mathrm{Hom}_R(E, -)$ to J yields the complex

$$\mathrm{Hom}_R(E, J) = \quad 0 \longrightarrow \mathrm{Hom}_R(E, Q(R)) \longrightarrow \mathrm{Hom}_R(E, E) \longrightarrow 0.$$

It is well-known that $\mathrm{Hom}_R(E, Q(R)) = 0$ and $\mathrm{Hom}_R(E, E) \cong R$. It follows that $\mathbf{R}\mathrm{Hom}_R(E, R) \simeq \mathrm{Hom}_R(E, J) \simeq \Sigma^{-1} R \in \mathcal{A}_E(R)$, by Proposition 3.2.

Next, consider the isomorphisms

$$\Sigma^{-1} E \simeq E \otimes^{\mathbf{L}}_R \Sigma^{-1} R \simeq E \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(E, R).$$

It follows that the morphism $\delta_R^E\colon E\otimes_R^{\mathbf{L}}\mathbf{R}\mathrm{Hom}_R(E,R)\to R$ is not isomorphism. Hence, we have $R\notin\mathcal{B}_E(R)$. One can also deduce this from Foxby Equivalence 3.6(b) since we have $\mathrm{supp}_R(R)=\mathrm{Spec}(R)\not\subseteq\mathrm{V}(\mathfrak{a})$.

Next, it is straightforward to show that $E\otimes_R^{\mathbf{L}}E\simeq\Sigma^1E\in\mathcal{B}_E(R)$; see [27, Example 6.4] and Proposition 3.2. From this, we have

$$\mathbf{R}\mathrm{Hom}_R(E,E\otimes_R^{\mathbf{L}}E)\simeq\mathbf{R}\mathrm{Hom}_R(E,\Sigma^1E)\simeq\Sigma^1\mathbf{R}\mathrm{Hom}_R(E,E)\simeq\Sigma^1R\not\simeq E$$

so $E\notin\mathcal{A}_E(R)$; one can also deduce this using $\text{co-supp}_R(E)$ as above.

The next result shows what happens when you do remove the support conditions from Foxby Equivalence 3.6. Again, note the lack of boundedness assumptions.

Corollary 3.8 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and let $X\in\mathcal{D}(R)$.*

(a) *One has $\mathbf{R}\mathrm{Hom}_R(M,X)\in\mathcal{A}_M(R)$ if and only if $\mathbf{R}\Gamma_{\mathfrak{a}}(X)\in\mathcal{B}_M(R)$. When these conditions are satisfied, one has $\mathbf{R}\Gamma_{\mathfrak{a}}(X)\simeq M\otimes_R^{\mathbf{L}}\mathbf{R}\mathrm{Hom}_R(M,X)$.*
(b) *One has $M\otimes_R^{\mathbf{L}}X\in\mathcal{B}_M(R)$ if and only if $\mathbf{L}\Lambda^{\mathfrak{a}}(X)\in\mathcal{A}_M(R)$. When these conditions are satisfied, one has $\mathbf{L}\Lambda^{\mathfrak{a}}(X)\simeq\mathbf{R}\mathrm{Hom}_R(M,M\otimes_R^{\mathbf{L}}X)$.*

Proof We prove part (a). From Fact 2.7, we have

$$\mathbf{R}\mathrm{Hom}_R(M,\mathbf{R}\Gamma_{\mathfrak{a}}(X))\simeq\mathbf{R}\mathrm{Hom}_R(M,X).\tag{3.8.1}$$

For the forward implication, assume $\mathbf{R}\mathrm{Hom}_R(M,X)\in\mathcal{A}_M(R)$. From Eq. 3.8.1, we have $\mathbf{R}\mathrm{Hom}_R(M,\mathbf{R}\Gamma_{\mathfrak{a}}(X))\in\mathcal{A}_M(R)$. Fact 2.5.2 implies that $\mathrm{supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X))\subseteq\mathrm{V}(\mathfrak{a})$, so Foxby Equivalence 3.6(b) implies that $\mathbf{R}\Gamma_{\mathfrak{a}}(X)\in\mathcal{B}_M(R)$.

For the converse, assume that $\mathbf{R}\Gamma_{\mathfrak{a}}(X)\in\mathcal{B}_M(R)$. By definition, this yields the second isomorphism in the next sequence.

$$M\otimes_R^{\mathbf{L}}\mathbf{R}\mathrm{Hom}_R(M,X)\simeq M\otimes_R^{\mathbf{L}}\mathbf{R}\mathrm{Hom}_R(M,\mathbf{R}\Gamma_{\mathfrak{a}}(X))\simeq\mathbf{R}\Gamma_{\mathfrak{a}}(X)\in\mathcal{B}_M(R)$$

The first isomorphism is again by Eq. 3.8.1. Fact 2.5.2 implies

$$\text{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(M,X))\subseteq\mathrm{supp}_R(M)\subseteq\mathrm{V}(\mathfrak{a})$$

so by Foxby Equivalence 3.6(b), we have $\mathbf{R}\mathrm{Hom}_R(M,X)\in\mathcal{A}_M(R)$, as desired. □

Example 3.9 Let k be a field, and set $R:=k[\![Y]\!]$ with $E:=E_R(k)$ and $\mathfrak{a}:=(Y)R$. From Example 3.7 we have $R\notin\mathcal{B}_E(R)$ and $\mathbf{R}\mathrm{Hom}_R(E,R)\in\mathcal{A}_E(R)$. On the other hand, Corollary 3.8(a) implies that

$$\Sigma^{-1}E\simeq\mathbf{R}\Gamma_{\mathfrak{a}}(R)\simeq E\otimes_R^{\mathbf{L}}\mathbf{R}\mathrm{Hom}_R(E,R)\in\mathcal{B}_E(R).$$

Note that this corroborates part of Proposition 3.2. Example 3.7 also shows that $E\notin\mathcal{A}_E(R)$ and $E\otimes_R^{\mathbf{L}}E\in\mathcal{B}_E(R)$. Corollary 3.8(a) implies that

$$\Sigma^1R\simeq\mathbf{R}\mathrm{Hom}_R(E,E\otimes_R^{\mathbf{L}}E)\simeq\mathbf{L}\Lambda^{\mathfrak{a}}(E)\in\mathcal{A}_E(R)$$

which again bears witness to Proposition 3.2.

Our next results document adic versions of some standard facts, starting with an augmentation of Proposition 3.2.

Proposition 3.10 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R. Then one has $R\in\mathcal{A}_M(R)$ if and only if R is $\mathfrak{a}$-adically complete.*

Proof For the forward implication, assume that we have $R \in \mathcal{A}_M(R)$. Foxby Equivalence 3.6(c) implies that $\operatorname{co-supp}_R(R) \subseteq \mathrm{V}(\mathfrak{a})$, so $R \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(R) \simeq \widehat{R}^{\mathfrak{a}}$ by Fact 2.5.3, thus R is $\mathfrak{a}$-adically complete. Conversely, if R is $\mathfrak{a}$-adically complete, then $R \cong \widehat{R}^{\mathfrak{a}} \in \mathcal{A}_M(R)$ by Proposition 3.2. □

The next remark is for use in the sequel.

Remark 3.11 Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and let $X, Y \in \mathcal{D}(R)$. Since $\widehat{R}^{\mathfrak{a}}$ is flat over R and the homology of K is $\mathfrak{a}$-torsion, the natural morphism $\iota\colon K \to K \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}}$ is an isomorphism in $\mathcal{D}(R)$. This explains the vertical isomorphism in the following commutative diagram in $\mathcal{D}(R)$:

$$\begin{array}{ccc}
K \otimes^{\mathbf{L}}_R X & \xrightarrow{K \otimes^{\mathbf{L}}_R \gamma^M_X} & K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X) \\
{\scriptstyle \iota \otimes^{\mathbf{L}}_R X}\downarrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle K \otimes^{\mathbf{L}}_R \omega_{MMX}} \\
K \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X & \xrightarrow[\simeq]{K \otimes^{\mathbf{L}}_R \chi^{\widehat{R}^{\mathfrak{a}}}_M \otimes^{\mathbf{L}}_R X} & K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, M) \otimes^{\mathbf{L}}_R X.
\end{array} \tag{3.11.1}$$

The morphism ω_{MMX} is tensor-evaluation. The lower horizontal morphism is an isomorphism since M is $\mathfrak{a}$-adically semidualizing. Similarly, we have the next commutative diagram where $\nu := \mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\mathrm{Hom}_R(\chi^{\widehat{R}^{\mathfrak{a}}}_M, Y))$.

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_R(K, M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, Y)) & \xrightarrow{\mathbf{R}\mathrm{Hom}_R(K, \xi^M_Y)} & \mathbf{R}\mathrm{Hom}_R(K, Y) \\
 & & {\scriptstyle \simeq}\uparrow{\scriptstyle \mathbf{R}\mathrm{Hom}_R(\iota, Y)} \\
{\scriptstyle \mathbf{R}\mathrm{Hom}_R(K, \theta_{MMY})}\downarrow & & \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R K, Y) \\
 & & {\scriptstyle \simeq}\uparrow \\
\mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, M), Y)) & \xrightarrow[\simeq]{\nu} & \mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, Y))
\end{array} \tag{3.11.2}$$

The unspecified isomorphism is adjointness, and θ_{MMY} is tensor-evaluation.

It is straightforward to show that the trivial semidualizing complex R has trivial Auslander and Bass classes: $\mathcal{A}_R(R) = \mathcal{D}_{\mathrm{b}}(R) = \mathcal{B}_R(R)$. Our next result generalizes this to the adic situation. The result applies, in particular, to the special case $M = \mathbf{R}\Gamma_{\mathfrak{a}}(R)$, by [34, Corollary 4.8]. Note that Foxby Equivalence 3.6 shows that this is as trivial as things get in this setting. Also, see [34, Section 5] for characterizations of the property $\mathrm{fd}_R(M) < \infty$ in this context.

Proposition 3.12 *Let M be an $\mathfrak{a}$-adic semidualizing complex with $\mathrm{fd}_R(M) < \infty$, where $\mathfrak{a}$ is a proper ideal of R.*

(a) *The class $\mathcal{A}_M(R)$ consists of all $X \in \mathcal{D}_b(R)$ such that $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, that is, the complexes $X \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(Z)$ for some $Z \in \mathcal{D}_b(R)$.*

(b) *The class $\mathcal{B}_M(R)$ consists of all $Y \in \mathcal{D}_b(R)$ with $\mathrm{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$, that is, the complexes $X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Z)$ for some $Z \in \mathcal{D}_b(R)$.*

Proof MGM Equivalence [32, Theorem 6.11] shows that $X \in \mathcal{D}_b(R)$ satisfies $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ if and only if we have $X \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(Z)$ for some $Z \in \mathcal{D}_b(R)$, and similarly for supp. (Parts of this are in Facts 2.5.2–2.5.3.) Thus, by Foxby Equivalence 3.6, we need only show the following: (1) if $X \in \mathcal{D}_b(R)$ satisfies $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, then $X \in \mathcal{A}_M(R)$, and (2) if $Y \in \mathcal{D}_b(R)$ satisfies $\operatorname{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$, then $Y \in \mathcal{B}_M(R)$.

(1) Let $X \in \mathcal{D}_b(R)$ be such that $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Since $\operatorname{fd}_R(M) < \infty$, we have $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_b(R)$. Also, Fact 2.5.2 implies that

$$\operatorname{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)) \subseteq \operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a}).$$

Thus, by Fact 2.8, to show that γ^M_X is an isomorphism, it suffices to show that $K \otimes^{\mathbf{L}}_R \gamma^M_X$ is an isomorphism. The morphism $K \otimes^{\mathbf{L}}_R \omega_{MMX}$ from Remark 3.11 is an isomorphism by Fact 2.12(a). It follows from the diagram (3.11.1) that $K \otimes^{\mathbf{L}}_R \gamma^M_X$ is an isomorphism, as desired.

(2) Let $Y \in \mathcal{D}_b(R)$ be such that $\operatorname{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$. Since M has finite projective dimension by [37, Theorem 6.1], we have $\mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathcal{D}_b(R)$. Thus, to complete the proof, one argues as in part (1) to show that the morphism ξ^M_Y is an isomorphism in $\mathcal{D}(R)$, using Eq. 3.11.2 and Fact 2.12(b). □

Given a semidualizing R-complex C, we know from [8, Proposition 4.4] that $\mathcal{A}_C(R)$ contains all complexes of finite flat dimension, and $\mathcal{B}_C(R)$ contains all complexes of finite injective dimension. The next result is our version of this fact in the adic setting.

Proposition 3.13 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R.*

(a) *The Auslander class $\mathcal{A}_M(R)$ contains all R-complexes $X \in \mathcal{D}_b(R)$ of finite flat dimension such that* $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.
(b) *The Bass class $\mathcal{B}_M(R)$ contains all R-complexes $Y \in \mathcal{D}_b(R)$ of finite injective dimension such that* $\operatorname{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$.

Proof We deal with part (a). Let $X \in \mathcal{D}_b(R)$ with $\operatorname{fd}_R(X) < \infty$ be such that $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The condition $\operatorname{fd}_R(X) < \infty$ implies that $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_b(R)$. To show that the morphism $\gamma^M_X \colon X \to \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)$ is an isomorphism in $\mathcal{D}(R)$, it suffices by Fact 2.8 to show that the induced morphism $K \otimes^{\mathbf{L}}_R \gamma^M_X$ is an isomorphism. This is accomplished using Fact 2.12(a) with the diagram (3.11.1) from Remark 3.11, as in the proof of Proposition 3.12. □

Corollary 3.14 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and fix an ideal $\mathfrak{b} \supseteq \mathfrak{a}$. Let L be the Koszul complex over R on a finite generating sequence for $\mathfrak{b}$.*

(a) *The Auslander class $\mathcal{A}_M(R)$ contains every R-complex of the form $\mathbf{L}\Lambda^{\mathfrak{b}}(F)$ and $L \otimes^{\mathbf{L}}_R F$ where $F \in \mathcal{D}_b(R)$ has finite flat dimension. In particular, we have $\widehat{R}^{\mathfrak{b}}, L \in \mathcal{A}_M(R)$.*
(b) *The Bass class $\mathcal{B}_M(R)$ contains every complex of the form $\mathbf{R}\Gamma_{\mathfrak{b}}(I)$ and $L \otimes^{\mathbf{L}}_R I$ where $I \in \mathcal{D}_b(R)$ has finite injective dimension.*

Proof We prove part (a). The complex $\mathbf{L}\Lambda^{\mathfrak{b}}(F)$ has finite flat dimension, by Fact 2.3. Since we also have $\operatorname{co-supp}_R(\mathbf{L}\Lambda^{\mathfrak{b}}(F)) \subseteq \mathrm{V}(\mathfrak{b}) \subseteq \mathrm{V}(\mathfrak{a})$ by Fact 2.5.2, we deduce from Proposition 3.13(a) that $\mathbf{L}\Lambda^{\mathfrak{b}}(F) \in \mathcal{A}_M(R)$, and similarly for $L \otimes^{\mathbf{L}}_R F$, using Fact 2.5.3. The conclusion $\widehat{R}^{\mathfrak{b}}, L \in \mathcal{A}_M(R)$ is from the special case $F = R$. □

Our next three results are versions of [39, Corollary 2.10 and Theorem 2.11] and [40, Theorem 1.1] for the adic semidualizing context. As with the previous results, a major difference is the inclusion of a (co)support condition.

Proposition 3.15 *Let M be $\mathfrak{a}$-adic semidualizing over R, where $\mathfrak{a}$ is a proper ideal of R, and let $X \in \mathcal{D}_b(R)$. The following conditions are equivalent.*

- (i) *One has* $\operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(M, X)) < \infty$ *and* $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.
- (ii) *There is a complex $F \in \mathcal{D}_b(R)$ with $\operatorname{fd}_R(F) < \infty$ such that $X \simeq M \otimes^{\mathbf{L}}_R F$ and* $\operatorname{co-supp}_R(F) \subseteq \mathrm{V}(\mathfrak{a})$.
- (iii) *There is a complex $G \in \mathcal{D}_b(R)$ with $\operatorname{fd}_R(G) < \infty$ such that $X \simeq M \otimes^{\mathbf{L}}_R G$.*

If these conditions hold, then $F \simeq \mathbf{R}\mathrm{Hom}_R(M, X) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(G)$ and $X \in \mathcal{B}_M(R)$.

Proof (i) $\Longrightarrow$ (iii). Assume that $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ and that the complex $G := \mathbf{R}\mathrm{Hom}_R(M, X) \in \mathcal{D}_b(R)$ satisfies $\operatorname{fd}_R(G) < \infty$. Since Fact 2.5.2 implies that

$$\operatorname{co-supp}_R(G) = \operatorname{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(M, X)) \subseteq \operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$$

we know from Proposition 3.13(a) that $G = \mathbf{R}\mathrm{Hom}_R(M, X) \in \mathcal{A}_M(R)$. The condition $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ implies that $X \in \mathcal{B}_M(R)$, by Foxby Equivalence 3.6(b). Thus, we have $X \simeq M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, X) = M \otimes^{\mathbf{L}}_R G$, as desired.

(iii) $\Longrightarrow$ (ii). Assume that there is a complex $G \in \mathcal{D}_b(R)$ with $\operatorname{fd}_R(G) < \infty$ such that $X \simeq M \otimes^{\mathbf{L}}_R G$. Set $F := \mathbf{L}\Lambda^{\mathfrak{a}}(G)$, which satisfies $\operatorname{fd}_R(F) < \infty$ and $\operatorname{co-supp}_R(F) \subseteq \mathrm{V}(\mathfrak{a})$, by Facts 2.3 and 2.5.2. The first and last isomorphisms in the next sequence are by assumption

$$M \otimes^{\mathbf{L}}_R F \simeq M \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(G) \simeq M \otimes^{\mathbf{L}}_R G \simeq X$$

and the second isomorphism is from Fact 2.7. Also, in the next sequence, the first isomorphism is by definition and the last one is from the previous display

$$\mathbf{L}\Lambda^{\mathfrak{a}}(G) \simeq F \simeq \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R F) \simeq \mathbf{R}\mathrm{Hom}_R(M, X).$$

The second isomorphism is because $F \in \mathcal{A}_M(R)$; see Proposition 3.13(a). This completes the proof of this implication and explains one of the additional claims in the statement of the proposition.

(ii) $\Longrightarrow$ (i). Assume that there is a complex $F \in \mathcal{D}_b(R)$ with $\operatorname{fd}_R(F) < \infty$ such that $X \simeq M \otimes^{\mathbf{L}}_R F$ and $\operatorname{co-supp}_R(F) \subseteq \mathrm{V}(\mathfrak{a})$. Proposition 3.13(a) implies that $F \in \mathcal{A}_M(R)$, so we have $X \simeq M \otimes^{\mathbf{L}}_R F \in \mathcal{B}_M(R)$ by Foxby Equivalence 3.6(b). Moreover, this implies that we have

$$F \simeq \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R F) \simeq \mathbf{R}\mathrm{Hom}_R(M, X)$$

so $\operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(M, X)) = \operatorname{fd}_R(F) < \infty$. The isomorphism $X \simeq M \otimes^{\mathbf{L}}_R F$ implies

$$\operatorname{supp}_R(X) = \operatorname{supp}_R(M \otimes^{\mathbf{L}}_R F) \subseteq \operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$$

by Fact 2.5.2. This completes the proof of this implication and explains the remaining claims in the statement of the proposition. □

The next two results are proved similarly to the previous one.

Proposition 3.16 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and let $X \in \mathcal{D}_b(R)$. The following conditions are equivalent.*

- (i) *One has* $\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(M, X)) < \infty$ *and* $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

(ii) *There is a complex* $P \in \mathcal{D}_b(R)$ *with* $\operatorname{pd}_R(P) < \infty$ *such that* $X \simeq M \otimes^{\mathbf{L}}_R P$ *and* $\operatorname{co-supp}_R(P) \subseteq \mathrm{V}(\mathfrak{a})$.
(iii) *There is a complex* $Q \in \mathcal{D}_b(R)$ *with* $\operatorname{pd}_R(Q) < \infty$ *such that* $X \simeq M \otimes^{\mathbf{L}}_R Q$.

If these conditions hold, then $P \simeq \mathbf{R}\mathrm{Hom}_R(M, X) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(Q)$ *and* $X \in \mathcal{B}_M(R)$.

Proposition 3.17 *Let* M *be an* $\mathfrak{a}$*-adic semidualizing* R*-complex, where* $\mathfrak{a}$ *is a proper ideal of* R*, and let* $Y \in \mathcal{D}_b(R)$*. The following conditions are equivalent.*

(i) *One has* $\operatorname{id}_R(M \otimes^{\mathbf{L}}_R X) < \infty$ *and* $\operatorname{co-supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$.
(ii) *There is a complex* $I \in \mathcal{D}_b(R)$ *with* $\operatorname{id}_R(I) < \infty$ *such that* $Y \simeq \mathbf{R}\mathrm{Hom}_R(M, I)$ *and* $\operatorname{supp}_R(I) \subseteq \mathrm{V}(\mathfrak{a})$.
(iii) *There is a complex* $J \in \mathcal{D}_b(R)$ *with* $\operatorname{id}_R(J) < \infty$ *such that* $Y \simeq \mathbf{R}\mathrm{Hom}_R(M, J)$.

When these conditions hold, one has $I \simeq M \otimes^{\mathbf{L}}_R Y \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(J)$ *and* $Y \in \mathcal{A}_M(R)$.

We conclude this section a technical, but useful result, *à la* [8, Proposition 4.8].

Lemma 3.18 *Let* M *be an* $\mathfrak{a}$*-adic semidualizing* R*-complex, where* $\mathfrak{a}$ *is a proper ideal of* R*, and let* $X \in \mathcal{D}_b(R)$.

(a) *If* $X \in \mathcal{A}_M(R)$*, then we have*
$$\inf(M) + \sup(X) \leqslant \sup(M \otimes^{\mathbf{L}}_R X) \leqslant \sup(X) + \sup(M) + n.$$
(b) *If* $X \in \mathcal{B}_M(R)$*, then we have*
$$\inf(X) - \sup(M) - 2n \leqslant \inf(\mathbf{R}\mathrm{Hom}_R(M, X)) \leqslant \inf(X) - \inf(M).$$

Proof (a) The condition $X \in \mathcal{A}_M(R)$ implies that $X \simeq \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)$. This explains the first step in the next display.

$$\sup(X) = \sup(\mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)) \geqslant \sup(M \otimes^{\mathbf{L}}_R X) - \sup(M) - n$$

The second step follows from [36, Proposition 3.1(a)]; the hypotheses of this result are satisfied since we have $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(R)$, and M is $\mathfrak{a}$-adically finite with $\operatorname{supp}_R(M) = \mathrm{V}(\mathfrak{a}) \supseteq \operatorname{supp}_R(M \otimes^{\mathbf{L}}_R X)$ by [38, Proposition 7.17]. This yields the second of the inequalities from the statement of the lemma. For the first one, we argue similarly, using Fact 2.1(b):

$$\sup(X) = \sup(\mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)) \leqslant \sup(M \otimes^{\mathbf{L}}_R X) - \inf(M)$$

(b) These inequalities are verified similarly, using [36, Proposition 3.6(a)]. □

4 Stability

In this section, we document various stability results (and special converses) for Foxby classes, including Theorem 1.2 from the introduction.

4.1 Sums and Products

Let M be an $\mathfrak{a}$-adic semidualizing R-complex. Then $\mathcal{A}_M(R)$ and $\mathcal{B}_M(R)$ always fail to be closed under arbitrary direct sums and products. The main reason for this is that if $N_i \in \mathcal{D}_{\mathrm{b}}(R)$, then in general we have $\bigoplus_i N_i, \prod_i N_i \notin \mathcal{D}_{\mathrm{b}}(R)$. However, the next example shows that this can fail for other reasons.

Example 4.1 Let k be a field, and set $R := k[\![Y]\!]$ with $E := E_R(k)$ and $\mathfrak{a} := (Y)R$. Then we have $R \in \mathcal{A}_E(R)$ and $E \in \mathcal{B}_E(R)$ by Proposition 3.2.

We claim that the direct sum $R^{(\mathbb{N})}$ is not contained in $\mathcal{A}_E(R)$. Indeed, from [35, Lemma 3.2] we know that the natural map $R^{(\mathbb{N})} \to \Lambda^{\mathfrak{a}}(R^{(\mathbb{N})})$ is not an isomorphism. Since $R^{(\mathbb{N})}$ is flat, this says that the natural morphism $R^{(\mathbb{N})} \to \mathbf{L}\Lambda^{\mathfrak{m}}(R^{(\mathbb{N})})$ is not an isomorphism in $\mathcal{D}(R)$, so $\text{co-supp}_R(R^{(\mathbb{N})}) \not\subseteq \text{V}(\mathfrak{a})$ by Fact 2.5.3. Thus, Foxby Equivalence 3.6(c) shows that $R^{(\mathbb{N})} \notin \mathcal{A}_E(R)$.

Similarly, the fact that the product $E^{\mathbb{N}}$ is not $\mathfrak{a}$-torsion implies that $\text{supp}_R(E^{\mathbb{N}}) \not\subseteq \text{V}(\mathfrak{a})$, so $E^{\mathbb{N}} \notin \mathcal{B}_E(R)$.

On the other hand, it is straightforward to show that we do have $R^{\mathbb{N}} \in \mathcal{A}_E(R)$ and $E^{(\mathbb{N})} \in \mathcal{B}_E(R)$ in this setting.[2] The next result augments this fact significantly. It is not clear that this result has even been documented in the case of semidualizing complexes (that is, the case $\mathfrak{a} = 0$).

Theorem 4.2 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and consider a set $\{N_i\}_{i\in I} \subseteq \mathcal{D}_b(R)$ such that there are integers j, t such that $j \leqslant \inf(N_i)$ and $\sup(N_i) \leqslant t$ for all $i \in I$.*

(a) *One has $N_i \in \mathcal{A}_M(R)$ for all $i \in I$ if and only if $\prod_i N_i \in \mathcal{A}_M(R)$.*
(b) *One has $N_i \in \mathcal{B}_M(R)$ for all $i \in I$ if and only if $\bigoplus_i N_i \in \mathcal{B}_M(R)$.*

Proof Note that the conditions $j \leqslant \inf(N_i)$ and $\sup(N_i) \leqslant t$ for all $i \in I$ guarantee that $\prod_i N_i, \bigoplus_i N_i \in \mathcal{D}_\text{b}(R)$.

(a) One implication follows from Proposition 3.5. For the converse, assume that we have $N_i \in \mathcal{A}_M(R)$ for all $i \in I$. As we have noted, we have $\prod_i N_i \in \mathcal{D}_\text{b}(R)$. We need to show next that $M \otimes^{\mathbf{L}}_R \prod_i N_i \in \mathcal{D}_\text{b}(R)$. Note that the condition $M, \prod_i N_i \in \mathcal{D}_\text{b}(R)$ implies $M \otimes^{\mathbf{L}}_R \prod_i N_i \in \mathcal{D}_+(R)$, so we need to show that $M \otimes^{\mathbf{L}}_R \prod_i N_i \in \mathcal{D}_-(R)$. To this end, we use the following isomorphism from [37, Lemma 4.5].

$$K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R \prod_i N_i \simeq \textstyle\prod_i (K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R N_i)$$

From this, we have the second step in the next display; the third step is routine.

$$\begin{aligned}
\sup\left(M \otimes^{\mathbf{L}}_R \prod_i N_i\right) &= \sup\left(K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R \prod_i N_i\right) - n \\
&= \sup\left(\prod_i (K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R N_i)\right) - n \\
&= \sup_i\{\sup(K \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R N_i))\} - n \\
&= \sup_i\{n + \sup(M \otimes^{\mathbf{L}}_R N_i)\} - n \\
&\leqslant \sup_i\{2n + \sup(M) + \sup(N_i)\} - n \\
&\leqslant n + \sup(M) + t
\end{aligned}$$

[2] We do not verify this explicitly since it follows directly from the next result using the conditions $R \in \mathcal{A}_E(R)$ and $E \in \mathcal{B}_E(R)$.

Fact 2.5.2 shows that the supports of $M \otimes^{\mathbf{L}}_R \prod_i N_i$ and $M \otimes^{\mathbf{L}}_R N_i$ are contained in $\mathrm{V}(\mathfrak{a})$; so, the first and fourth steps are from [37, Lemma 3.1(a)]. The fifth step is from Lemma 3.18(a), and the last one is by assumption.

Next, we need to show that the morphism

$$\gamma^M_{\prod_i N_i} \colon \prod_i N_i \to \mathbf{R}\mathrm{Hom}_R\left(M, M \otimes^{\mathbf{L}}_R \prod_i N_i\right)$$

is an isomorphism. We consider the following commutative diagram in $\mathcal{D}(R)$.

$$\begin{array}{ccc}
\prod_i N_i \gamma^M_{\prod_i N_i} & \xrightarrow{\quad =\quad} & \mathbf{R}\mathrm{Hom}_R\left(M, M \otimes^{\mathbf{L}}_R \prod_i N_i\right) \\
{\scriptstyle \prod_i \gamma^M_{N_i}} \big\downarrow \simeq & & \big\downarrow \simeq \\
\prod_i \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R N_i) \simeq & \xrightarrow{\quad =\quad} & \mathbf{R}\mathrm{Hom}_R\left(M, \prod_i M \otimes^{\mathbf{L}}_R N_i\right)
\end{array}$$

The unspecified vertical isomorphism is from [37, Theorem 4.7(b)], and the unspecified horizontal one is standard for products. We conclude that $\gamma^M_{\prod_i N_i}$ is an isomorphism, so $\prod_i N_i \in \mathcal{A}_M(R)$, as desired.

(b) This is verified similarly, using Lemma 3.18(b) with [37, Lemma 3.2(a), Lemma 4.6, and Theorem 4.8(b)]. □

4.2 Finite Flat Dimension

We now turn our attention to stability results for Foxby classes with respect to $F \otimes^{\mathbf{L}}_R -$ where F is an R-complex of finite flat dimension. Again, it is worth noting the lack of a priori boundedness assumptions on X in many of these (and subsequent) results, beginning with Theorem 1.2 from the introduction. Recall that, given a ring homomorphism $\varphi \colon R \to S$, we let $\varphi^* \colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ denote the induced map. See [37, Proposition 5.6(c)] for perspective on the condition $\varphi^*(\mathrm{supp}_S(F)) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$.

Theorem 4.3 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R. Let $F \in \mathcal{D}_b(R)$ be such that $\mathrm{fd}_R(F) < \infty$, and let $X \in \mathcal{D}(R)$. If $X \in \mathcal{B}_M(R)$, then $X \otimes^{\mathbf{L}}_R F \in \mathcal{B}_M(R)$ and $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The converse of this statement holds when at least one of the following conditions is satisfied.*

(1) *F is $\mathfrak{a}$-adically finite such that $\mathrm{supp}_R(F) = \mathrm{V}(\mathfrak{a})$.*
(2) *There is a homomorphism $\varphi \colon R \to S$ of commutative noetherian rings with $\mathfrak{a}S \neq S$ such that $F \in \mathcal{D}_b(S)$ is $\mathfrak{a}S$-adically finite over S with $\varphi^*(\mathrm{supp}_S(F)) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, and we have $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$.*
(3) *F is a flat R-module with $\mathrm{supp}_R(F) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, e.g., F is faithfully flat, e.g., free.*

Proof Let $\mathbf{R}\mathrm{Hom}_R(M, X) \otimes^{\mathbf{L}}_R F \xrightarrow{\omega_{MXF}} \mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F)$ be the natural tensor-evaluation morphism. From Fact 2.12(a), we know that the induced morphisms $K \otimes^{\mathbf{L}}_R \omega_{MXF}$ and $M \otimes^{\mathbf{L}}_R \omega_{MXF}$ are isomorphisms in $\mathcal{D}(R)$. Furthermore, we have the following commutative diagram in $\mathcal{D}(R)$.

$$\begin{array}{ccc} M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,X) \otimes^{\mathbf{L}}_R F & & \\ {\scriptstyle M\otimes^{\mathbf{L}}_R \omega_{MXF}} \downarrow \simeq & \overset{\xi^M_X \otimes^{\mathbf{L}}_R F}{\searrow} & \\ M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F) & \xrightarrow{\xi^M_{X\otimes^{\mathbf{L}}_R F}} & X \otimes^{\mathbf{L}}_R F \end{array} \tag{4.3.1}$$

Note that if $X \otimes^{\mathbf{L}}_R F \in \mathcal{D}_{\mathrm{b}}(R)$ and $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ and at least one of the conditions (1)–(3) holds, then $X \in \mathcal{D}_{\mathrm{b}}(R)$, by [36, 3.13(b), 4.1(c), and 5.2(c)]. Since we have $\mathcal{B}_M(R) \subseteq \mathcal{D}_{\mathrm{b}}(R)$, we assume without loss of generality that $X \in \mathcal{D}_{\mathrm{b}}(R)$.

Claim 1: If $\mathbf{R}\mathrm{Hom}_R(M,X) \in \mathcal{D}_{\mathrm{b}}(R)$, then $\mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F) \in \mathcal{D}_{\mathrm{b}}(R)$. Assume that $\mathbf{R}\mathrm{Hom}_R(M,X) \in \mathcal{D}_{\mathrm{b}}(R)$. Since Fact 2.5.2 implies that we have

$$\text{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F)) \subseteq \mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$$

it suffices to show that $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F) \in \mathcal{D}_{\mathrm{b}}(R)$, by Fact 2.6. The next isomorphism from the first paragraph of this proof

$$K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F) \xrightarrow[\simeq]{K\otimes^{\mathbf{L}}_R \omega_{MXF}} K \otimes^{\mathbf{L}}_R (\mathbf{R}\mathrm{Hom}_R(M,X) \otimes^{\mathbf{L}}_R F)$$

implies that it suffices for us to show that $K \otimes^{\mathbf{L}}_R (\mathbf{R}\mathrm{Hom}_R(M,X) \otimes^{\mathbf{L}}_R F) \in \mathcal{D}_{\mathrm{b}}(R)$. The conditions $\mathbf{R}\mathrm{Hom}_R(M,X) \in \mathcal{D}_{\mathrm{b}}(R)$ and $\mathrm{fd}_R(K), \mathrm{fd}_R(F) < \infty$ guarantee that $K \otimes^{\mathbf{L}}_R (\mathbf{R}\mathrm{Hom}_R(M,X) \otimes^{\mathbf{L}}_R F) \in \mathcal{D}_{\mathrm{b}}(R)$, so the claim is established.

Claim 2: If $\mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F) \in \mathcal{D}_{\mathrm{b}}(R)$ and any of the conditions (1)–(3) hold, then $\mathbf{R}\mathrm{Hom}_R(M,X) \in \mathcal{D}_{\mathrm{b}}(R)$. Assume that $\mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F) \in \mathcal{D}_{\mathrm{b}}(R)$ and at least one of the conditions (1)–(3) holds. As in the proof of Claim 1, this yields $(K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,X)) \otimes^{\mathbf{L}}_R F \in \mathcal{D}_{\mathrm{b}}(R)$.

We show how each of the conditions (1)–(3) implies that $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,X) \in \mathcal{D}_{\mathrm{b}}(R)$. In cases (1) and (3), this is from [36, Theorem 3.13(b) and Proposition 5.2(c)]. In case (2), due to the assumption $X \in \mathcal{D}_{\mathrm{b}}(R)$ from the first paragraph of this proof, we have $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,X) \in \mathcal{D}^{\mathrm{f}}(R)$ by Lemma 2.2. Thus, in this case the boundedness of $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,X)$ is from [36, Theorem 4.1(c)].

Now, the condition $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,X) \in \mathcal{D}_{\mathrm{b}}(R)$ implies that $\mathbf{R}\mathrm{Hom}_R(M,X) \in \mathcal{D}_{\mathrm{b}}(R)$ by Facts 2.5.2 and 2.6. This establishes Claim 2.

Claim 3: If $X \in \mathcal{B}_M(R)$, then $X \otimes^{\mathbf{L}}_R F \in \mathcal{B}_M(R)$ and $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Assume that $X \in \mathcal{B}_M(R)$. Foxby Equivalence 3.6(b) implies that $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, and Claim 1 implies that $\mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F) \in \mathcal{D}_{\mathrm{b}}(R)$. Since X is in $\mathcal{B}_M(R)$, the morphism ξ^M_X is an isomorphism, hence so are $\xi^M_X \otimes^{\mathbf{L}}_R F$ and $\xi^M_{X\otimes^{\mathbf{L}}_R F}$, because of Eq. 4.3.1. Thus, we have $X \otimes^{\mathbf{L}}_R F \in \mathcal{B}_M(R)$, and Claim 3 is established.

We complete the proof by assuming that $X \otimes^{\mathbf{L}}_R F \in \mathcal{B}_M(R)$ and $\mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ and at least one of the conditions (1)–(3) holds, and we prove that $X \in \mathcal{B}_M(R)$. Our assumptions imply that $\mathbf{R}\mathrm{Hom}_R(M, X \otimes^{\mathbf{L}}_R F) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, so Claim 2 implies that $\mathbf{R}\mathrm{Hom}_R(M,X) \in \mathcal{D}_{\mathrm{b}}(R)$. Furthermore, the morphism $\xi^M_{X\otimes^{\mathbf{L}}_R F}$ is an isomorphism, hence so is $\xi^M_X \otimes^{\mathbf{L}}_R F$, because of Eq. 4.3.1.

We show how each of the conditions (1)–(3) implies that ξ^M_X is an isomorphism as well. Note that the domain and co-domain of ξ^M_X have their supports contained in $\mathrm{V}(\mathfrak{a})$, one by assumption and the other by Fact 2.5.2. Thus, following the logic of Fact 2.8, we conclude from Fact 2.5.2 and [36, Proposition 5.2(b)] that ξ^M_X is an isomorphism in cases (1) and (3). In case (2), we use [36, Theorem 4.1(b)] similarly; for this, we need to show that the complexes $K \otimes^{\mathbf{L}}_R X$ and $K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, X)$ are in $\mathcal{D}^{\mathrm{f}}(R)$. The first of these is from assumption (2), and the second one is by Lemma 2.2, since we have $\mathbf{R}\mathrm{Hom}_R(M, X) \in \mathcal{D}_{\mathrm{b}}(R)$. □

Our next result is a version of Theorem 4.3 for Auslander classes.

Theorem 4.4 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, with $\mathfrak{a}$ a proper ideal of R. Let $F \in \mathcal{D}_b(R)$ be such that $\mathrm{fd}_R(F) < \infty$, and let $X \in \mathcal{D}(R)$. If $X \in \mathcal{A}_M(R)$, then $\mathbf{L}\Lambda^{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R F) \in \mathcal{A}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The converse of this statement holds when at least one of the conditions* (1)–(3) *from* Theorem 4.3 *holds.*

Proof By Fact 2.7 we have the following isomorphism in $\mathcal{D}(R)$

$$M \otimes^{\mathbf{L}}_R X \otimes^{\mathbf{L}}_R F \simeq M \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R F). \tag{4.4.1}$$

For the forward implication, assume that $X \in \mathcal{A}_M(R)$. Foxby Equivalence 3.6(c) implies that $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ and that $M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$. Thus, we have $M \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R F) \simeq M \otimes^{\mathbf{L}}_R X \otimes^{\mathbf{L}}_R F \in \mathcal{B}_M(R)$, by Eq. 4.4.1 and Theorem 4.3. Fact 2.5.2 implies that $\text{co-supp}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R F)) \subseteq \mathrm{V}(\mathfrak{a})$, so another application of Foxby Equivalence 3.6(c) implies that $\mathbf{L}\Lambda^{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R F) \in \mathcal{A}_M(R)$.

The converse is handled similarly, as follows. Assume that $\mathbf{L}\Lambda^{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R F) \in \mathcal{A}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Assume also that at least one of the conditions (1)–(3) from Theorem 4.3 is satisfied.

Assume for this paragraph that condition (2) from Theorem 4.3 is satisfied. Then we have $\mathrm{supp}_R(F) \subseteq \mathrm{V}(\mathfrak{a})$ by [37, Lemma 5.3], so Fact 2.5.2 implies that $\mathrm{supp}_R(X \otimes^{\mathbf{L}}_R F) \subseteq \mathrm{supp}_R(F) \subseteq \mathrm{V}(\mathfrak{a})$. The fact that we have $\mathbf{L}\Lambda^{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R F) \in \mathcal{D}_{\mathrm{b}}(R)$ implies that $X \otimes^{\mathbf{L}}_R F \in \mathcal{D}_{\mathrm{b}}(R)$ by [36, Corollary 3.10(b)]; so $X \in \mathcal{D}_{\mathrm{b}}(R)$ by [36, Theorem 3.13(b)]. From Lemma 2.2, it follows that $K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}(R)$.

Foxby Equivalence 3.6(c) and the isomorphism (4.4.1) imply that $M \otimes^{\mathbf{L}}_R X \otimes^{\mathbf{L}}_R F \simeq M \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R F) \in \mathcal{B}_M(R)$. Since $\mathrm{supp}_R(M \otimes^{\mathbf{L}}_R X) \subseteq \mathrm{V}(\mathfrak{a})$, Theorem 4.3 implies that $M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$; in case (2), this uses the condition $K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}(R)$ from the previous paragraph. Another application of Foxby Equivalence 3.6(c) implies that $X \in \mathcal{A}_M(R)$. □

The presence of $\mathbf{L}\Lambda^{\mathfrak{a}}$ in the previous result may be a bit unsettling. However, it is a necessary consequence of the co-support condition in Foxby Equivalence 3.6(c); we see in the next example that it is unavoidable in general, even over a very nice ring. It can be gotten around, though, in the special case $F \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, as we show in the subsequent corollary.

Example 4.5 Let k be a field, and set $R := k[\![Y]\!]$ with $E := E_R(k)$ and $\mathfrak{a} := (Y)R$. Since R is Gorenstein and local, we have $\mathrm{fd}_R(E) < \infty$. Also, we have $R \in \mathcal{A}_E(R)$ by Proposition 3.2, but $E \otimes^{\mathbf{L}}_R R \simeq E \notin \mathcal{A}_E(R)$ by Example 3.7.

Corollary 4.6 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R. Let $F \in \mathcal{D}^f_b(R)$ be such that $\mathrm{fd}_R(F) < \infty$, and let $X \in \mathcal{D}(R)$. If $X \in \mathcal{A}_M(R)$, then $X \otimes^{\mathbf{L}}_R F \in \mathcal{A}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The converse of this statement holds when at least one of the following conditions is satisfied.*

(1) $\mathrm{supp}_R(F) = \mathrm{V}(\mathfrak{a})$.
(2) $\mathrm{supp}_R(F) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$ *and* $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$.

(3) *F is a flat R-module with* $\operatorname{supp}_R(F) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$*, e.g., F is free.*

Proof If $X \in \mathcal{A}_M(R)$, then $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ by Foxby Equivalence 3.6(c). So we assume without loss of generality that $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Because of this, our assumptions on F imply that $\text{co-supp}_R(X \otimes^{\mathbf{L}}_R F) \subseteq \text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, by Lemma 2.13. Fact 2.5.3 implies that $X \otimes^{\mathbf{L}}_R F \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R F)$, so the desired conclusions follow from Theorem 4.4. □

Remark 4.7 Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and let $X \in \mathcal{D}(R)$.

The Koszul complex K satisfies condition (1) of Theorem 4.3 and Corollary 4.6. Thus, we have $X \in \mathcal{B}_M(R)$ if and only if $K \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ and $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$; and we have $X \in \mathcal{A}_M(R)$ if and only if $K \otimes^{\mathbf{L}}_R X \in \mathcal{A}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

One can similarly use Theorem 4.3 to conclude that we have $X \in \mathcal{B}_M(R)$ if and only if $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \in \mathcal{B}_M(R)$ and $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$; however this true for more trivial reasons. Indeed, If $X \in \mathcal{B}_M(R)$, then Foxby Equivalence 3.6(b) implies $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, so we have $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq X \in \mathcal{B}_M(R)$ by Fact 2.5.3. Conversely, if $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \in \mathcal{B}_M(R)$ and $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, then Fact 2.5.3 implies that $X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(X) \in \mathcal{B}_M(R)$. Similarly, we have $X \in \mathcal{A}_M(R)$ if and only if $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \in \mathcal{A}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

It is not clear that the converse statements of this section have been documented in the case of semidualizing complexes (that is, the case $\mathfrak{a} = 0$). We write this out explicitly for Theorem 4.3 and leave the remaining cases for the interested reader.

Corollary 4.8 *Let C be a semidualizing R-complex. Let $F \in \mathcal{D}_b(R)$ be such that* $\operatorname{fd}_R(F) < \infty$*, and let $X \in \mathcal{D}(R)$. If $X \in \mathcal{B}_C(R)$, then $X \otimes^{\mathbf{L}}_R F \in \mathcal{B}_C(R)$. The converse of this statement holds when at least one of the following conditions holds.*

(1) *$F \in \mathcal{D}^f_b(R)$ satisfies* $\operatorname{supp}_R(F) = \operatorname{Spec}(R)$.
(2) *There is a homomorphism $\varphi\colon R \to S$ of commutative noetherian rings such that $F \in \mathcal{D}^f_b(S)$ satisfies* $\varphi^*(\operatorname{supp}_S(F)) \supseteq \text{m-Spec}(R)$*, and $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$, where K is the Koszul complex over R on a generating sequence for $\mathfrak{a}$.*
(3) *F is a faithfully flat R-module.*

4.3 Finite Projective Dimension

We now turn our attention to stability with respect to $\mathbf{R}\mathrm{Hom}_R(P, -)$ where P is an R-complex of finite projective dimension.

Theorem 4.9 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R. Let $P \in \mathcal{D}_b(R)$ be such that* $\operatorname{pd}_R(P) < \infty$*, and let $X \in \mathcal{D}(R)$. If $X \in \mathcal{A}_M(R)$, then* $\mathbf{R}\mathrm{Hom}_R(P, X) \in \mathcal{A}_M(R)$ *and* $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. *The converse of this statement holds when at least one of the following conditions is satisfied.*

(1) *P is $\mathfrak{a}$-adically finite such that* $\operatorname{supp}_R(P) = \mathrm{V}(\mathfrak{a})$.
(2) *P is $\mathfrak{a}$-adically finite with* $\operatorname{supp}_R(P) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$ *and $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$ where K is the Koszul complex over R on a generating sequence for $\mathfrak{a}$.*
(3) *P is a projective R-module with* $\operatorname{supp}_R(P) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$*, e.g., P is faithfully projective, e.g., free.*

Proof The proof of this result is very similar to that of Theorem 4.3, so we only sketch it, highlighting the differences. If $X \in \mathcal{A}_M(R)$, then Foxby Equivalence 3.6(c) implies that $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Thus, we assume throughout that $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. From this, if $\mathbf{R}\mathrm{Hom}_R(P, X) \in \mathcal{D}_\mathrm{b}(R)$ and at least one of the conditions (1)–(3) holds, then we have $X \in \mathcal{D}_\mathrm{b}(R)$, by [36, 3.11(b), 4.5(c), and 5.1(e)]. Thus, we assume throughout that $X \in \mathcal{D}_\mathrm{b}(R)$.

The following isomorphism is tensor-evaluation [10, Proposition 2.2(vi)].

$$K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(P, X) \simeq \mathbf{R}\mathrm{Hom}_R(P, K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R X)$$

Thus, if $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_\mathrm{b}(R)$, then $K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(P, X) \in \mathcal{D}_\mathrm{b}(R)$; and by Fact 2.6, the condition $\operatorname{supp}_R(M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(P, X)) \subseteq \operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$ implies that we have $M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(P, X) \in \mathcal{D}_\mathrm{b}(R)$. Conversely, if $M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(P, X) \in \mathcal{D}_\mathrm{b}(R)$ and at least one of the conditions (1)–(3) is satisfied, then the complex $\mathbf{R}\mathrm{Hom}_R(P, K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R X)$ is in $\mathcal{D}_\mathrm{b}(R)$, and it follows from Lemma 2.2 and [36, 3.9(b), 4.3(c), and 5.1(e)] that $K \otimes^{\mathbf{L}}_R M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_\mathrm{b}(R)$; from this, we have $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_\mathrm{b}(R)$ by Fact 2.6. Thus, we assume throughout that $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_\mathrm{b}(R)$.

Next, we consider the following commutative diagram in $\mathcal{D}(R)$

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_R(P, X) & \xrightarrow[\simeq]{\gamma^M_{\mathbf{R}\mathrm{Hom}_R(P,X)}} & \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(P, X)) \\
\Big\downarrow {\scriptstyle \mathbf{R}\mathrm{Hom}_R(P, \gamma^M_X)} & & \simeq \Big\downarrow {\scriptstyle \mathbf{R}\mathrm{Hom}_R(M, \omega_{MPX})} \\
\mathbf{R}\mathrm{Hom}_R(P, \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)) & \xrightarrow{\simeq} & \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_R(P, M \otimes^{\mathbf{L}}_R X))
\end{array}$$

wherein ω_{MPX} is tensor-evaluation [37, Theorem 4.3(b)] and the unspecified isomorphism is "swap", i.e., a composition of adjointness isomorphisms. Thus, if γ^M_X is an isomorphism, then so is $\gamma^M_{\mathbf{R}\mathrm{Hom}_R(P,X)}$. Conversely, assume that $\gamma^M_{\mathbf{R}\mathrm{Hom}_R(P,X)}$ is an isomorphism and at least one of the conditions (1)–(3) holds. It follows that $\mathbf{R}\mathrm{Hom}_R(P, \gamma^M_X)$ is an isomorphism. Since the source and target of γ^M_X both have their co-supports contained in $\mathrm{V}(\mathfrak{a})$, Fact 2.5.2 and [36, Proposition 5.1(d)] show that γ^M_X is an isomorphism in cases (1) and (3), respectively; see Fact 2.8. In case (2), we use [36, Theorem 4.3(b)] similarly; for this, we need to show that $K \otimes^{\mathbf{L}}_R X, K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X) \in \mathcal{D}^\mathrm{f}(R)$. The first of these is from assumption (2), and the second one is by Lemma 2.2, as we have $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_\mathrm{b}(R)$. □

Theorem 4.10 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, with $\mathfrak{a}$ a proper ideal of R. Let $X \in \mathcal{D}(R)$ and $P \in \mathcal{D}_b(R)$ be such that $\mathrm{pd}_R(P) < \infty$. If $X \in \mathcal{B}_M(R)$, then $\mathbf{R}\Gamma_\mathfrak{a}(\mathbf{R}\mathrm{Hom}_R(P, X)) \in \mathcal{B}_M(R)$ and $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The converse of this statement holds when at least one of the conditions* (1)–(3) *from* Theorem 4.9 *holds.*

Proof Again, we sketch the proof. By Foxby Equivalence 3.6(b), we assume without loss of generality that $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. From Fact 2.7, we have the first isomorphism in the next sequence

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\Gamma_\mathfrak{a}(\mathbf{R}\mathrm{Hom}_R(P, X))) &\simeq \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_R(P, X)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(P, \mathbf{R}\mathrm{Hom}_R(M, X)).
\end{aligned}$$

The second isomorphism is swap.

Now, for the forward implication, assume that $X \in \mathcal{B}_M(R)$. Then we have $\mathbf{R}\mathrm{Hom}_R(M, X) \in \mathcal{A}_M(R)$, by Foxby Equivalence 3.6(b). Theorem 4.9 implies that $\mathbf{R}\mathrm{Hom}_R(P, \mathbf{R}\mathrm{Hom}_R(M, X)) \in \mathcal{A}_M(X)$. From the above isomorphisms, it follows that we have $\mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(P, X))) \in \mathcal{A}_M(X)$. Fact 2.5.2 implies that we also have $\mathrm{supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(P, X))) \subseteq \mathrm{V}(\mathfrak{a})$, so we conclude that $\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(P, X)) \in \mathcal{B}_M(R)$ by Foxby Equivalence 3.6(b). This completes the proof of the forward implication.

Assume for this paragraph that $\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(P, X)) \in \mathcal{D}_b(R)$ and condition (2) from Theorem 4.9 is satisfied. Fact 2.5.2 implies that

$$\text{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(P, X)) \subseteq \mathrm{supp}_R(P) \subseteq \mathrm{V}(\mathfrak{a})$$

so we have $\mathbf{R}\mathrm{Hom}_R(P, X) \in \mathcal{D}_b(R)$ by [36, Corollary 3.15(b)]; we conclude that $X \in \mathcal{D}_b(R)$ by [36, Theorem 4.5(c)]. In particular, we have $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, X) \in \mathcal{D}^{\mathrm{f}}(R)$ by Lemma 2.2.

Now, for the converse, assume that $\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(P, X)) \in \mathcal{B}_M(R)$ and at least one of the conditions (1)–(3) from Theorem 4.9 holds. Foxby Equivalence 3.6(b) and the isomorphisms above imply that

$$\mathbf{R}\mathrm{Hom}_R(P, \mathbf{R}\mathrm{Hom}_R(M, X)) \simeq \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(P, X))) \in \mathcal{A}_M(X).$$

Since $\text{co-supp}_R(\mathbf{R}\mathrm{Hom}_R(M, X)) \subseteq \mathrm{V}(\mathfrak{a})$ by Fact 2.5.2, we have $\mathbf{R}\mathrm{Hom}_R(M, X) \in \mathcal{A}_M(R)$ by Theorem 4.9; in case (2), this uses the condition $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, X) \in \mathcal{D}^{\mathrm{f}}(R)$ from the previous paragraph. A final application of Foxby Equivalence 3.6(b) implies that $X \in \mathcal{B}_M(R)$. □

The next example shows that one cannot drop the $\mathbf{R}\Gamma_{\mathfrak{a}}$ from Theorem 4.10, even when P is free. See, however, Corollary 4.12 for the special case $P \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

Example 4.11 Let k be a field, and set $R := k[\![Y]\!]$ with $E := E_R(k)$. Then we have $\mathbf{R}\mathrm{Hom}_R(R^{(\mathbb{N})}, E) \simeq E^{\mathbb{N}} \notin \mathcal{B}_E(R)$ and $E \in \mathcal{B}_E(R)$, by Example 4.1.

Corollary 4.12 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, with $\mathfrak{a}$ a proper ideal of R. Let $P \in \mathcal{D}^f_b(R)$ be such that $\mathrm{pd}_R(P) < \infty$, and let $X \in \mathcal{D}(R)$. If $X \in \mathcal{B}_M(R)$, then one has $\mathbf{R}\mathrm{Hom}_R(P, X) \in \mathcal{B}_M(R)$ and $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The converse of this statement holds when at least one of the following conditions is satisfied.*

(1) $\mathrm{supp}_R(P) = \mathrm{V}(\mathfrak{a})$.
(2) $\mathrm{supp}_R(P) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, *and* $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$.
(3) *P is a projective R-module with* $\mathrm{supp}_R(P) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, *e.g., P is free.*

Proof Again, assume without loss of generality that $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Because of this, our assumptions on P imply that $\mathrm{supp}_R(\mathbf{R}\mathrm{Hom}_R(P, X)) \subseteq \mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, by Lemma 2.13. Fact 2.5.3 implies that $\mathbf{R}\mathrm{Hom}_R(P, X) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(P, X))$. Thus, the desired conclusions follow from Theorem 4.10. □

4.4 Finite Injective Dimension

The next three results are verified like earlier ones.[3] For perspective in these results, we recall the following.

[3] In Corollary 4.17, use [38, Proposition 3.16] to conclude that $\mathrm{supp}_R(\mathbf{R}\mathrm{Hom}_R(X, I)) \subseteq \mathrm{V}(\mathfrak{a})$.

Remark 4.13 Let J be an injective R-module, with Matlis decomposition $J \cong \bigoplus_{\mathfrak{p}\in\mathrm{Spec}(R)} E_R(R/\mathfrak{p})^{(\mu_\mathfrak{p})}$. Then one has

$$\mathrm{supp}_R(J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mu_\mathfrak{p} \neq \emptyset\}$$
$$\text{co-supp}_R(J) = \{` \in{}' \mathrm{Spec}(R) \mid \text{there is a } \mathfrak{p} \in \mathrm{Spec}(R) \text{ such that } ` \subseteq{}' \mathfrak{p} \text{ and } \mu_\mathfrak{p} \neq \emptyset\}$$

by [38, Propositions 3.8 and 6.3]. From this, one verifies readily that, if $\mathfrak{a}$ is a proper ideal of R, then

(a) $\mathrm{supp}_R(J) \subseteq \text{co-supp}_R(J)$, and
(b) $\mathrm{V}(\mathfrak{a}) \subseteq \text{co-supp}_R(J)$ if and only if $\mathrm{supp}_R(J) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$.

Theorem 4.14 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R. Let $I \in \mathcal{D}_b(R)$ be such that $\mathrm{id}_R(I) < \infty$, and let $X \in \mathcal{D}(R)$. If $X \in \mathcal{B}_M(R)$, then $\mathbf{R}\mathrm{Hom}_R(X, I) \in \mathcal{A}_M(R)$ and $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The converse of this statement holds when I is an injective R-module with $\text{co-supp}_R(I) \supseteq \mathrm{V}(\mathfrak{a})$, e.g., I is faithfully injective.*

Theorem 4.15 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R. Let $X \in \mathcal{D}(R)$, and let $I \in \mathcal{D}_b(R)$ be such that $\mathrm{id}_R(I) < \infty$. If $X \in \mathcal{A}_M(R)$, then $\mathbf{R}\Gamma_\mathfrak{a}(\mathbf{R}\mathrm{Hom}_R(X, I)) \in \mathcal{B}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The converse of this statement holds when I is an injective R-module with $\text{co-supp}_R(I) \supseteq \mathrm{V}(\mathfrak{a})$, e.g., I is faithfully injective.*

The next example shows that one cannot avoid the $\mathbf{R}\Gamma_\mathfrak{a}$ in Theorem 4.15, even in very nice situations.

Example 4.16 Let k be a field, and set $R := k[\![Y]\!]$ with $E := E_R(k)$ and $I := E_R(R)$. Remark 4.13 shows that we have $\mathrm{supp}_R(I) = \{0\} \not\subseteq \mathrm{V}(\mathfrak{a})$. Then Foxby Equivalence 3.6(b) implies that $\mathbf{R}\mathrm{Hom}_R(R, I) \simeq I \notin \mathcal{B}_E(R)$, even though by Proposition 3.2 implies that we have $R \in \mathcal{A}_E(R)$.

Corollary 4.17 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, with $\mathfrak{a}$ a proper ideal of R. Let $X \in \mathcal{D}^f(R)$ and $I \in \mathcal{D}_b(R)$ be such that $\mathrm{id}_R(I) < \infty$ and $\mathrm{supp}_R(I) \subseteq \mathrm{V}(\mathfrak{a})$. If $X \in \mathcal{A}_M(R)$, then $\mathbf{R}\mathrm{Hom}_R(X, I) \in \mathcal{B}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The converse of this statement holds when I is an injective R-module such that $\text{co-supp}_R(I) \supseteq \mathrm{V}(\mathfrak{a})$.*

Remark 4.18 One has to be a bit careful with the converse in Corollary 4.17 to make sure that one satisfies both assumptions $\mathrm{supp}_R(I) \subseteq \mathrm{V}(\mathfrak{a}) \subseteq \text{co-supp}_R(I)$. For instance, this will fail in general when I is faithfully injective, by Remark 4.13.

Remark 4.19 One can use the results in this section in a variety of combinations. For instance, combining Theorems 4.3 and 4.14, one obtains the following.

Let M be an $\mathfrak{a}$-adic semidualizing R-complex, with $\mathfrak{a}$ a proper ideal of R. Let $F \in \mathcal{D}_\mathrm{b}(R)$ be such that $\mathrm{fd}_R(F) < \infty$, and let $X \in \mathcal{D}(R)$. Let I be an injective R-module with $\text{co-supp}_R(I) \supseteq \mathrm{V}(\mathfrak{a})$, e.g., I is faithfully injective, and set $J = \mathbf{R}\mathrm{Hom}_R(F, I)$. If $X \in \mathcal{B}_M(R)$, then $\mathbf{R}\mathrm{Hom}_R(X, I) \in \mathcal{A}_M(R)$ and $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. The converse holds if at least one of the conditions (1)–(3) from Theorem 4.3 holds.

Indeed, one has $\mathrm{id}_R(J) < \infty$, so the forward implication follows directly from Theorem 4.14 Also, by definition and Hom-tensor adjointness, we have

$$\mathbf{R}\mathrm{Hom}_R(X, J) \simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_R(F, I)) \simeq \mathbf{R}\mathrm{Hom}_R(X \otimes^{\mathbf{L}}_R F, I)$$

so the converse follows by applying first Theorem 4.14 and then Theorem 4.3. We leave other variations on this theme to the interested reader.

5 Base Change

This section focuses on some transfer properties for Foxby classes. It contains Theorem 1.3 from the introduction.

Notation 5.1 In this section, let $\varphi\colon R \to S$ be a homomorphism of commutative noetherian rings, where $\mathfrak{a}$ is a proper ideal of R such that $\mathfrak{a}S \neq S$, and let $Q\colon \mathcal{D}(S) \to \mathcal{D}(R)$ be the forgetful functor.

If C is a semidualizing R-complex, then [8, Theorem 5.1] says that $S \in \mathcal{A}_C(R)$ if and only if $S \otimes^{\mathbf{L}}_R C$ is a semidualizing S-complex. Proposition 3.10 shows that things are not so simple for adic semidualizing complexes. Specifically, assume that R is not $\mathfrak{a}$-adically complete, and let M be an $\mathfrak{a}$-adic semidualizing R-complex. Then $M \simeq R \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}$-adically semidualizing over R, but $R \notin \mathcal{A}_M(R)$ by Proposition 3.10. As one might expect, the missing ingredient involves co-support, embodied in the completeness condition in our next result.

Theorem 5.2 *Let $\varphi\colon R \to S$ be a homomorphism of commutative noetherian rings, where $\mathfrak{a}$ is a proper ideal of R such that $\mathfrak{a}S \neq S$. Let M be $\mathfrak{a}$-adic semidualizing over R. Then $S \in \mathcal{A}_M(R)$ if and only if $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically semidualizing over S and S is $\mathfrak{a}S$-adically complete.*

Proof For this paragraph, assume that S is $\mathfrak{a}S$-adically complete and $S \otimes^{\mathbf{L}}_R M \in \mathcal{D}_{\mathrm{b}}(S)$. Since M is $\mathfrak{a}$-adically semidualizing, we have $\operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$, and hence $\operatorname{supp}_S(S \otimes^{\mathbf{L}}_R M) \subseteq \mathrm{V}(\mathfrak{a}S)$ by [37, Lemma 5.7]. Thus, the morphism $\chi^S_{S\otimes^{\mathbf{L}}_R M} = \chi^{\circ S^{\mathfrak{a}S}}_{S\otimes^{\mathbf{L}}_R M}$ is defined, and we have the following commutative diagram in $\mathcal{D}(S)$:

$$
\begin{array}{ccc}
S\gamma^M_S & \xrightarrow{\quad \chi^S_{S\otimes^{\mathbf{L}}_R M} \quad} & \\
{\scriptstyle -}\big\downarrow & & \searrow \\
\mathbf{R}\mathrm{Hom}_R(M, S\otimes^{\mathbf{L}}_R M) & \xrightarrow[-]{\simeq} & \mathbf{R}\mathrm{Hom}_S(S\otimes^{\mathbf{L}}_R M, S\otimes^{\mathbf{L}}_R M).
\end{array}
\tag{5.2.1}
$$

The unspecified isomorphism is from Hom-tensor adjointness.

Now, for the forward implication, assume that $S \in \mathcal{A}_M(R)$. Foxby Equivalence 3.6(c) implies that $\operatorname{co-supp}_R(S) \subseteq \mathrm{V}(\mathfrak{a})$, so we conclude that $\operatorname{co-supp}_S(S) \subseteq \mathrm{V}(\mathfrak{a}S)$ by [37, Lemma 5.4]. It follows from Fact 2.5.3 that the natural morphism $S \to \mathbf{L}\Lambda^{\mathfrak{a}S}(S)$ is an isomorphism in $\mathcal{D}(S)$, i.e., S is $\mathfrak{a}S$-adically complete. Furthermore, the condition $S \in \mathcal{A}_M(R)$ implies by definition that $S \otimes^{\mathbf{L}}_R M \in \mathcal{D}_{\mathrm{b}}(R)$ and γ^M_S is an isomorphism. In particular, we are in the situation of the first paragraph of this proof, and the diagram (5.2.1) implies that $\chi^S_{S\otimes^{\mathbf{L}}_R M}$ is an isomorphism. Since M is $\mathfrak{a}$-adically finite over R, [37, Theorem 5.10] implies that $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically finite over S, so $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically semidualizing over S, as desired.

For the converse, assume that $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically semidualizing over S and S is $\mathfrak{a}S$-adically complete. In particular, we are in the situation of the first paragraph of this proof, and the morphism $\chi^S_{S\otimes^{\mathbf{L}}_R M}$ is an isomorphism in $\mathcal{D}(S)$. The diagram (5.2.1) implies that γ^M_S is an isomorphism, so $S \in \mathcal{A}_M(R)$, as desired. □

Theorem 5.2 gives some perspective on the $\mathfrak{a}S$-adically semidualizing condition for $S\otimes^{\mathbf{L}}_R M$ in the next few results, as does [34, Theorem 5.6], which says that $S\otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically semidualizing over S whenever $\operatorname{fd}_R(S) < \infty$.

Proposition 5.3 *Let $\varphi\colon R \to S$ be a homomorphism of commutative noetherian rings, where $\mathfrak{a}$ is a proper ideal of R such that $\mathfrak{a}S \neq S$. Let M be an $\mathfrak{a}$-adic semidualizing R-complex. Assume that $S \otimes^{\mathbf{L}}_R M$ is $\mathfrak{a}S$-adically semidualizing over S, and let Y be an S-complex.*

(a) *One has $Y \in \mathcal{A}_{S\otimes^{\mathbf{L}}_R M}(S)$ if and only if $Q(Y) \in \mathcal{A}_M(R)$.*
(b) *One has $Y \in \mathcal{B}_{S\otimes^{\mathbf{L}}_R M}(S)$ if and only if $Q(Y) \in \mathcal{B}_M(R)$.*

Proof Argue as in the proof of [8, Proposition 5.3]. □

Remark 5.4 Let $\varphi\colon R \to S$ be a homomorphism of commutative noetherian rings, where $\mathfrak{a}$ is a proper ideal of R such that $\mathfrak{a}S \neq S$. For perspective in some of our subsequent results, note that [37, Proposition 5.6(a)] implies that $\operatorname{supp}_R(S) = \varphi^*(\operatorname{Spec}(S))$.

Also, we have $K \otimes^{\mathbf{L}}_R S \in \mathcal{D}^{\mathrm{f}}(R)$ if and only if the induced map $R/\mathfrak{a} \to S/\mathfrak{a}S$ is module-finite, where K is the Koszul complex over R on a generating sequence for $\mathfrak{a}$. (For instance, this is satisfied when S is module-finite over R or when $S = \widehat{R}^{\mathfrak{b}}$ for some ideal $\mathfrak{b} \subseteq \mathfrak{a}$.) Indeed, we have $\mathrm{H}_0(K \otimes^{\mathbf{L}}_R S) \cong S/\mathfrak{a}S$; thus, if $K \otimes^{\mathbf{L}}_R S \in \mathcal{D}^{\mathrm{f}}(R)$, then $S/\mathfrak{a}S$ is finitely generated over R, hence over $R/\mathfrak{a}R$. For the converse, note that each module $\mathrm{H}_i(K \otimes^{\mathbf{L}}_R S)$ is finitely generated over S, hence over $S/\mathfrak{a}S$; thus, if the induced map $R/\mathfrak{a} \to S/\mathfrak{a}S$ is module-finite these homology modules are finitely generated over $R/\mathfrak{a}$, hence over R.

It follows that S is $\mathfrak{a}$-adically finite over R if and only if $\operatorname{supp}_R(S) \subseteq \mathrm{V}(\mathfrak{a})$ and the induced map $R/\mathfrak{a} \to S/\mathfrak{a}S$ is module-finite.

5.1 Base Change for Bass Classes

Here is Theorem 1.3 from the introduction.

Theorem 5.5 *Let $\varphi\colon R \to S$ be a homomorphism of commutative noetherian rings, where $\mathfrak{a}$ is a proper ideal of R such that $\mathfrak{a}S \neq S$. Let M be an $\mathfrak{a}$-adic semidualizing R-complex, and assume that $\operatorname{fd}_R(S) < \infty$. Let $X \in \mathcal{D}(R)$ be given, and consider the following conditions.*

(i) $X \in \mathcal{B}_M(R)$.
(ii) $S \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ *and* $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.
(iii) $S \otimes^{\mathbf{L}}_R X \in \mathcal{B}_{S\otimes^{\mathbf{L}}_R M}(S)$ *and* $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

Then we have (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii). *The conditions* (i)–(iii) *are equivalent when at least one of the following conditions is satisfied.*

(1) *S is $\mathfrak{a}$-adically finite over R such that* $\operatorname{supp}_R(S) = \mathrm{V}(\mathfrak{a})$.
(2) $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \operatorname{m-Spec}(R)$, *and* $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$, *where K is the Koszul complex over R on a generating sequence for $\mathfrak{a}$.*
(3) *S is flat over R with* $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \operatorname{m-Spec}(R)$, *e.g., S is faithfully flat.*

Proof The implication (i) $\Longrightarrow$ (ii) and its conditional converse follow from Theorem 4.3. The equivalence (ii) $\Longleftrightarrow$ (iii) is from Proposition 5.3(a). □

The point of the next result is to shift the support conditions in Theorem 5.5.

Corollary 5.6 *Let* $\varphi\colon R \to S$ *be a homomorphism of commutative noetherian rings, where* $\mathfrak{a}$ *is a proper ideal of* R *such that* $\mathfrak{a}S \neq S$. *Let* M *be an* $\mathfrak{a}$*-adic semidualizing* R*-complex, and assume that* $\operatorname{fd}_R(S) < \infty$. *Let* $X \in \mathcal{D}(R)$ *be such that* $\operatorname{supp}_R(X) \subseteq \operatorname{supp}_R(S)$. *Consider the following conditions.*

(i) $X \in \mathcal{B}_M(R)$.
(ii) $S \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$.
(iii) $S \otimes^{\mathbf{L}}_R X \in \mathcal{B}_{S\otimes^{\mathbf{L}}_R M}(S)$.

Then we have (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii). *The conditions* (i)–(iii) *are equivalent when at least one of the following conditions is satisfied.*

(1) S *is* $\mathfrak{a}$*-adically finite such that* $\operatorname{supp}_R(S) = \operatorname{V}(\mathfrak{a})$.
(2) $\operatorname{supp}_R(S) \supseteq \operatorname{V}(\mathfrak{a}) \bigcap \operatorname{m-Spec}(R)$, *and* $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$, *where* K *is the Koszul complex over* R *on a generating sequence for* $\mathfrak{a}$.
(3) S *is flat over* R *with* $\operatorname{supp}_R(S) \supseteq \operatorname{V}(\mathfrak{a}) \bigcap \operatorname{m-Spec}(R)$, *e.g.,* S *is faithfully flat.*

Proof Conditions (ii) and (iii) are equivalent by Proposition 5.3(a), and the implication (i) $\Longrightarrow$ (ii) is from Theorem 4.3. By Theorem 5.5, it remains to assume that $S \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$, and show that $\operatorname{supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$. The assumption $\operatorname{supp}_R(X) \subseteq \operatorname{supp}_R(S)$ explains the first step in the next display, and the second step is from Fact 2.5.2.

$$\operatorname{supp}_R(X) = \operatorname{supp}_R(X) \bigcap \operatorname{supp}_R(S) = \operatorname{supp}_R(S \otimes^{\mathbf{L}}_R X) \subseteq \operatorname{V}(\mathfrak{a})$$

The last step here is from Foxby Equivalence 3.6(b), since $S \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$. □

Corollary 5.7 *Let* M *be an* $\mathfrak{a}$*-adic semidualizing* R*-complex, where* $\mathfrak{a}$ *is a proper ideal of* R. *Given an* R*-complex* $X \in \mathcal{D}(R)$, *the following conditions are equivalent.*

(i) $X \in \mathcal{B}_M(R)$.
(ii) $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ *and* $\operatorname{supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$.
(iii) $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X \in \mathcal{B}_{\widehat{R}^{\mathfrak{a}}\otimes^{\mathbf{L}}_R M}(\widehat{R}^{\mathfrak{a}})$ *and* $\operatorname{supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$.

Proof The completion $\widehat{R}^{\mathfrak{a}}$ is flat over R with $\operatorname{supp}_R(\widehat{R}^{\mathfrak{a}}) \supseteq \operatorname{V}(\mathfrak{a}) \bigcap \operatorname{m-Spec}(R)$. Thus, the desired result follows from Theorem 5.5, using condition (3). □

5.2 Base Change for Auslander Classes

Theorem 5.8 *Let* $\varphi\colon R \to S$ *be a homomorphism of commutative noetherian rings, where* $\mathfrak{a}$ *is a proper ideal of* R *such that* $\mathfrak{a}S \neq S$. *Let* M *be an* $\mathfrak{a}$*-adic semidualizing* R*-complex, and assume that* $\operatorname{fd}_R(S) < \infty$. *Let* $X \in \mathcal{D}(R)$ *be given, and consider the following conditions.*

(i) $X \in \mathcal{A}_M(R)$.
(ii) $\mathbf{L}\Lambda^{\mathfrak{a}}(S \otimes^{\mathbf{L}}_R X) \in \mathcal{A}_M(R)$ *and* $\operatorname{co-supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$.
(iii) $\mathbf{L}\Lambda^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_R X) \in \mathcal{A}_{S\otimes^{\mathbf{L}}_R M}(S)$ *and* $\operatorname{co-supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$.

Then we have (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii). *The conditions* (i)–(iii) *are equivalent when at least one of the following conditions is satisfied.*

(1) *S is $\mathfrak{a}$-adically finite over R such that* $\mathrm{supp}_R(S) = \mathrm{V}(\mathfrak{a})$.
(2) $\mathrm{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$*, and* $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$*, where K is the Koszul complex over R on a generating sequence for* $\mathfrak{a}$.
(3) *S is flat over R with* $\mathrm{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$*, e.g., S is faithfully flat.*

Proof By [37, Lemma 5.2], we have $Q(\mathbf{L}\Lambda^{\mathfrak{a}S}(S \otimes^{\mathbf{L}}_R X)) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(S \otimes^{\mathbf{L}}_R X)$ in $\mathcal{D}(R)$. Thus, one verifies the desired conclusions as in the proof of Theorem 5.5, using Theorem 4.4 and Proposition 5.3(a). □

One might expect a version of Corollary 5.6 to follow here. The key point of the proof of such a result would be to assume that $\text{co-supp}_R(X) \subseteq \mathrm{supp}_R(S)$ and $\mathbf{L}\Lambda^{\mathfrak{a}}(S \otimes^{\mathbf{L}}_R X) \in \mathcal{A}_M(R)$, and then show that $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. However, the next example shows that this implication fails in general.

Example 5.9 Let k be a field, and consider the localized polynomial ring $R = k[Y]_{Yk[Y]}$. Set $\mathfrak{a} := YR$ and $E := E_R(k)$. Since $\widehat{R}^{\mathfrak{a}}$ is faithfully flat over R and R is not $\mathfrak{a}$-adically complete, we have

$$\text{co-supp}_R(R) = \mathrm{Spec}(R) = \mathrm{supp}_R(\widehat{R}^{\mathfrak{a}}) \not\subseteq \mathrm{V}(\mathfrak{a})$$

by [38, Proposition 6.10]. On the other hand, we have

$$\mathbf{L}\Lambda^{\mathfrak{a}}(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R R) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(\widehat{R}^{\mathfrak{a}}) \simeq \widehat{R}^{\mathfrak{a}} \in \mathcal{A}_E(R)$$

by Proposition 3.2.

The next result is proved like Corollary 5.7, using Theorem 5.8.

Corollary 5.10 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and let $X \in \mathcal{D}(R)$ be given. Then the following conditions are equivalent.*

(i) $X \in \mathcal{A}_M(R)$.
(ii) $\mathbf{L}\Lambda^{\mathfrak{a}}(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X) \in \mathcal{A}_M(R)$ *and* $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.
(iii) $\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X) \in \mathcal{A}_{\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R M}(\widehat{R}^{\mathfrak{a}})$ *and* $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

The next two results show how to remove the derived local homology from the previous two results, in the presence of extra finiteness conditions.

Corollary 5.11 *Let $\varphi\colon R \to S$ be a homomorphism of commutative noetherian rings, where $\mathfrak{a}$ is a proper ideal of R such that $\mathfrak{a}S \neq S$. Let M be an $\mathfrak{a}$-adic semidualizing R-complex, and assume that S is module-finite over R with* $\mathrm{fd}_R(S) < \infty$*. Let $X \in \mathcal{D}(R)$ be given, and consider the following conditions.*

(i) $X \in \mathcal{A}_M(R)$.
(ii) $S \otimes^{\mathbf{L}}_R X \in \mathcal{A}_M(R)$ *and* $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.
(iii) $S \otimes^{\mathbf{L}}_R X \in \mathcal{A}_{S \otimes^{\mathbf{L}}_R M}(S)$ *and* $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

Then we have (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii). *The conditions* (i)–(iii) *are equivalent when at least one of the following conditions is satisfied.*

(1) $\operatorname{supp}_R(S) = \mathrm{V}(\mathfrak{a})$.
(2) $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, *and* $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f(R)$, *where* K *is the Koszul complex over* R *on a generating sequence for* $\mathfrak{a}$.
(3) S *is flat over* R *with* $\operatorname{supp}_R(S) \supseteq \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, *e.g.,* S *is faithfully flat.*

Proof The module-finite assumption on S says that $S \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. Thus, the desired conclusions follow from Corollary 4.6 and Proposition 5.3(a). □

Corollary 5.12 *Let* M *be an* $\mathfrak{a}$*-adic semidualizing* R*-complex, where* $\mathfrak{a}$ *is a proper ideal of* R*. Given an* R*-complex* $X \in \mathcal{D}^f(R)$*, the following conditions are equivalent.*

(i) $X \in \mathcal{A}_M(R)$.
(ii) $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X \in \mathcal{A}_M(R)$ *and* $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.
(iii) $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X \in \mathcal{A}_{\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R M}(\widehat{R}^{\mathfrak{a}})$ *and* $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

Proof Assume without loss of generality that $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Then [36, Theorem 4.2(c)] implies that $X \in \mathcal{D}_{\mathrm{b}}(R)$ if and only if $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(R)$. Thus, we assume without loss of generality that $X \in \mathcal{D}_{\mathrm{b}}(R)$, that is, $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. It follows that $\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R}^{\mathfrak{a}})$, so each homology module $\mathrm{H}_i(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X)$ is $\mathfrak{a}$-adically complete over R and $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-adically complete over $\widehat{R}^{\mathfrak{a}}$. Fact 2.5.3 thus implies that $\mathbf{L}\Lambda^{\mathfrak{a}}(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X$ in $\mathcal{D}(R)$, and $\mathbf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. Thus, the desired conclusions follow from Corollary 5.10. □

5.3 Local-Global Behavior

To keep the notation under control in the next few results, we write $U^{-1}X$ for $(U^{-1}R) \otimes^{\mathbf{L}}_R X$, and similarly for $U^{-1}M$, $X_{\mathfrak{p}}$, etc. Note that in each result, each localization of M is appropriately adically semidualizing over the localized ring by [34, Theorem 5.7]; this is why we restrict to localizations that are well-behaved with respect to $\mathfrak{a}$. In turn, this is why we need to assume that $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$: for instance, if $\mathfrak{n} \in \text{m-Spec}(R) \smallsetminus \mathrm{V}(\mathfrak{a})$, then $X = R/\mathfrak{n}$ satisfies condition (iv) in the theorem, but not condition (i).

Theorem 5.13 *Let* M *be an* $\mathfrak{a}$*-adic semidualizing* R*-complex, where* $\mathfrak{a}$ *is a proper ideal of* R*, and let* $X \in \mathcal{D}_b(R)$ *be such that* $\operatorname{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$*. Then the following conditions are equivalent.*

(i) $X \in \mathcal{B}_M(R)$.
(ii) *for each multiplicatively closed subset* $U \subseteq R$ *such that* $U^{-1}\mathfrak{a} \neq U^{-1}R$*, we have* $U^{-1}X \in \mathcal{B}_{U^{-1}M}(U^{-1}R)$.
(iii) *For all* $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$*, we have* $X_{\mathfrak{p}} \in \mathcal{B}_{M_{\mathfrak{p}}}(R_{\mathfrak{p}})$.
(iv) *For all* $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$*, we have* $X_{\mathfrak{m}} \in \mathcal{B}_{M_{\mathfrak{m}}}(R_{\mathfrak{m}})$.

Proof In light of Theorem 5.5, it suffices to prove the implication (iv) $\Longrightarrow$ (i). Assume that for all $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, we have $X_{\mathfrak{m}} \in \mathcal{B}_{M_{\mathfrak{m}}}(R_{\mathfrak{m}})$. From Proposition 5.3(b) we have $X_{\mathfrak{m}} \in \mathcal{B}_M(R)$ for all such $\mathfrak{m}$. Note that we have

$$-\infty < \inf(X) \leqslant \inf(X_{\mathfrak{m}}) \qquad \text{and} \qquad \sup(X_{\mathfrak{m}}) \leqslant \sup(X) < \infty$$

for all $\mathfrak{m}$. Thus, Theorem 4.2(b) implies that

$$X \otimes^{\mathbf{L}}_R \left(\bigoplus_{\mathfrak{m}} R_{\mathfrak{m}}\right) \simeq \bigoplus_{\mathfrak{m}} X_{\mathfrak{m}} \in \mathcal{B}_M(R)$$

where the sums are taken over all $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$. Since the module $\bigoplus_{\mathfrak{m}} R_{\mathfrak{m}}$ is flat over R with support containing $\mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, it follows from Theorem 4.3 that we have $X \in \mathcal{B}_M(R)$, as desired. □

The next example shows why we need $X \in \mathcal{D}_{\mathrm{b}}(R)$ in the previous result.

Example 5.14 Let k be a field and consider the polynomial ring $R := k[X]$ with $\mathfrak{a} = 0$. Let $\{\mathfrak{m}_i\}_{i\in\mathbb{Z}}$ be a set of distinct maximal ideals of R, and set $X := \bigoplus_{i\in\mathbb{Z}} \Sigma^i R/\mathfrak{m}_i$. Then for each $\mathfrak{m} \in \text{m-Spec}(R)$, we have $X_{\mathfrak{m}} \simeq \Sigma^i \kappa(\mathfrak{m})$ if $\mathfrak{m} = \mathfrak{m}_i$ for some i, and $X_{\mathfrak{m}} \simeq 0$ otherwise. In particular, this implies $X_{\mathfrak{m}} \in \mathcal{D}_{\mathrm{b}}(R_{\mathfrak{m}}) = \mathcal{B}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}})$ for all $\mathfrak{m}$. On the other hand, we have $X \notin \mathcal{D}_{\mathrm{b}}(R) = \mathcal{B}_R(R)$. Note that we have $\mathrm{supp}_R(X) = \{\mathfrak{m}_i\}_{i\in\mathbb{Z}}$, which is trivially contained in $\mathrm{Spec}(R) = \mathrm{V}(0)$. So, the failure here is not due to any absence of a support condition.

Theorem 5.15 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and let $X \in \mathcal{D}_b(R)$ be such that* $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. *Then the following conditions are equivalent.*

(i) $X \in \mathcal{A}_M(R)$.
(ii) *for each multiplicatively closed subset $U \subseteq R$ such that $U^{-1}\mathfrak{a} \neq U^{-1}R$, we have* $\mathbf{L}\Lambda^{U^{-1}\mathfrak{a}}(U^{-1}X) \in \mathcal{A}_{U^{-1}M}(U^{-1}R)$.
(iii) *For all* $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$, *we have* $\mathbf{L}\Lambda^{\mathfrak{a}\mathfrak{p}}(X_{\mathfrak{p}}) \in \mathcal{A}_{M_{\mathfrak{p}}}(R_{\mathfrak{p}})$.
(iv) *For all* $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, *we have* $\mathbf{L}\Lambda^{\mathfrak{a}\mathfrak{m}}(X_{\mathfrak{m}}) \in \mathcal{A}_{M_{\mathfrak{m}}}(R_{\mathfrak{m}})$.

Proof Again, it suffices to prove the implication (iv) $\Longrightarrow$ (i). Assume that for all $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, we have $\mathbf{L}\Lambda^{\mathfrak{a}\mathfrak{m}}(X_{\mathfrak{m}}) \in \mathcal{A}_{M_{\mathfrak{m}}}(R_{\mathfrak{m}})$. Foxby Equivalence 3.6(c) over $R_{\mathfrak{m}}$ implies that we have $M_{\mathfrak{m}} \otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} \mathbf{L}\Lambda^{\mathfrak{a}\mathfrak{m}}(X_{\mathfrak{m}}) \in \mathcal{B}_{M_{\mathfrak{m}}}(R_{\mathfrak{m}})$ for all such $\mathfrak{m}$. By Fact 2.7 we have the following isomorphisms in $\mathcal{D}(R_{\mathfrak{m}})$

$$M_{\mathfrak{m}} \otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} \mathbf{L}\Lambda^{\mathfrak{a}\mathfrak{m}}(X_{\mathfrak{m}}) \simeq M_{\mathfrak{m}} \otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} X_{\mathfrak{m}} \simeq (M \otimes^{\mathbf{L}}_R X)_{\mathfrak{m}} \tag{5.15.1}$$

so we conclude that $(M \otimes^{\mathbf{L}}_R X)_{\mathfrak{m}} \in \mathcal{B}_{M_{\mathfrak{m}}}(R_{\mathfrak{m}})$ for all $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$. Moreover, we have $\mathrm{supp}_R(M \otimes^{\mathbf{L}}_R X) \subseteq \mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$, by Fact 2.5.2.

Claim: we have $M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$. In light of the previous paragraph, it suffices by Theorem 5.13 to show that $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(R)$. Since the conditions $M, X \in \mathcal{D}_{\mathrm{b}}(R)$ imply that $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}_+(R)$, it suffices to show that $\sup(M \otimes^{\mathbf{L}}_R X) < \infty$. To this end, the first two steps in the next sequence are from Facts 2.3 and 2.1(b).

$$\begin{aligned}
\sup(\mathbf{L}\Lambda^{\mathfrak{a}\mathfrak{m}}(X_{\mathfrak{m}})) &= \sup(\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{m}}}(\mathbf{R}\Gamma_{\mathfrak{a}_{\mathfrak{m}}}(R_{\mathfrak{m}}), X_{\mathfrak{m}})) \\
&\leqslant \sup(X_{\mathfrak{m}}) - \inf(\mathbf{R}\Gamma_{\mathfrak{a}_{\mathfrak{m}}}(R_{\mathfrak{m}})) \\
&= \sup(X_{\mathfrak{m}}) - \inf(\mathbf{R}\Gamma_{\mathfrak{a}}(R)_{\mathfrak{m}}) \\
&\leqslant \sup(X) - \inf(\mathbf{R}\Gamma_{\mathfrak{a}}(R)) \\
&\leqslant \sup(X) + \mathrm{depth}_{\mathfrak{a}}(R)
\end{aligned}$$

The third step is from the standard isomorphism $\mathbf{R}\Gamma_{\mathfrak{a}_{\mathfrak{m}}}(R_{\mathfrak{m}}) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R)_{\mathfrak{m}}$; one can verify this via the Čech complex over R. The fourth step is routine, and the fifth one is Grothendieck's standard non-vanishing result for local cohomology.

For all $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$ we have $\mathbf{L}\Lambda^{\mathfrak{a}_\mathfrak{m}}(X_\mathfrak{m}) \in \mathcal{A}_{M_\mathfrak{m}}(R_\mathfrak{m})$. Thus, for all such $\mathfrak{m}$, the first step in the next sequence is from Lemma 3.18(a).

$$\begin{aligned}\sup(M_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} \mathbf{L}\Lambda^{\mathfrak{a}_\mathfrak{m}}(X_\mathfrak{m})) &\leqslant \sup(\mathbf{L}\Lambda^{\mathfrak{a}_\mathfrak{m}}(X_\mathfrak{m})) + \sup(M_\mathfrak{m}) + n\\ &\leqslant \sup(X) + \operatorname{depth}_\mathfrak{a}(R) + \sup(M_\mathfrak{m}) + n\\ &\leqslant \sup(X) + \operatorname{depth}_\mathfrak{a}(R) + \sup(M) + n\end{aligned}$$

The second step is from the preceding paragraph, and the third step is routine. This explains the last step in the next sequence.

$$\begin{aligned}\sup(M \otimes^{\mathbf{L}}_R X) &= \sup\left\{\sup((M \otimes^{\mathbf{L}}_R X)_\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Supp}_R(M \otimes^{\mathbf{L}}_R X) \bigcap \text{m-Spec}(R)\right\}\\ &= \sup\left\{\sup((M \otimes^{\mathbf{L}}_R X)_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)\right\}\\ &= \sup\left\{\sup(M_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} \mathbf{L}\Lambda^{\mathfrak{a}_\mathfrak{m}}(X_\mathfrak{m})) \mid \mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)\right\}\\ &\leqslant \sup(X) + \operatorname{depth}_\mathfrak{a}(R) + \sup(M) + n\end{aligned}$$

The first step here is routine, where the "large support" of an R-complex Y is $\operatorname{Supp}_R(Y) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid Y_\mathfrak{p} \not\simeq 0\}$. For the second step, use the fact that we have $\operatorname{supp}_R(M \otimes^{\mathbf{L}}_R X) \subseteq \operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$, which implies that $\operatorname{Supp}_R(M \otimes^{\mathbf{L}}_R X) \subseteq \mathrm{V}(\mathfrak{a})$, by [38, Proposition 3.15(a)]. The third step is from the isomorphism (5.15.1). This establishes the Claim.

To complete the proof, note that $\operatorname{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ and $M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$, by the Claim. So, Foxby Equivalence 3.6(c) implies that $X \in \mathcal{A}_M(R)$, as desired. □

The next example shows that one cannot replace $\mathbf{L}\Lambda^{U^{-1}\mathfrak{a}}(U^{-1}X)$ with $U^{-1}X$ in the previous result.

Example 5.16 Let k be a field and consider the power series ring $R = k[\![Y, Z]\!]$ with $\mathfrak{a} = YR$ and $U = \{1, Z, Z^2, \ldots\}$. We first show that $U^{-1}R$ is not $U^{-1}\mathfrak{a}$-adically complete.[4] Suppose by way of contradiction that $U^{-1}R$ were $U^{-1}\mathfrak{a}$-adically complete. Since $U^{-1}\mathfrak{a} = YU^{-1}R$, it follows that we have

$$\sum_{i=0}^{\infty} \frac{1}{Z^i} Y^i \in U^{-1}R = k[\![Y, Z]\!][Z^{-1}].$$

This means that there is a power series $f \in R$ and an integer $m \geqslant 0$ such that

$$\sum_{i=0}^{\infty} \frac{1}{Z^i} Y^i = \frac{f}{Z^m}.$$

Clearing the denominator Z^m, we find that

$$\sum_{i=0}^{\infty} \frac{Z^m}{Z^i} Y^i = f \in R = k[\![Y, Z]\!]$$

which is impossible.

Now, Proposition 3.2 shows that $R \in \mathcal{A}_{\mathbf{R}\Gamma_\mathfrak{a}(R)}(R)$. On the other hand, we have just shown that $U^{-1}R$ is not $U^{-1}\mathfrak{a}$-adically complete, so we have $U^{-1}R \notin \mathcal{A}_{U^{-1}\mathbf{R}\Gamma_\mathfrak{a}(R)}(U^{-1}R)$ by Proposition 3.10.

[4] This may be well-known, but we include a short proof for the sake of completeness.

Independent of this example, the derived local homology in Theorem 5.15 is still a bit ugly. Of course, since localization is a tensor product, and tensor products respect supports (not cosupports), this is inevitable. On the other hand, Homs respect cosupports, so it makes sense to consider co-localization as well. Here, we have only limited results.

Proposition 5.17 *Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and assume that $\dim(R) < \infty$. Let $U \subseteq R$ be multiplicatively closed such that $U^{-1}\mathfrak{a} \neq U^{-1}R$, and let $X \in \mathcal{D}_b(R)$ be given.*

(a) *If $X \in \mathcal{A}_M(R)$, then $\mathbf{R}\mathrm{Hom}_R(U^{-1}R, X) \in \mathcal{A}_{U^{-1}M}(U^{-1}R)$.*
(b) *If $X \in \mathcal{B}_M(R)$, then $\mathbf{R}\Gamma_{U^{-1}\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(U^{-1}R, X)) \in \mathcal{B}_{U^{-1}M}(U^{-1}R)$.*

Proof We prove part (b); the proof of part (a) is easier. Let $Q\colon \mathcal{D}(U^{-1}R) \to \mathcal{D}(R)$ be the forgetful functor. The fact that $U^{-1}R$ is flat over R implies that we have natural isomorphisms $Q \circ \mathbf{R}\Gamma_{U^{-1}\mathfrak{a}} \simeq \mathbf{R}\Gamma_{\mathfrak{a}} \circ Q$ of functors $\mathcal{D}(U^{-1}R) \to \mathcal{D}(R)$; this is easily verified using the Čech complex.

Assume that $X \in \mathcal{B}_M(R)$. The assumption $\dim(R) < \infty$ implies that we have $\mathrm{pd}_R(U^{-1}R) \leqslant \dim(R) < \infty$ by [33, Theorem II.3.2.6]. Thus, the isomorphism from the preceding paragraph conspires with Theorem 4.10 to show that

$$Q(\mathbf{R}\Gamma_{U^{-1}\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(U^{-1}R, X))) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(U^{-1}R, X)) \in \mathcal{B}_M(R).$$

The desired conclusion $\mathbf{R}\Gamma_{U^{-1}\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(U^{-1}R, X)) \in \mathcal{B}_{U^{-1}M}(U^{-1}R)$ now follows from Proposition 5.3(b). □

Question 5.18 Let M be an $\mathfrak{a}$-adic semidualizing R-complex, where $\mathfrak{a}$ is a proper ideal of R, and assume that $\dim(R) < \infty$. Let $X \in \mathcal{D}_b(R)$ be given.

(a) If $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{m}, X) \in \mathcal{A}_{M_\mathfrak{m}}(R_\mathfrak{m})$ for every $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, must we have $X \in \mathcal{A}_M(R)$?
(b) If $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathbf{R}\Gamma_{\mathfrak{a}R_\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{m}, X)) \in \mathcal{B}_{M_\mathfrak{m}}(R_\mathfrak{m})$ for every $\mathfrak{m} \in \mathrm{V}(\mathfrak{a}) \bigcap \text{m-Spec}(R)$, must we have $X \in \mathcal{B}_M(R)$?

Remark 5.19 One may be tempted to attempt to answer Question 5.18(a) as in the proof of Theorem 5.13. Following this logic, one concludes that

$$\inf(X) - \dim(R) \leqslant \inf(\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{m}, X))$$
$$\sup(\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{m}, X)) \leqslant \sup(X)$$

for all $\mathfrak{m} \in \text{m-Spec}(R) \bigcap \mathrm{V}(\mathfrak{a})$ and that

$$\mathbf{R}\mathrm{Hom}\left(\bigoplus_{\mathfrak{m}} R_\mathfrak{m}, X\right) \simeq \prod_{\mathfrak{m}} \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{m}, X) \in \mathcal{A}_M(R).$$

However, Theorem 4.9 does not allow us to conclude that $X \in \mathcal{A}_M(R)$.

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