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AB-Contexts and Stability for Gorenstein Flat Modules with Respect to Semidualizing Modules

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Abstract We investigate the properties of categories of G_C-flat R-modules where C is a semidualizing module over a commutative noetherian ring R. We prove that the category of all G_C-flat R-modules is part of a weak AB-context, in the terminology of Hashimoto. In particular, this allows us to deduce the existence of certain Auslander-Buchweitz approximations for R-modules of finite G_C-flat dimension. We also prove that two procedures for building R-modules from complete resolutions by certain subcategories of G_C-flat R-modules yield only the modules in the original subcategories.

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1 Introduction

Auslander and Bridger [1, 2] introduce the modules of finite G-dimension over a commutative noetherian ring R, in part, to identify a class of finitely generated R-modules with particularly nice duality properties with respect to R. They are exactly the R-modules which admit a finite resolution by modules of G-dimension 0. As a special case, the duality theory for these modules recovers the well-known duality theory for finitely generated modules over a Gorenstein ring.

This notion has been extended in several directions. For instance, Enochs et al. [8, 10] introduce the Gorenstein projective modules and the Gorenstein flat modules; these are analogues of modules of G-dimension 0 for the non-finitely generated arena. Foxby [11], Golod [13] and Vasconcelos [25] focus on finitely generated modules, but consider duality with respect to a semidualizing module C. Recently, Holm and Jørgensen [17] have unified these approaches with the G_C-projective modules and the G_C-flat modules. For background and definitions, see Sections 2 and 3.

The purpose of this paper is to use cotorsion flat modules in order to further study the G_C-flat modules, which are more technically challenging to investigate than the G_C-projective modules. Cotorsion flat modules have been successfully used to investigate flat modules, for instance in the work of Xu [27], and this paper shows how they are similarly well-suited for studying the G_C-flat modules.

More specifically, an R-module is *C-flat C-cotorsion* when it is isomorphic to an R-module of the form $F \otimes_R C$ where F is flat and cotorsion. We let $\mathcal{F}_C^{\mathrm{cot}}(R)$ denote the category of all C-flat C-cotorsion R-modules, and we let $\operatorname{res}\widehat{\mathcal{F}_C^{\mathrm{cot}}(R)}$ denote the category of all R-modules admitting a finite resolution by C-flat C-cotorsion R-modules. The first step of our analysis is carried out in Section 4 where we investigate the fundamental properties of these categories; see Theorem I(b) for some of the conclusions from this section.

Section 5 contains our analysis of the category of G_C-flat modules, denoted $\mathcal{GF}_C(R)$. This section culminates in the following theorem. In the terminology of Hashimoto [15], it says that the triple $(\mathcal{GF}_C(R), \operatorname{res}\widehat{\mathcal{F}_C^{\mathrm{cot}}(R)}, \mathcal{F}_C^{\mathrm{cot}}(R))$ satisfies the axioms for a weak AB-context. The proof of this result is in (5.9).

Theorem I *Let C be a semidualizing R-module.*

(a) *$\mathcal{GF}_C(R)$ is closed under extensions, kernels of epimorphisms and summands.*
(b) *$\operatorname{res}\widehat{\mathcal{F}_C^{cot}(R)}$ is closed under cokernels of monomorphisms, extensions and summands, and $\operatorname{res}\widehat{\mathcal{F}_C^{cot}(R)} \subseteq \operatorname{res}\widehat{\mathcal{GF}_C(R)}$.*
(c) *$\mathcal{F}_C^{cot}(R) = \mathcal{GF}_C(R) \cap \operatorname{res}\widehat{\mathcal{F}_C^{cot}(R)}$, and $\mathcal{F}_C^{cot}(R)$ is an injective cogenerator for $\mathcal{GF}_C(R)$.*

In conjunction with [15, (1.12.10)], this result implies many of the conclusions of [3] for the triple $(\mathcal{GF}_C(R), \operatorname{res}\widehat{\mathcal{F}_C^{\mathrm{cot}}(R)}, \mathcal{F}_C^{\mathrm{cot}}(R))$. For instance, we conclude that every module M of finite G_C-flat dimension fits in an exact sequence

$$0 \to Y \to X \to M \to 0$$

such that X is in $\mathcal{GF}_C(R)$ and Y is in $\operatorname{res}\widehat{\mathcal{F}_C^{\mathrm{cot}}(R)}$. Such "approximations" have been very useful, for instance, in the study of modules of finite G-dimension. See Corollary 5.10 for this and other conclusions.

In Section 6 we apply these techniques to continue our study of stability properties of Gorenstein categories, initiated in [23]. For each subcategory $\mathcal{X}$ of the category of R-modules, let $\mathcal{G}^1(\mathcal{X})$ denote the category of all R-modules isomorphic to $\operatorname{Coker}(\partial_1^X)$ for some exact complex X in $\mathcal{X}$ such that the complexes $\operatorname{Hom}_R(X', X)$ and $\operatorname{Hom}_R(X, X')$ are exact for each module X' in $\mathcal{X}$. This definition is a modification of the construction of G_C-projective R-modules. Inductively, set $\mathcal{G}^{n+1}(\mathcal{X}) = \mathcal{G}(\mathcal{G}^n(\mathcal{X}))$ for each $n \geqslant 1$. The techniques of this paper allow us to prove the following G_C-flat versions of some results of [23]; see Corollary 6.10 and Theorem 6.14.

Theorem II *Let C be a semidualizing R-module and let $n \geqslant 1$.*

(a) *We have $\mathcal{G}^n(\mathcal{GF}_C(R) \cap \mathcal{B}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$.*
(b) *If $\dim(R) < \infty$, then $\mathcal{G}^n(\mathcal{F}_C^{\mathrm{cot}}(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$.*

Here $\mathcal{B}_C(R)$ is the Bass class associated to C, and $\mathcal{F}_C(R)^\perp$ is the category of all R-modules N such that $\operatorname{Ext}_R^{\geqslant 1}(F \otimes_R C, N) = 0$ for each flat R-module F. In particular, when $C = R$ this result yields $\mathcal{G}^n(\mathcal{GF}(R)) = \mathcal{GF}(R)$ and, when $\dim(R)$ is finite, $\mathcal{G}^n(\mathcal{F}^{\mathrm{cot}}(R)) = \mathcal{GF}(R) \cap \mathcal{F}(R)^\perp$.

2 Modules, Complexes and Resolutions

We begin with some notation and terminology for use throughout this paper.

Definition 2.1 Throughout this work R is a commutative noetherian ring and $\mathcal{M}(R)$ is the category of R-modules. We use the term "subcategory" to mean a "full, additive subcategory $\mathcal{X} \subseteq \mathcal{M}(R)$ such that, for all R-modules M and N, if $M \cong N$ and $M \in \mathcal{X}$, then $N \in \mathcal{X}$." Write $\mathcal{P}(R)$, $\mathcal{F}(R)$ and $\mathcal{I}(R)$ for the subcategories of projective, flat and injective R-modules, respectively.

Definition 2.2 We fix subcategories $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{W}$, and $\mathcal{V}$ of $\mathcal{M}(R)$ such that $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$. Write $\mathcal{X} \perp \mathcal{Y}$ if $\operatorname{Ext}_R^{\geqslant 1}(X, Y) = 0$ for each $X \in \mathcal{X}$ and each $Y \in \mathcal{Y}$. For an R-module M, write $M \perp \mathcal{Y}$ (resp., $\mathcal{X} \perp M$) if $\operatorname{Ext}_R^{\geqslant 1}(M, Y) = 0$ for each $Y \in \mathcal{Y}$ (resp., if $\operatorname{Ext}_R^{\geqslant 1}(X, M) = 0$ for each $X \in \mathcal{X}$). Set

$$\mathcal{X}^\perp = \text{the subcategory of } R\text{-modules } M \text{ such that } \mathcal{X} \perp M.$$

We say $\mathcal{W}$ is a *cogenerator* for $\mathcal{X}$ if, for each $X \in \mathcal{X}$, there is an exact sequence

$$0 \to X \to W \to X' \to 0$$

such that $W \in \mathcal{W}$ and $X' \in \mathcal{X}$; and $\mathcal{W}$ is an *injective cogenerator* for $\mathcal{X}$ if $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{X} \perp \mathcal{W}$. The terms *generator* and *projective generator* are defined dually.

We say that $\mathcal{X}$ is *closed under extensions* when, for every exact sequence

$$0 \to M' \to M \to M'' \to 0 \qquad (*)$$

if $M', M'' \in \mathcal{X}$, then $M \in \mathcal{X}$. We say that $\mathcal{X}$ is *closed under kernels of monomorphisms* when, for every exact sequence ($*$), if $M', M \in \mathcal{X}$, then $M'' \in \mathcal{X}$. We say that $\mathcal{X}$ is *closed under cokernels of epimorphisms* when, for every exact sequence ($*$), if $M, M'' \in \mathcal{X}$, then $M' \in \mathcal{X}$. We say that $\mathcal{X}$ is *closed under summands* when, for every exact sequence ($*$), if $M \in \mathcal{X}$ and Eq. $*$ splits, then $M', M'' \in \mathcal{X}$. We say that $\mathcal{X}$ is *closed under products* when, for every set $\{M_\lambda\}_{\lambda \in \Lambda}$ of modules in $\mathcal{X}$, we have $\prod_{\lambda \in \Lambda} M_\lambda \in \mathcal{X}$.

Definition 2.3 We employ the notation from [5] for R-complexes. In particular, R-complexes are indexed homologically

$$M = \cdots \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots$$

with nth homology module denoted $\mathrm{H}_n(M)$. We frequently identify R-modules with R-complexes concentrated in degree 0.

Let M, N be R-complexes. For each integer i, let $\Sigma^i M$ denote the complex with $(\Sigma^i M)_n = M_{n-i}$ and $\partial_n^{\Sigma^i M} = (-1)^i \partial^M_{n-i}$. Let $\mathrm{Hom}_R(M, N)$ and $M \otimes_R N$ denote the associated Hom complex and tensor product complex, respectively. A morphism $\alpha \colon M \to N$ is a *quasiisomorphism* when each induced map $\mathrm{H}_n(\alpha) \colon \mathrm{H}_n(M) \to \mathrm{H}_n(N)$ is bijective. Quasiisomorphisms are designated by the symbol $\simeq$.

The complex M is $\mathrm{Hom}_R(\mathcal{X}, -)$*-exact* if the complex $\mathrm{Hom}_R(X, M)$ is exact for each $X \in \mathcal{X}$. Dually, the complex M is $\mathrm{Hom}_R(-, \mathcal{X})$*-exact* if $\mathrm{Hom}_R(M, X)$ is exact for each $X \in \mathcal{X}$, and M is $- \otimes_R \mathcal{X}$*-exact* if $M \otimes_R X$ is exact for each $X \in \mathcal{X}$.

Definition 2.4 When $X_{-n} = 0 = \mathrm{H}_n(X)$ for all $n > 0$, the natural morphism $X \to \mathrm{H}_0(X) = M$ is a quasiisomorphism, that is, the following sequence is exact

$$X^+ = \cdots \xrightarrow{\partial^X_2} X_1 \xrightarrow{\partial^X_1} X_0 \to M \to 0.$$

In this event, X is an $\mathcal{X}$*-resolution* of M if each X_n is in $\mathcal{X}$, and X^+ is the *augmented* $\mathcal{X}$*-resolution* of M associated to X. We write "projective resolution" in lieu of "$\mathcal{P}$-resolution", and we write "flat resolution" in lieu of "$\mathcal{F}$-resolution". The $\mathcal{X}$*-projective dimension* of M is the quantity

$$\mathcal{X}\text{-}\mathrm{pd}_R(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

The modules of $\mathcal{X}$-projective dimension 0 are the nonzero modules of $\mathcal{X}$. We set

$$\mathrm{res}\,\widehat{\mathcal{X}} = \text{the subcategory of } R\text{-modules } M \text{ with } \mathcal{X}\text{-}\mathrm{pd}_R(M) < \infty.$$

One checks easily that $\mathrm{res}\,\widehat{\mathcal{X}}$ is additive and contains $\mathcal{X}$. Following established conventions, we set $\mathrm{pd}_R(M) = \mathcal{P}\text{-}\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M) = \mathcal{F}\text{-}\mathrm{pd}_R(M)$.

The term $\mathcal{Y}$*-coresolution* is defined dually. The $\mathcal{Y}$*-injective dimension* of M is denoted $\mathcal{Y}\text{-}\operatorname{id}_R(M)$, and the *augmented* $\mathcal{Y}$*-coresolution* associated to a $\mathcal{Y}$-coresolution Y is denoted ${}^{+}Y$. We write "injective resolution" for "$\mathcal{I}$-coresolution", and we set

$$\operatorname{cores}\widehat{\mathcal{Y}} = \text{the subcategory of } R\text{-modules } N \text{ with } \mathcal{Y}\text{-}\operatorname{id}_R(N) < \infty$$

which is additive and contains $\mathcal{Y}$.

Definition 2.5 A $\mathcal{Y}$-coresolution Y is $\mathcal{X}$*-proper* if the augmented resolution ${}^{+}Y$ is $\operatorname{Hom}_R(,-\mathcal{X})$-exact. We set

$$\operatorname{cores}\widetilde{\mathcal{Y}} = \text{the subcategory of } R\text{-modules admitting a } \mathcal{Y}\text{-proper } \mathcal{Y}\text{-coresolution.}$$

One checks readily that $\operatorname{cores}\widetilde{\mathcal{Y}}$ is additive and contains $\mathcal{Y}$. The term $\mathcal{Y}$*-proper* $\mathcal{X}$*-resolution* is defined dually.

Definition 2.6 An $\mathcal{X}$*-precover* of an R-module M is an R-module homomorphism $\varphi\colon X \to M$ where $X \in \mathcal{X}$ such that, for each $X' \in X$, the homomorphism $\operatorname{Hom}_R(X',\varphi)\colon \operatorname{Hom}_R(X',X) \to \operatorname{Hom}_R(X',M)$ is surjective. An $\mathcal{X}$-precover $\varphi\colon X \to M$ is an $\mathcal{X}$*-cover* if, every endomorphism $f\colon X \to X$ such that $\varphi = \varphi f$ is an automorphism. The terms *preenvelope* and *envelope* are defined dually.

The next three lemmata have standard proofs; see [3, proofs of (2.1) and (2.3)].

Lemma 2.7 *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence of R-modules.*

(a) *If* $M_3 \perp \mathcal{W}$, *then* $M_1 \perp \mathcal{W}$ *if and only if* $M_2 \perp \mathcal{W}$. *If* $M_1 \perp \mathcal{W}$ *and* $M_2 \perp \mathcal{W}$, *then* $M_3 \perp \mathcal{W}$ *if and only if the given sequence is* $\operatorname{Hom}_R(-,\mathcal{W})$*-exact.*
(b) *If* $\mathcal{V} \perp M_1$, *then* $\mathcal{V} \perp M_2$ *if and only if* $\mathcal{V} \perp M_3$. *If* $\mathcal{V} \perp M_2$ *and* $\mathcal{V} \perp M_3$, *then* $\mathcal{V} \perp M_1$ *if and only if the given sequence is* $\operatorname{Hom}_R(\mathcal{V},-)$*-exact.*
(c) *If* $\operatorname{Tor}^R_{\geqslant 1}(M_3,\mathcal{V}) = 0$, *then* $\operatorname{Tor}^R_{\geqslant 1}(M_1,\mathcal{V}) = 0$ *if and only if* $\operatorname{Tor}^R_{\geqslant 1}(M_2,\mathcal{V}) = 0$. *If* $\operatorname{Tor}^R_{\geqslant 1}(M_1,\mathcal{V}) = 0 = \operatorname{Tor}^R_{\geqslant 1}(M_2,\mathcal{V})$, *then* $\operatorname{Tor}^R_{\geqslant 1}(M_3,\mathcal{V}) = 0$ *if and only if the given sequence is* $-\otimes_R \mathcal{V}$*-exact.*

Lemma 2.8 *If* $\mathcal{X} \perp \mathcal{Y}$, *then* $\mathcal{X} \perp \operatorname{res}\widehat{\mathcal{Y}}$ *and* $\operatorname{cores}\widehat{\mathcal{X}} \perp \mathcal{Y}$.

Lemma 2.9 *Let X be an exact R-complex.*

(a) *Assume* $X_i \perp \mathcal{V}$ *for all i. If X is* $\operatorname{Hom}_R(-,\mathcal{V})$*-exact, then* $\operatorname{Ker}(\partial^X_i) \perp \mathcal{V}$ *for all i. Conversely, if* $\operatorname{Ker}(\partial^X_i) \perp \mathcal{V}$ *for all i or if* $X_i = 0$ *for all* $i \ll 0$, *then X is* $\operatorname{Hom}_R(-,\mathcal{V})$*-exact.*
(b) *Assume* $\mathcal{V} \perp X_i$ *for all i. If X is* $\operatorname{Hom}_R(\mathcal{V},-)$*-exact, then* $\mathcal{V} \perp \operatorname{Ker}(\partial^X_i)$ *for all i. Conversely, if* $\mathcal{V} \perp \operatorname{Ker}(\partial^X_i)$ *for all i or if* $X_i = 0$ *for all* $i \gg 0$, *then X is* $\operatorname{Hom}_R(\mathcal{V},-)$*-exact.*
(c) *Assume* $\operatorname{Tor}^R_{\geqslant 1}(X_i,\mathcal{V}) = 0$ *for all i. If the complex X is* $-\otimes_R \mathcal{V}$*-exact, then* $\operatorname{Tor}^R_{\geqslant 1}(\operatorname{Ker}(\partial^X_i),\mathcal{V}) = 0$ *for all i. Conversely, if* $\operatorname{Tor}^R_{\geqslant 1}(\operatorname{Ker}(\partial^X_i),\mathcal{V}) = 0$ *for all i or if* $X_i = 0$ *for all* $i \ll 0$, *then X is* $-\otimes_R \mathcal{V}$*-exact.*

A careful reading of the proofs of [23, (2.1), (2.2)] yields the next result.

Lemma 2.10 *Assume that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$. If M has an $\mathcal{X}$-coresolution that is $\mathcal{W}$-proper and $M \perp \mathcal{W}$, then M is in* $\operatorname{cores}\widetilde{\mathcal{W}}$.

3 Categories of Interest

This section contains definitions of and basic facts about the categories to be investigated in this paper.

Definition 3.1 An R-module M is *cotorsion* if $\mathcal{F}(R) \perp M$. We set

$$\mathcal{F}^{\text{cot}}(R) = \text{the subcategory of flat cotorsion } R\text{-modules.}$$

Definition 3.2 The *Pontryagin dual* or *character module* of an R-module M is the R-module $M^* = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

One implication in the following lemma is from [27, (3.1.4)], and the others are established similarly.

Lemma 3.3 *Let M be an R-module.*

(a) *The Pontryagin dual M^* is R-flat if and only if M is R-injective.*
(b) *The Pontryagin dual M^* is R-injective if and only if M is R-flat.*

Semidualizing modules, defined next, form the basis for our categories of interest.

Definition 3.4 A finitely generated R-module C is *semidualzing* if the natural homothety morphism $R \to \operatorname{Hom}_R(C, C)$ is an isomorphism and $\operatorname{Ext}_R^{\geqslant 1}(C, C) = 0$. An R-module D is *dualizing* if it is semidualizing and has finite injective dimension.

Let C be a semidualizing R-module. We set

$$\mathcal{P}_C(R) = \text{the subcategory of modules } P \otimes_R C \text{ where } P \text{ is } R\text{-projective}$$

$$\mathcal{F}_C(R) = \text{the subcategory of modules } F \otimes_R C \text{ where } F \text{ is } R\text{-flat}$$

$$\mathcal{F}_C^{\text{cot}}(R) = \text{the subcategory of modules } F \otimes_R C \text{ where } F \text{ is flat and cotorsion}$$

$$\mathcal{I}_C(R) = \text{the subcategory of modules } \operatorname{Hom}_R(C, I) \text{ where } I \text{ is } R\text{-injective.}$$

Modules in $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$, $\mathcal{F}_C^{\text{cot}}(R)$ and $\mathcal{I}_C(R)$ are called *C-projective*, *C-flat*, *C-flat C-cotorsion*, and *C-injective*, respectively. An R-module M is *C-cotorsion* if $\mathcal{F}_C(R) \perp M$.

Remark 3.5 We justify the terminology "C-flat C-cotorsion" in Lemma 4.3 where we show that M is C-flat C-cotorsion if and only if it is C-flat and C-cotorsion.

The following categories were introduced by Foxby [12], Avramov and Foxby [4], and Christensen [6], though the idea goes at least back to Vasconcelos [25].

Definition 3.6 Let C be a semidualizing R-module. The *Auslander class* of C is the subcategory $\mathcal{A}_C(R)$ of R-modules M such that

(1) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0 = \mathrm{Ext}^{\geqslant 1}_R(C, C \otimes_R M)$, and
(2) The natural map $M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* of C is the subcategory $\mathcal{B}_C(R)$ of R-modules M such that

(1) $\mathrm{Ext}^{\geqslant 1}_R(C, M) = 0 = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, M))$, and
(2) The natural evaluation map $C \otimes_R \mathrm{Hom}_R(C, M) \to M$ is an isomorphism.

Fact 3.7 Let C be a semidualizing R-module. The categories $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ are closed under extensions, kernels of epimorphisms and cokernels of monomorphism; see [18, Cor. 6.3]. The category $\mathcal{A}_C(R)$ contains all modules of finite flat dimension and those of finite $\mathcal{I}_C$-injective dimension, and the category $\mathcal{B}_C(R)$ contains all modules of finite injective dimension and those of finite $\mathcal{F}_C$-projective dimension by [18, Cors. 6.1 and 6.2].

Arguing as in [5, (3.2.9)], we see that $M \in \mathcal{A}_C(R)$ if and only if $M^* \in \mathcal{B}_C(R)$, and $M \in \mathcal{B}_C(R)$ if and only if $M^* \in \mathcal{A}_C(R)$. Similarly, we have $M \in \mathcal{B}_C(R)$ if and only if $\mathrm{Hom}_R(C, M) \in \mathcal{A}_C(R)$ by [24, (2.8.a)]. From [18, Thm. 6.1] we know that every module in $\mathcal{B}_C(R)$ has a $\mathcal{P}_C$-proper $\mathcal{P}_C$-resolution.

The next definitions are due to Holm and Jørgensen [17] in this generality.

Definition 3.8 Let C be a semidualizing R-module. A *complete $\mathcal{I}_C\mathcal{I}$-resolution* is a complex Y of R-modules satisfying the following:

(1) Y is exact and $\mathrm{Hom}_R(\mathcal{I}_C, -)$-exact, and
(2) Y_i is C-injective when $i \geqslant 0$ and Y_i is injective when $i < 0$.

An R-module H is *G_C-injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-resolution Y such that $H \cong \mathrm{Coker}(\partial_1^Y)$, in which case Y is a *complete $\mathcal{I}_C\mathcal{I}$-resolution of H*. We set

$$\mathcal{GI}_C(R) = \text{the subcategory of } \mathrm{G}_C\text{-injective } R\text{-modules.}$$

In the special case $C = R$, we write $\mathcal{GI}(R)$ in place of $\mathcal{GI}_R(R)$.

A *complete $\mathcal{FF}_C$-resolution* is a complex Z of R-modules satisfying the following.

(1) Z is exact and $- \otimes_R \mathcal{I}_C$-exact.
(2) Z_i is flat if $i \geqslant 0$ and Z_i is C-flat if $i < 0$.

An R-module M is *G_C-flat* if there exists a complete $\mathcal{FF}_C$-resolution Z such that $M \cong \mathrm{Coker}(\partial_1^Z)$, in which case Z is a *complete $\mathcal{FF}_C$-resolution of M*. We set

$$\mathcal{GF}_C(R) = \text{the subcategory of } \mathrm{G}_C\text{-flat } R\text{-modules.}$$

In the special case $C = R$, we set $\mathcal{GF}(R) = \mathcal{GF}_R(R)$, and $\mathrm{Gfd} = \mathcal{GF}\text{-}\mathrm{pd}$.

A *complete $\mathcal{PP}_C$-resolution* is a complex X of R-modules satisfying the following.

(1) X is exact and $\mathrm{Hom}_R(-, \mathcal{P}_C)$-exact.
(2) X_i is projective if $i \geqslant 0$ and X_i is C-projective if $i < 0$.

An R-module M is *G_C-projective* if there exists a complete $\mathcal{PP}_C$-resolution X such that $M \cong \mathrm{Coker}(\partial_1^X)$, in which case X is a *complete $\mathcal{PP}_C$-resolution of M*. We set

$$\mathcal{GP}_C(R) = \text{the subcategory of } \mathrm{G}_C\text{-projective } R\text{-modules.}$$

Fact 3.9 Let C be a semidualizing R-module. Flat R-modules and C-flat R-modules are G_C-flat by [17, (2.8.c)]. It is straightforward to show that an R-module M is G_C-flat if and only the following conditions hold:

(1) M admits an augmented $\mathcal{F}_C$-coresolution that is $- \otimes_R \mathcal{I}_C$-exact, and
(2) $\mathrm{Tor}^R_{\geqslant 1}(M, \mathcal{I}_C) = 0$.

Let $R \ltimes C$ denote the trivial extension of R by C, defined to be the R-module $R \ltimes_R C = R \oplus C$ with ring structure given by $(r, c)(r', c') = (rr', rc' + r'c)$. Each R-module M is naturally an $R \ltimes C$-module via the natural surjection $R \ltimes C \to R$. Within this protocol we have $M \in \mathcal{GI}_C(R)$ if and only if $M \in \mathcal{GI}(R \ltimes C)$ and $M \in \mathcal{GF}_C(R)$ if and only if $M \in \mathcal{GF}(R \ltimes C)$ by [17, (2.13) and (2.15)]. Also [17, (2.16)] implies $\mathcal{GF}_C\text{-}\mathrm{pd}_R(M) = \mathrm{Gfd}_{R \ltimes C}(M)$.

The next definition, from [23], is modeled on the construction of $\mathcal{GI}(R)$.

Definition 3.10 Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$. A *complete $\mathcal{X}$-resolution* is an exact complex X in $\mathcal{X}$ that is $\mathrm{Hom}_R(\mathcal{X}, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{X})$-exact.[1] Such a complex is a *complete $\mathcal{X}$-resolution* of $\mathrm{Coker}(\partial_1^X)$. We set

$$\mathcal{G}(\mathcal{X}) = \text{the subcategory of } R\text{-modules with a complete } \mathcal{X}\text{-resolution.}$$

Set $\mathcal{G}^0(\mathcal{X}) = \mathcal{X}, \mathcal{G}^1(\mathcal{X}) = \mathcal{G}(\mathcal{X})$ and $\mathcal{G}^{n+1}(\mathcal{X}) = \mathcal{G}(\mathcal{G}^n(\mathcal{X}))$ for $n \geqslant 1$.

Fact 3.11 Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$. Using a resolution of the form $0 \to X \to 0$, one sees that $\mathcal{X} \subseteq \mathcal{G}(\mathcal{X})$ and so $\mathcal{G}^n(\mathcal{X}) \subseteq \mathcal{G}^{n+1}(\mathcal{X})$ for each $n \geqslant 0$. If C is a semidualizing R-module, then $\mathcal{G}^n(\mathcal{I}_C(R)) = \mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ for each $n \geqslant 1$; see [23, (4.4)].

The final definition of this section is for use in the proof of Theorem II.

Definition 3.12 Let C be a semidualizing R-module, and let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$. A *$\mathcal{P}_C\mathcal{F}_C^{cot}$-complete $\mathcal{X}$-resolution* is an exact complex X in $\mathcal{X}$ that is $\mathrm{Hom}_R(\mathcal{P}_C, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{F}_C^{\mathrm{cot}})$-exact. Such a complex is a *$\mathcal{P}_C\mathcal{F}_C^{cot}$-complete $\mathcal{X}$-resolution* of $\mathrm{Coker}(\partial_1^X)$. We set

$$\mathcal{H}_C(\mathcal{X}) = \text{the subcategory of } R\text{-modules with a } \mathcal{P}_C\mathcal{F}_C^{\mathrm{cot}}\text{-complete } \mathcal{X}\text{-resolution.}$$

Set $\mathcal{H}_C^0(\mathcal{X}) = \mathcal{X}, \mathcal{H}_C^1(\mathcal{X}) = \mathcal{H}_C(\mathcal{X})$ and $\mathcal{H}_C^{n+1}(\mathcal{X}) = \mathcal{H}_C(\mathcal{H}_C^n(\mathcal{X}))$ for each $n \geqslant 1$.

[1] In the literature, these complexes are sometimes called "totally acyclic".

Remark 3.13 Let C be a semidualizing R-module, and let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$. Let X be an exact complex in $\mathcal{X}$ that is $\operatorname{Hom}_R(C,-)$-exact and $\operatorname{Hom}_R(-,\mathcal{F}_C^{\mathrm{cot}})$-exact. Hom-tensor adjointness implies that X is $\operatorname{Hom}_R(\mathcal{P}_C,-)$-exact and hence a $\mathcal{P}_C\mathcal{F}_C^{\mathrm{cot}}$-complete $\mathcal{X}$-resolution, as is the complex $\Sigma^i X$ for each $i \in \mathbb{Z}$. It follows that $\operatorname{Coker}(\partial_i^X) \in \mathcal{H}_C(\mathcal{X})$ for each i.

Using a resolution of the form $0 \to X \to 0$, one sees that $\mathcal{X} \subseteq \mathcal{H}_C(\mathcal{X})$ and so $\mathcal{H}_C^n(\mathcal{X}) \subseteq \mathcal{H}_C^{n+1}(\mathcal{X})$ for each $n \geqslant 0$. Furthermore, if $\mathcal{F}_C(R) \subseteq \mathcal{X}$, then $\mathcal{G}(\mathcal{X}) \subseteq \mathcal{H}_C(\mathcal{X})$ and so $\mathcal{G}^n(\mathcal{X}) \subseteq \mathcal{H}_C^n(\mathcal{X})$ for each $n \geqslant 1$.

4 Modules of Finite $\mathcal{F}_C^{\mathrm{cot}}$-projective Dimension

This section contains the fundamental properties of the modules of finite $\mathcal{F}_C^{\mathrm{cot}}$-projective dimension. The first two results allow us to deduce information for these modules from the modules of finite $\mathcal{I}_C(R)$-injective dimension.

Lemma 4.1 *Let M be an R-module, and let C be a semidualizing R-module.*

(a) *The Pontryagin dual M^* is C-flat if and only if M is C-injective.*
(b) *The Pontryagin dual M^* is C-injective if and only if M is C-flat.*
(c) *If $\operatorname{Tor}^R_{\geqslant 1}(C,M) = 0$, then M^* is C-cotorsion.*
(d) *If M is C-injective, then M^* is C-flat and C-cotorsion.*

Proof (a) Assume that M is C-injective, so there exists an injective R-module I such that $M \cong \operatorname{Hom}_R(C,I)$. This yields the first isomorphism in the following sequence while the second is from Hom-evaluation [7, Prop. 2.1(ii)]:

$$M^* \cong \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_R(C,I),\mathbb{Q}/\mathbb{Z}) \cong C \otimes_R \operatorname{Hom}_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z}).$$

Since I is injective, Lemma 3.3(b) implies that $\operatorname{Hom}_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z})$ is flat. Hence, the displayed isomorphisms imply that M^* is C-flat.

Conversely, assume that M^* is C-flat, so there exists a flat R-module F such that $M^* \cong F \otimes_R C$. As F is flat it is in $\mathcal{A}_C(R)$, and this yields the first isomorphism in the next sequence, while the third isomorphism is Hom-tensor adjointness

$$F \cong \operatorname{Hom}_R(C, F\otimes_R C) \cong \operatorname{Hom}_R(C,\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbb{Z}}(C\otimes_R M,\mathbb{Q}/\mathbb{Z}).$$

This module is flat, and so Lemma 3.3(a) implies that $C \otimes_R M$ is injective. From [18, Thm. 1] we conclude that M is C-injective.

(b) This is proved similarly.

(c) Let P be a projective resolution of M. Our Tor-vanishing hypothesis implies that there is a quasiisomorphism $C \otimes_R P \simeq C \otimes_R M$. For each flat R-module F, this yields a quasiisomorphism

$$F \otimes_R C \otimes_R P \simeq F \otimes_R C \otimes_R M.$$

Because $\mathbb{Q}/\mathbb{Z}$ is injective over $\mathbb{Z}$, this provides the third quasiisomorphism in the next sequence, while the second quasiisomorphism is Hom-tensor adjointness

$$\begin{aligned} \mathrm{Hom}_R(F \otimes_R C, P^*) &\simeq \mathrm{Hom}_R(F \otimes_R C, \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z})) \\ &\simeq \mathrm{Hom}_{\mathbb{Z}}(F \otimes_R C \otimes_R P, \mathbb{Q}/\mathbb{Z}) \qquad (*) \\ &\simeq \mathrm{Hom}_{\mathbb{Z}}(F \otimes_R C \otimes_R M, \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

Since $\mathbb{Q}/\mathbb{Z}$ is injective over $\mathbb{Z}$, there are quasiisomorphisms

$$M^* \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z}) \simeq P^*.$$

By Lemma 3.3(a), it follows that P^* is an injective resolution of M^* over R. In particular, taking cohomology in the displayed sequence ($*$) yields isomorphisms

$$\begin{aligned} \mathrm{Ext}^i_R(F \otimes_R C, M^*) &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(F \otimes_R C, P^*)) \\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_{\mathbb{Z}}(F \otimes_R C \otimes_R M, \mathbb{Q}/\mathbb{Z})). \end{aligned}$$

This is 0 when $i \neq 0$ because $\mathrm{Hom}_{\mathbb{Z}}(F \otimes_R C \otimes_R M, \mathbb{Q}/\mathbb{Z})$ is a module. Hence, the desired conclusion.

(d) Since M is C-injective, it is in $\mathcal{A}_C(R)$ by Fact 3.7, and so $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0$. Hence M is C-cotorsion by part (c), and it is C-flat by part (a). □

Lemma 4.2 *Let M be an R-module, and let C be a semidualizing R-module.*

(a) *There is an equality $\mathcal{I}_C\text{-}\mathrm{id}_R(M^*) = \mathcal{F}_C\text{-}\mathrm{pd}_R(M)$.*
(b) *There is an equality $\mathcal{F}_C\text{-}\mathrm{pd}_R(M^*) = \mathcal{I}_C\text{-}\mathrm{id}_R(M)$.*

Proof We prove part (a); the proof of part (b) is similar.

For the inequality $\mathcal{I}_C\text{-}\mathrm{id}_R(M^*) \leqslant \mathcal{F}_C\text{-}\mathrm{pd}_R(M)$, assume that $\mathcal{F}_C\text{-}\mathrm{pd}_R(M) < \infty$. Let X be a $\mathcal{F}_C(R)$-resolution of M such that $X_i = 0$ for all $i > \mathcal{F}_C\text{-}\mathrm{pd}_R(M)$. It follows from Lemma 4.1(b) that the complex X^* is an $\mathcal{I}_C$-coresolution of M^* such that $X_i^* = 0$ for all $i > \mathcal{F}_C\text{-}\mathrm{pd}_R(M)$. The desired inequality now follows.

For the reverse inequality, assume that $j = \mathcal{I}_C\text{-}\mathrm{id}_R(M^*) < \infty$. Fact 3.7 implies that M^* is in $\mathcal{A}_C(R)$, and hence also implies that $M \in \mathcal{B}_C(R)$. This condition implies that M has a proper $\mathcal{P}_C$-resolution Z by Fact 3.7. In particular, this is an $\mathcal{F}_C$-resolution of M, so Lemma 4.1(b) implies that Z^* is an $\mathcal{I}_C$-coresolution of M^*.

We claim that Z^* is a proper $\mathcal{I}_C$-coresolution of M^*. Let I be an injective R-module. By assumption, the complex $\mathrm{Hom}_R(C, Z^+)$ is exact. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, we have $(Z^*)^+ \cong (Z^+)^* = \mathrm{Hom}_{\mathbb{Z}}(Z^+, \mathbb{Q}/\mathbb{Z})$, and this explains the first isomorphism in the next sequence

$$\begin{aligned} \mathrm{Hom}_R((Z^*)^+, \mathrm{Hom}_R(C, I)) &\cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathbb{Z}}(Z^+, \mathbb{Q}/\mathbb{Z}), \mathrm{Hom}_R(C, I)) \\ &\cong \mathrm{Hom}_R(C \otimes_R \mathrm{Hom}_{\mathbb{Z}}(Z^+, \mathbb{Q}/\mathbb{Z}), I) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(C, Z^+), \mathbb{Q}/\mathbb{Z}), I). \end{aligned}$$

The second isomorphism is Hom-tensor adjointness, and the third isomorphism is Hom-evaluation [7, Prop. 2.1(ii)]. Since $\mathrm{Hom}_R(C, Z^+)$ is exact, we conclude that the complex $\mathrm{Hom}_R(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(C, Z^+), \mathbb{Q}/\mathbb{Z}), I)$ is also exact because $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module and I is an injective R-module. This shows that $(Z^*)^+$ is $\mathrm{Hom}_R(-, \mathcal{I}_C)$-exact, and establishes the claim.

From [24, (3.3.b)] we know that $\mathrm{Ker}((\partial^Z_{j+1})^*) \cong \mathrm{Coker}(\partial^Z_{j+1})^*$ is in $\mathcal{I}_C(R)$. Lemma 4.1(b) implies $\mathrm{Coker}(\partial^Z_{j+1}) \in \mathcal{F}_C(R)$. It follows that the truncated complex

$$Z' : 0 \to \mathrm{Coker}(\partial^Z_{j+1}) \to Z_{j-1} \to \cdots \to Z_0 \to 0$$

is an $\mathcal{F}_C$-resolution of M such that $Z'_i = 0$ for all $i > j$. The desired inequality now follows, and hence the equality. □

The next three lemmata document properties of $\mathcal{F}_C^{\mathrm{cot}}(R)$ for use in the sequel. The first of these contains the characterization of C-flat C-cotorsion modules mentioned in Remark 3.5.

Lemma 4.3 *Let C and M be R-modules with C semidualizing. The following conditions are equivalent:*

(i) $M \in \mathcal{F}_C^{cot}(R)$;
(ii) $M \in \mathcal{F}_C(R)$ *and* $\mathcal{F}_C(R) \perp M$;
(iii) $M \in \mathcal{B}_C(R)$ *and* $\mathrm{Hom}_R(C, M) \in \mathcal{F}^{cot}(R)$;
(iv) $\mathrm{Hom}_R(C, M) \in \mathcal{F}^{cot}(R)$.

In particular, we have $\mathcal{F}_C(R) \perp \mathcal{F}_C^{cot}(R)$.

Proof (i) $\Longleftrightarrow$ (ii). It suffices to show, for each flat R-module F, that $\mathcal{F}(R) \perp F$ if and only if $\mathcal{F}_C(R) \perp F \otimes_R C$. Let F' be a flat R-module. It suffices to show that

$$\mathrm{Ext}^i_R(F' \otimes_R C, F \otimes_R C) \cong \mathrm{Ext}^i_R(F', F)$$

for each i. From [26, (1.11.a)] we have the first isomorphism in the next sequence

$$\mathrm{Ext}^i_R(C, F \otimes_R C) \cong \mathrm{Ext}^i_R(C, C) \otimes_R F \cong \begin{cases} R \otimes_R F \cong F & \text{if } i \neq 0 \\ 0 \otimes_R F \cong 0 & \text{if } i = 0 \end{cases}$$

and the second isomorphism is from the fact that C is semidualizing. Let P be a projective resolution of C. The previous display provides a quasiisomorphism

$$\mathrm{Hom}_R(P, F \otimes_R C) \simeq F.$$

Let P' be a projective resolution of F'. Hom-tensor adjointness yields the first quasiisomorphism in the next sequence

$$\begin{aligned} \mathrm{Hom}_R(P' \otimes_R P, F \otimes_R C) &\simeq \mathrm{Hom}_R(P', \mathrm{Hom}_R(P, F \otimes_R C)) \\ &\simeq \mathrm{Hom}_R(P', F) \end{aligned}$$

and the second quasiisomorphism is from the previous display, because P' is a bounded below complex of projective R-modules. Since F' is flat, we conclude that

$P' \otimes_R P$ is a projective resolution of $F' \otimes_R C$. It follows that we have

$$\begin{aligned}\operatorname{Ext}^i_R(F' \otimes_R C, F \otimes_R C) &\cong \operatorname{H}_{-i}(\operatorname{Hom}_R(P' \otimes_R P, F \otimes_R C))\\ &\cong \operatorname{H}_{-i}(\operatorname{Hom}_R(P', F))\\ &\cong \operatorname{Ext}^i_R(F', F)\end{aligned}$$

as desired.

(i) $\Longrightarrow$ (iii). Assume that $M \in \mathcal{F}^{\text{cot}}_C(R)$, that is, that $M \cong C \otimes_R F$ for some $F \in \mathcal{F}^{\text{cot}}(R) \subseteq \mathcal{A}_C(R)$. Then

$$\operatorname{Hom}_R(C, M) \cong \operatorname{Hom}_R(C, C \otimes_R F) \cong F \in \mathcal{F}^{\text{cot}}_C(R)$$

and $M \in \mathcal{F}^{\text{cot}}_C(R) \subseteq \mathcal{F}_C(R) \subseteq \mathcal{B}_C(R)$.

(iii) $\Longrightarrow$ (i). If $M \in \mathcal{B}_C(R)$ and $\operatorname{Hom}_R(C, M) \in \mathcal{F}^{\text{cot}}(R)$, then there is an isomorphism $M \cong C \otimes_R \operatorname{Hom}_R(C, M) \in \mathcal{F}^{\text{cot}}_C(R)$.

(iii) $\Longleftrightarrow$ (iv). This is from Fact 3.7 because $\mathcal{F}^{\text{cot}}(R) \subseteq \mathcal{A}_C(R)$.

The conclusion $\mathcal{F}_C(R) \perp \mathcal{F}^{\text{cot}}_C(R)$ follows from the implication (i) $\Longrightarrow$ (ii). □

Lemma 4.4 *If C is a semidualzing R-module, then the category $\mathcal{F}^{cot}_C(R)$ is closed under products, extensions and summands.*

Proof Consider a set $\{F_\lambda\}_{\lambda\in\Lambda}$ of modules in $\mathcal{F}^{\text{cot}}(R)$. From [9, (3.2.24)] we have $\prod_\lambda F_\lambda \in \mathcal{F}^{\text{cot}}(R)$ and so $C \otimes_R (\prod_\lambda F_\lambda) \in \mathcal{F}^{\text{cot}}_C(R)$. Hence, we have

$$\textstyle\prod_\lambda (C \otimes_R F_\lambda) \cong C \otimes_R (\prod_\lambda F_\lambda) \in \mathcal{F}^{\text{cot}}_C(R)$$

where the isomorphism comes from the fact that C is finitely presented. Thus $\mathcal{F}^{\text{cot}}_C(R)$ is closed under products.

By Lemma 2.7(b), the category of C-cotorsion R-modules is closed under extensions, and it is closed under summands by the additivity of Ext. The category $\mathcal{F}_C(R)$ is closed under extensions and summands by [18, Props. 5.1(a) and 5.2(a)]. The result now follows from Lemma 4.3. □

Note that the hypotheses of the next lemma are satisfied when $M \in \mathcal{F}_C(R)^\perp \cap \mathcal{B}_C(R)$.

Lemma 4.5 *Let C be a semidualizing R-module, and let M be a C-cotorsion R-module such that the natural evaluation map $C \otimes_R \operatorname{Hom}_R(C, M) \to M$ is bijective.*

(a) *The module M has an $\mathcal{F}^{cot}_C$-cover, and every C-flat cover of M is an $\mathcal{F}^{cot}_C$-cover of M with C-cotorsion kernel.*
(b) *Each $\mathcal{F}^{cot}_C$-precover of M is surjective.*
(c) *Assume further that $\operatorname{Tor}^R_{\geqslant 1}(C, \operatorname{Hom}_R(C, M)) = 0$. Then M has an $\mathcal{F}_C$-proper $\mathcal{F}^{cot}_C$-resolution such that $\operatorname{Ker}(\partial^X_{i-1})$ is C-cotorsion for each i.*

Proof (a) The module M has a C-flat cover $\varphi\colon F \otimes_R C \to M$ by [18, Prop. 5.3(a)], and $\operatorname{Ker}(\varphi)$ is C-cotorsion by [27, (2.1.1)]. Furthermore, the bijectivity of the evaluation map $C \otimes_R \operatorname{Hom}_R(C, M) \to M$ implies that there is a projective R-module P and a surjective map $\varphi'\colon P \otimes_R C \twoheadrightarrow M$ by [24, (2.2.a)]. The fact that φ is a precover provides a map $f\colon P \otimes_R C \to F \otimes_R C$ such that $\varphi' = \varphi f$. Hence, the surjectivity of φ' implies that φ is surjective. It follows from Lemma 2.7(a) that $F \otimes_R C$ is C-cotorsion, and so $F \otimes_R C \in \mathcal{F}_C^{\text{cot}}(R)$ by Lemma 4.3. Since φ is a C-flat cover and $\mathcal{F}_C^{\text{cot}}(R) \subseteq \mathcal{F}_C(R)$, we conclude that φ is an $\mathcal{F}_C^{\text{cot}}$-cover.

(b) This follows as in part (a) because M has a surjective $\mathcal{F}_C^{\text{cot}}$-cover.

(c) Using parts (a) and (b), the argument of [18, Thm. 2] shows how to construct a resolution with the desired properties. □

The final three results of this section contain our main conclusions for $\operatorname{res}\widehat{\mathcal{F}_C^{\text{cot}}(R)}$. The first of these extends Lemma 4.3.

Proposition 4.6 *Let C and M be R-modules with C semidualizing, and let $n \geqslant 0$. The following conditions are equivalent:*

(i) $\mathcal{F}_C^{cot}\text{-}\operatorname{pd}_R(M) \leqslant n$;
(ii) $M \in \mathcal{B}_C(R)$ *and* $\mathcal{F}^{cot}\text{-}\operatorname{pd}_R(\operatorname{Hom}_R(C, M)) \leqslant n$;
(iii) $\mathcal{F}^{cot}\text{-}\operatorname{pd}_R(\operatorname{Hom}_R(C, M)) \leqslant n$;
(iv) $M \cong C \otimes_R K$ *for some R-module K such that* $\mathcal{F}^{cot}\text{-}\operatorname{pd}_R(K) \leqslant n$;
(v) $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leqslant n$ *and* $\mathcal{F}_C(R) \perp M$.

Proof (i) $\Longrightarrow$ (ii) Since $\mathcal{F}_C^{\text{cot}}\text{-}\operatorname{pd}_R(M) \leqslant n < \infty$, we have $M \in \mathcal{B}_C(R)$ by Fact 3.7. Let X be an $\mathcal{F}_C^{\text{cot}}$-resolution of M such that $X_i = 0$ when $i > n$. for each i, let $F_i \in \mathcal{F}^{\text{cot}}(R)$ such that $X_i \cong F_i \otimes_R C$. Since each F_i is in $\mathcal{A}_C(R)$, we have

$$\operatorname{Hom}_R(C, X)_i \cong \operatorname{Hom}_R(C, X_i) \cong \operatorname{Hom}_R(C, F_i \otimes_R C) \cong F_i.$$

A standard argument using the conditions $M, X_i \in \mathcal{B}_C(R)$ shows that $\operatorname{Hom}_R(C, X)$ is an $\mathcal{F}^{\text{cot}}$-resolution of $\operatorname{Hom}_R(C, M)$ such that $\operatorname{Hom}_R(C, X)_i = 0$ when $i > n$. The inequality $\mathcal{F}^{\text{cot}}\text{-}\operatorname{pd}_R(\operatorname{Hom}_R(C, M)) \leqslant n$ then follows.

(ii) $\Longrightarrow$ (iv) The condition $M \in \mathcal{B}_C(R)$ implies $M \cong C \otimes_R \operatorname{Hom}_R(C, M)$, and so $K = \operatorname{Hom}_R(C, M)$ satisfies the desired conclusions.

(iv) $\Longrightarrow$ (v) Let F be an $\mathcal{F}^{\text{cot}}$-resolution of K such that $F_i = 0$ when $i > n$. Using the condition $K, F_i \in \mathcal{A}_C(R)$, a standard argument shows that $C \otimes_R F$ is an $\mathcal{F}_C^{\text{cot}}$-resolution of $C \otimes_R K \cong M$. Hence, this resolution yields $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leqslant \mathcal{F}_C^{\text{cot}}\text{-}\operatorname{pd}_R(M) \leqslant n$. By Lemma 4.3, we have $\mathcal{F}_C(R) \perp \mathcal{F}_C^{\text{cot}}(R)$, and so Lemma 2.8 implies $\mathcal{F}_C(R) \perp \operatorname{res}\widehat{\mathcal{F}_C^{\text{cot}}(R)}$; in particular $\mathcal{F}_C(R) \perp M$.

(v) $\Longrightarrow$ (i) The assumption $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leqslant n$ implies $M \in \mathcal{B}_C(R)$ by Fact 3.7, and so $\operatorname{Ext}_R^{\geqslant 1}(C, M) = 0$. Lemma 4.5(c) implies that M has an $\mathcal{F}_C$-proper $\mathcal{F}_C^{\text{cot}}$-resolution X such that $K_i = \operatorname{Ker}(\partial_{i-1}^X)$ is C-cotorsion for each i. In particular, the truncated complex

$$X' = 0 \to K_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

is exact and $\operatorname{Hom}_R(C, -)$-exact. Since $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leqslant n$, the proof of the implication (i) $\Longrightarrow$ (ii) shows that $\operatorname{fd}_R(\operatorname{Hom}_R(C, M)) \leqslant n$. Since each R-module $\operatorname{Hom}_R(C, X_i)$ is flat by Lemma 4.3, the exact complex $\operatorname{Hom}_R(C, X')$ is a truncation of an augmented flat resolution of $\operatorname{Hom}_R(C, M)$. It follows that $\operatorname{Hom}_R(C, K_n)$ is flat, and so $K_n \in \mathcal{F}_C(R)$ by [18, Thm. 1]. Hence X' is an augmented $\mathcal{F}_C^{\text{cot}}$-resolution of M, and so $\mathcal{F}_C^{\text{cot}}\text{-}\operatorname{pd}_R(M) \leqslant n$.

(ii) $\Longleftrightarrow$ (iii) follows from Fact 3.7 because $\operatorname{res}\widehat{\mathcal{F}^{\text{cot}}(R)} \subseteq \mathcal{A}_C(R)$. □

Lemma 4.7 *Let C be a semidualizing R-module. If $\mathcal{F}_C^{cot}\text{-}\operatorname{pd}_R(M) < \infty$, then any bounded $\mathcal{F}_C^{cot}$-resolution X of M is $\mathcal{F}_C$-proper.*

Proof Observe that $\mathcal{F}_C(R) \perp X_i$ for all i and $\mathcal{F}_C(R) \perp M$ by Proposition 4.6. So, the complex X^+ is exact and such that $(X^+)_i = 0$ for $i \gg 0$ and $\mathcal{F}_C(R) \perp (X^+)_i$. Hence, Lemma 2.9(b) implies that X^+ is $\operatorname{Hom}_R(\mathcal{F}_C, -)$-exact. □

Proposition 4.8 *Let C be a semidualizing R-module. The category $\operatorname{res}\widehat{\mathcal{F}_C^{cot}(R)}$ is closed under extensions, cokernels of monomorphisms and summands.*

Proof Consider an exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

such that $\mathcal{F}_C^{\text{cot}}\text{-}\operatorname{pd}_R(M_1)$ and $\mathcal{F}_C^{\text{cot}}\text{-}\operatorname{pd}_R(M_3)$ are finite. To show that $\operatorname{res}\widehat{\mathcal{F}_C^{\text{cot}}(R)}$ is closed under extensions we need to show that $\mathcal{F}_C^{\text{cot}}\text{-}\operatorname{pd}_R(M_2)$ is finite.

The condition $\mathcal{F}_C^{\text{cot}}\text{-}\operatorname{pd}_R(M_1) < \infty$ implies $\mathcal{I}_C\text{-}\operatorname{id}(M_1^*) = \mathcal{F}_C\text{-}\operatorname{pd}_R(M_1) < \infty$ by Lemma 4.2(a) and Proposition 4.6; and similarly $\mathcal{I}_C\text{-}\operatorname{id}(M_3^*) < \infty$. From [24, (3.4)] we know that the category of R-modules of finite $\mathcal{I}_C$-injective dimension is closed under extensions. Using the dual exact sequence

$$0 \to M_3^* \to M_2^* \to M_1^* \to 0$$

we conclude that $\mathcal{I}_C\text{-}\operatorname{id}(M_2^*)$ is finite. Lemma 4.2(a) implies that $\mathcal{F}_C\text{-}\operatorname{pd}_R(M_2)$ is finite.

Since $\mathcal{F}_C^{\text{cot}}\text{-}\operatorname{pd}_R(M_1) < \infty$, Proposition 4.6 implies $\mathcal{F}_C(R) \perp M_1$; and similarly $\mathcal{F}_C(R) \perp M_3$. Thus, we have $\mathcal{F}_C(R) \perp M_2$ by Lemma 2.7(b). Combining this with the previous paragraph, Proposition 4.6 implies that $\mathcal{F}_C^{\text{cot}}\text{-}\operatorname{pd}_R(M_2) < \infty$.

The proof of the fact that $\operatorname{res}\widehat{\mathcal{F}_C^{\text{cot}}(R)}$ is closed under cokernels of monomorphisms is similar. The fact that $\operatorname{res}\widehat{\mathcal{F}_C^{\text{cot}}(R)}$ is closed under summands is even easier to prove using the natural isomorphism $(M_1 \oplus M_2)^* \cong M_1^* \oplus M_2^*$. □

5 Weak AB-Context

Let C be a semidualizing R-module. The point of this section is to show that the triple $(\mathcal{GF}_C(R), \operatorname{res}\widehat{\mathcal{F}_C^{\text{cot}}(R)}, \mathcal{F}_C^{\text{cot}}(R))$ is a weak AB-context, and to document the immediate consequences; see Theorem I and Corollary 5.10. We begin the section with two results modeled on [16, (3.22) and (3.6)].

Lemma 5.1 *If C is a semidualizing R-module, then $\mathcal{GF}_C(R) \perp \operatorname{res}\widehat{\mathcal{F}_C^{cot}(R)}$.*

Proof By Lemma 2.8 it suffices to show $\mathcal{GF}_C(R) \perp \mathcal{F}_C^{\text{cot}}(R)$. Fix modules $M \in \mathcal{GF}_C(R)$ and $N \in \mathcal{F}_C^{\text{cot}}(R)$. By Lemma 4.1, we know that the Pontryagin dual N^* is C-injective. Hence, for $i \geqslant 1$, the vanishing in the next sequence is from Fact 3.9

$$\operatorname{Ext}_R^i(M, N^{**}) \cong \operatorname{Ext}_R^i(M, \operatorname{Hom}_{\mathbb{Z}}(N^*, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}_R^i(M, N^*), \mathbb{Q}/\mathbb{Z}) = 0.$$

The second isomorphism is a form of Hom-tensor adjointness using the fact that $\mathbb{Q}/\mathbb{Z}$ is injective over $\mathbb{Z}$. To finish the proof, it suffices to show that N is a summand of N^{**}; then the last sequence shows $\operatorname{Ext}_R^{\geqslant 1}(M, N) = 0$. Write $N \cong C \otimes_R F$ for some flat cotorsion R-module F, and use Hom-tensor adjointness to conclude

$$N^* \cong \operatorname{Hom}_{\mathbb{Z}}(C \otimes_R F, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}_R(C, \operatorname{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})).$$

Lemma 3.3(b) implies that $\operatorname{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is injective, so the proof of Lemma 4.1(a) explains the second isomorphism in the next sequence

$$N^{**} \cong \operatorname{Hom}_R(C, \operatorname{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}))^* \cong C \otimes_R \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong C \otimes_R F^{**}.$$

The proof of [16, (3.22)] shows that F is a summand of F^{**}, and it follows that $N \cong C \otimes_R F$ is a summand of $C \otimes_R F^{**} \cong N^{**}$, as desired. □

Lemma 5.2 *Let C be a semidualizing R-module. If M is an R-module, then M is in $\mathcal{GF}_C(R)$ if and only if its Pontryagin dual M^* is in $\mathcal{GI}_C(R)$.*

Proof Consider the trivial extension $R \ltimes C$ from Fact 3.9. By [16, (3.6)] we know that M is in $\mathcal{GF}(R \ltimes C)$ if and only if M^* is in $\mathcal{GI}(R \ltimes C)$. Also M is in $\mathcal{GF}(R \ltimes C)$ if and only if M is in $\mathcal{GF}_C(R)$, and M^* is in $\mathcal{GI}(R \ltimes C)$ if and only if M^* is in $\mathcal{GI}_C(R)$ by Fact 3.9. Hence, the equivalence. □

The following result establishes Theorem I(a).

Proposition 5.3 *Let C be a semidualizing R-module. The category $\mathcal{GF}_C(R)$ is closed under kernels of epimorphisms, extensions and summands.*

Proof The result dual to [26, (2.8)] says that $\mathcal{GI}_C(R)$ is closed under cokernels of monomorphisms, extensions and summands. To see that $\mathcal{GF}_C(R)$ is closed under summands, let $M \in \mathcal{GF}_C(R)$ and assume that N is a direct summand of M. It follows that the Pontryagin dual N^* is a direct summand of M^*. Lemma 5.2 implies that M^* is in $\mathcal{GI}_C(R)$ which is closed under summands. We conclude that $N^* \in \mathcal{GI}_C(R)$, and so $N \in \mathcal{GF}_C(R)$. Hence $\mathcal{GF}_C(R)$ is closed under summands, and the other properties are verified similarly. □

The next four results put the finishing touches on Theorem I.

Lemma 5.4 *Let C be a semidualizing R-module. If X is a complete $\mathcal{FF}_C$-resolution, then* $\operatorname{Coker}(\partial_n^X) \in \mathcal{GF}_C(R)$ *for each $n \in \mathbb{Z}$.*

Proof Write $M_n = \operatorname{Coker}(\partial_n^X)$, and note that $M_1 \in \mathcal{GF}_C(R)$ by definition. Fact 3.9 implies that $X_n \in \mathcal{GF}_C(R)$ for each $n \in \mathbb{Z}$. Since M_1 is in $\mathcal{GF}_C(R)$, an induction argument using Proposition 5.3 shows $M_n \in \mathcal{GF}_C(R)$ for each $n \geqslant 1$.

Now assume $n \leqslant 0$. Lemma 2.9(c), implies $\operatorname{Tor}^R_{\geqslant 1}(M_n, \mathcal{I}_C) = 0$. By construction, the following sequence is exact and $- \otimes_R \mathcal{I}_C$-exact

$$0 \to M_n \to X_{n-2} \to X_{n-3} \cdots$$

with each $X_{n-i} \in \mathcal{GF}_C(R)$, and so $M_n \in \mathcal{GF}_C(R)$ by Fact 3.9. □

Lemma 5.5 *Let C be a semidualizing R-module. If $M \in \mathcal{F}_C(R)$, then there is an exact sequence $0 \to M \to M_1 \to M_2 \to 0$ with $M_1 \in \mathcal{F}_C^{cot}(R)$ and $M_2 \in \mathcal{F}_C(R)$.*

Proof Since M is C-flat, we know from [18, Thm. 1] that $\operatorname{Hom}_R(C,M)$ is flat. By [27, (3.1.6)] there is a cotorsion flat module F containing $\operatorname{Hom}_R(C,M)$ such that the quotient $F/\operatorname{Hom}_R(C,M)$ is flat. Consider the exact sequence

$$0 \to \operatorname{Hom}_R(C,M) \to F \to F/\operatorname{Hom}_R(C,M) \to 0.$$

Since $F/\operatorname{Hom}_R(C,M)$ is flat, an application of $C \otimes_R -$ yields an exact sequence

$$0 \to C \otimes_R \operatorname{Hom}_R(C,M) \to C \otimes_R F \to C \otimes_R (F/\operatorname{Hom}_R(C,M)) \to 0.$$

Because M is C-flat, it is in $\mathcal{B}_C(R)$ and so $C \otimes_R \operatorname{Hom}_R(C,M) \cong M$. With $M_1 = C \otimes_R F$ and $M_2 = C \otimes_R (F/\operatorname{Hom}_R(C,M))$ this yields the desired sequence. □

Lemma 5.6 *Let C be a semidualizing R-module. Each module $M \in \mathcal{GF}_C(R)$ admits an injective $\mathcal{F}_C^{cot}$-preenvelope $\alpha\colon M \to Y$ such that* $\operatorname{Coker}(\alpha) \in \mathcal{GF}_C(R)$.

Proof Let $M \in \mathcal{GF}_C(R)$ with complete $\mathcal{FF}_C$-resolution X. By definition, this says that M is a submodule of the C-flat R-module X_{-1}, and Lemma 5.4 implies that $X_{-1}/M \in \mathcal{GF}_C(R)$. Since X_{-1} is C-flat, Lemma 5.5 yields an exact sequence

$$0 \to X_{-1} \to Z \to Z/X_{-1} \to 0$$

with $Z \in \mathcal{F}_C^{\mathrm{cot}}(R)$ and $Z/X_{-1} \in \mathcal{F}_C(R)$. It follows that Z/X_{-1} is in $\mathcal{GF}_C(R)$. Since X_{-1}/M is also in $\mathcal{GF}_C(R)$, and $\mathcal{GF}_C(R)$ is closed under extensions by Proposition 5.3, the following exact sequence shows that Z/M is also in $\mathcal{GF}_C(R)$

$$0 \to X_{-1}/M \to Z/M \to Z/X_{-1} \to 0.$$

In particular, Lemma 5.1 implies $Z/M \perp \mathcal{F}_C^{\mathrm{cot}}(R)$, and it follows that the next sequence is $\operatorname{Hom}_R(-,\mathcal{F}_C^{\mathrm{cot}})$-exact by Lemma 2.7(a).

$$0 \to M \to C \otimes_R F \to Z/M \to 0$$

The conditions $Z \in \mathcal{F}_C^{\mathrm{cot}}(R)$ and $Z/M \in \mathcal{GF}_C(R)$ then implies that the inclusion $M \to Z$ is an $\mathcal{F}_C^{\mathrm{cot}}$-preenvelope whose cokernel is in $\mathcal{GF}_C(R)$. □

Proposition 5.7 *Let C be a semidualizing R-module. The category $\mathcal{F}_C^{cot}(R)$ is an injective cogenerator for the category $\mathcal{GF}_C(R)$. In particular, every module in $\mathcal{GF}_C(R)$ admits a $\mathcal{F}_C^{cot}$-proper $\mathcal{F}_C^{cot}$-coresolution, and so $\mathcal{GF}_C(R) \subseteq$ cores $\widehat{\mathcal{F}_C^{cot}(R)}$.*

Proof Lemmas 5.1 and 5.6 imply that $\mathcal{F}_C^{\mathrm{cot}}(R)$ is an injective cogenerator for $\mathcal{GF}_C(R)$. The remaining conclusions follow immediately. □

Lemma 5.8 *If C is a semidualizing R-module, then there is an equality $\mathcal{F}_C^{cot}(R) = \mathcal{GF}_C(R) \cap$ res $\widehat{\mathcal{F}_C^{cot}(R)}$.*

Proof The containment $\mathcal{F}_C^{\mathrm{cot}}(R) \subseteq \mathcal{GF}_C(R) \cap \operatorname{res}\widehat{\mathcal{F}_C^{\mathrm{cot}}(R)}$ is straightforward; see Definition 2.4 and Fact 3.9. For the reverse containment, let $M \in \mathcal{GF}_C(R) \cap \operatorname{res}\widehat{\mathcal{F}_C^{\mathrm{cot}}(R)}$. Truncate a bounded $\mathcal{F}_C^{\mathrm{cot}}$-resolution to obtain an exact sequence

$$0 \to K \to F \otimes_R C \to M \to 0$$

with $F \in \mathcal{F}^{\mathrm{cot}}(R)$ and such that $\mathcal{F}_C^{\mathrm{cot}}\text{-}\mathrm{pd}_R(K) < \infty$. We have $\mathrm{Ext}_R^1(M, K) = 0$ by Lemma 5.1, so this sequence splits. Hence M is a summand of $F \otimes_R C \in \mathcal{F}_C^{\mathrm{cot}}(R)$. Lemma 4.4 implies that $\mathcal{F}_C^{\mathrm{cot}}(R)$ is closed under summands, so $M \in \mathcal{F}_C^{\mathrm{cot}}(R)$. □

5.9 *Proof of Theorem 1* Part (a) is in Proposition 5.3. Since $\mathcal{F}_C^{\mathrm{cot}}(R) \subseteq \mathcal{GF}_C(R)$ by Fact 3.9, we have $\mathrm{res}\,\widehat{\mathcal{F}_C^{\mathrm{cot}}(R)} \subseteq \mathrm{res}\,\widehat{\mathcal{GF}_C(R)}$. With this, part (b) follows from Proposition 4.8. Proposition 5.7 and Lemma 5.8 justify part (c).

Here is the list of immediate consequences of Theorem I and [15, (1.12.10)]. For part (a), recall that $\mathrm{add}(\mathcal{X})$ is the subcategory of all R-modules isomorphic to a direct summand of a finite direct sum of modules in $\mathcal{X}$.

Corollary 5.10 *Let C be a semidualizing R-module and let $M \in \mathrm{res}\,\widehat{\mathcal{GF}_C(R)}$.*

(a) *If $\mathcal{X}$ is an injective cogenerator for $\mathcal{GF}_C(R)$, then $\mathrm{add}(\mathcal{X}) = \mathcal{F}_C^{cot}(R)$.*

(b) *There exists an exact sequence $0 \to Y \to X \to M \to 0$ with $X \in \mathcal{GF}_C(R)$ and $Y \in \mathrm{res}\,\widehat{\mathcal{F}_C^{cot}(R)}$.*

(c) *There exists an exact sequence $0 \to M \to Y \to X \to 0$ with $X \in \mathcal{GF}_C(R)$ and $Y \in \mathrm{res}\,\widehat{\mathcal{F}_C^{cot}(R)}$.*

(d) *The following conditions are equivalent:*
   (i) $M \in \mathcal{GF}_C(R)$;
   (ii) $\mathrm{Ext}_R^{\geqslant 1}(M, \mathrm{res}\,\widehat{\mathcal{F}_C^{cot}}) = 0$;
   (iii) $\mathrm{Ext}_R^1(M, \mathrm{res}\,\widehat{\mathcal{F}_C^{cot}}) = 0$;
   (iv) $\mathrm{Ext}_R^{\geqslant 1}(M, \mathcal{F}_C^{cot}) = 0$.

   Thus, the surjection $X \to M$ from (b) is a $\mathcal{GF}_C$-precover of M.

(e) *The following conditions are equivalent:*
   (i) $M \in \mathrm{res}\,\widehat{\mathcal{F}_C^{cot}(R)}$;
   (ii) $\mathrm{Ext}_R^{\geqslant 1}(\mathcal{GF}_C, M) = 0$;
   (iii) $\mathrm{Ext}_R^1(\mathcal{GF}_C, M) = 0$;
   (iv) $\sup\{i \geqslant 0 \mid \mathrm{Ext}_R^i(\mathcal{GF}_C, M) \neq 0\} < \infty$ *and* $\mathrm{Ext}_R^{\geqslant 1}(\mathcal{F}_C^{cot}, M) = 0$.

   Thus, the injection $M \to Y$ from (c) is a $\mathrm{res}\,\widehat{\mathcal{F}_C^{cot}}$-preenvelope of M.

(f) *There are equalities*

$$\begin{aligned}\mathcal{GF}_C\text{-}\mathrm{pd}_R(M) &= \sup\{i \geqslant 0 \mid \mathrm{Ext}_R^i(M, \mathrm{res}\,\widehat{\mathcal{F}_C^{cot}}) \neq 0\}\\ &= \sup\{i \geqslant 0 \mid \mathrm{Ext}_R^i(M, \mathcal{F}_C^{cot}) \neq 0\}\end{aligned}$$

(g) *There is an inequality $\mathcal{GF}_C\text{-}\mathrm{pd}_R(M) \leqslant \mathcal{F}_C^{cot}\text{-}\mathrm{pd}_R(M)$ with equality when $\mathcal{F}_C^{cot}\text{-}\mathrm{pd}_R(M) < \infty$.*

(h) *The category $\mathrm{res}\,\widehat{\mathcal{GF}_C(R)}$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms.*

For the next result recall that the triple $(\mathcal{GF}_C(R), \mathrm{res}\,\widehat{\mathcal{F}_C^{\mathrm{cot}}(R)}, \mathcal{F}_C^{\mathrm{cot}}(R))$ is an AB-context if it is a weak AB-context and such that $\mathrm{res}\,\widehat{\mathcal{GF}_C(R)} = \mathcal{M}(R)$.

Proposition 5.11 *Assume that* $\dim(R)$ *is finite, and let C be a semidualizing R-module. The triple* $(\mathcal{GF}_C(R), \operatorname{res}\widehat{\mathcal{F}_C^{cot}(R)}, \mathcal{F}_C^{cot}(R))$ *is an AB-context if and only if C is dualizing for R.*

Proof Assume first that $(\mathcal{GF}_C(R), \operatorname{res}\widehat{\mathcal{F}_C^{\mathrm{cot}}(R)}, \mathcal{F}_C^{\mathrm{cot}}(R))$ is an AB-context. Recall that every maximal ideal of the trivial extension $R \ltimes C$ is of the form $\mathfrak{m} \ltimes C$ for some maximal ideal $\mathfrak{m} \subset R$, and there is an isomorphism $(R \ltimes C)/(\mathfrak{m} \ltimes C) \cong R/\mathfrak{m}$. With Fact 3.9, this yields the equality in the next sequence

$$\begin{aligned}\operatorname{Gfd}_{(R\ltimes C)_{\mathfrak{m}\ltimes C}}((R \ltimes C)_{\mathfrak{m}\ltimes C}/(\mathfrak{m} \ltimes C)_{\mathfrak{m}\ltimes C}) &\leqslant \operatorname{Gfd}_{R\ltimes C}((R \ltimes C)/(\mathfrak{m} \ltimes C)) \\ &= \mathcal{GF}_C\text{-}\operatorname{pd}_R(R/\mathfrak{m}) < \infty.\end{aligned}$$

The first inequality follows from [5, (5.1.3)], and the finiteness is by assumption. Using [5, (1.2.7),(1.4.9),(5.1.11)] we deduce that the following ring is Gorenstein

$$(R \ltimes C)_{\mathfrak{m}\ltimes C} \cong R_{\mathfrak{m}} \ltimes C_{\mathfrak{m}}$$

and so [21, (7)] implies that $C_{\mathfrak{m}}$ is dualizing for $R_{\mathfrak{m}}$. (This also follows from [6, (8.1)] and [17, (3.1)].) Since this is true for each maximal ideal of R and $\dim(R) < \infty$, we conclude that C is dualizing for R by [14, (5.8.2)].

Conversely, assume that C is dualizing for R. Using Theorem I, it suffices to show that each R-module M has $\mathcal{GF}_C\text{-}\operatorname{pd}_R(M) < \infty$. Since C is dualizing, the trivial extension $R \ltimes C$ is Gorenstein by [21, (7)]. Also, we have $\dim(R \ltimes C) = \dim(R) < \infty$ as $\operatorname{Spec}(R \ltimes C)$ is in bijection with $\operatorname{Spec}(R)$. Thus, in the next sequence

$$\mathcal{GF}_C\text{-}\operatorname{pd}_R(M) = \operatorname{Gfd}_{R\ltimes C}(M) < \infty$$

the finiteness is from [9, (12.3.1)] and the equality is from Fact 3.9. □

To end this section, we prove a complement to [26, (3.6)] which establishes the existence of certain approximations. For this, we need the following preliminary result which compares to Lemma 5.8.

Lemma 5.12 *If C is a semidualizing R-module, then there is an equality* $\mathcal{F}_C(R) = \mathcal{GF}_C(R) \cap \operatorname{res}\widehat{\mathcal{F}_C(R)}$.

Proof The containment $\mathcal{F}_C(R) \subseteq \mathcal{GF}_C(R) \cap \operatorname{res}\widehat{\mathcal{F}_C(R)}$ is from Definition 2.4 and Fact 3.9. For the reverse containment, let $M \in \mathcal{GF}_C(R) \cap \operatorname{res}\widehat{\mathcal{F}_C(R)}$. Let $n \geqslant 1$ be an integer with $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leqslant n$. We show by induction on n that M is C-flat.

For the base case $n = 1$, there is an exact sequence

$$0 \to X_1 \to X_0 \to M \to 0 \tag{†}$$

with $X_1, X_0 \in \mathcal{F}_C(R)$. Lemma 5.5 provides an exact sequence

$$0 \to X_1 \to Y_1 \to Y_2 \to 0 \tag{‡}$$

with $Y_1 \in \mathcal{F}_C^{\mathrm{cot}}(R)$ and $Y_2 \in \mathcal{F}_C(R)$. Consider the following pushout diagram whose top row is Eq. † and whose leftmost column is Eq. ‡.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & V & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_2 & \xrightarrow{\cong} & Y_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad (*)$$

Since M is in $\mathcal{GF}_C(R)$ and Y_1 is in $\mathcal{F}_C^{\mathrm{cot}}(R)$, Lemma 5.1 implies $\mathrm{Ext}_R^1(M, Y_1) = 0$. Hence, the middle row of Eq. $*$ splits. The subcategory $\mathcal{F}_C(R)$ is closed under extensions and summands by [18, Props. 5.1(a) and 5.2(a)]. Hence, the middle column of Eq. $*$ shows that $V \in \mathcal{F}_C(R)$, so the fact that the middle row of Eq. $*$ splits implies that $M \in \mathcal{F}_C(R)$, as desired.

For the induction step, assume that $n \geqslant 2$. Truncate a bounded $\mathcal{F}_C$-resolution of M to find an exact sequence

$$0 \to K \to Z \to M \to 0$$

such that $Z \in \mathcal{F}_C(R)$ and $\mathcal{F}_C\text{-}\mathrm{pd}_R(K) \leqslant n - 1$. By induction, we conclude that $K \in \mathcal{F}_C(R)$. Hence, the displayed sequence implies $\mathcal{F}_C\text{-}\mathrm{pd}_R(M) \leqslant 1$, and the base case implies that $M \in \mathcal{F}_C(R)$. □

Proposition 5.13 *Let C be a semidualizing R-module and assume that* $\dim(R)$ *is finite. If $M \in \mathcal{GF}_C(R)$, then there exists an exact sequence*

$$0 \to K \to X \to M \to 0$$

such that $K \in \mathcal{F}_C(R)$ and $X \in \mathcal{GP}_C(R)$.

Proof Since M is in $\mathcal{GF}_C(R)$ and $\dim(R) < \infty$, we know that $\mathcal{GP}_C\text{-}\mathrm{pd}_R(M) < \infty$ by [22, (3.3.c)]. Hence, from [26, (3.6)] there is an exact sequence

$$0 \to K \to X \to M \to 0$$

with $K \in \mathrm{res}\,\widehat{\mathcal{P}_C(R)}$ and $X \in \mathcal{GP}_C(R)$. From [22, (3.3.a)] we have $X \in \mathcal{GP}_C(R) \subseteq \mathcal{GF}_C(R)$. Since $\mathcal{GF}_C(R)$ is closed under kernels of epimorphisms by Proposition 5.3, the displayed sequence implies that $K \in \mathcal{GF}_C(R)$. The containment $\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$ implies $K \in \mathrm{res}\,\widehat{\mathcal{P}_C(R)} \subseteq \mathrm{res}\,\widehat{\mathcal{F}_C(R)}$, and so Lemma 5.12 says $K \in \mathcal{F}_C(R)$. Thus, the displayed sequence has the desired properties. □

6 Stability of Categories

This section contains our analysis of the categories $\mathcal{G}^n(\mathcal{F}_C(R))$ and $\mathcal{G}^n(\mathcal{F}_C^{\mathrm{cot}}(R))$; see Definition 3.10. We draw many of our conclusions from the known behavior for $\mathcal{G}^n(\mathcal{I}_C(R))$ using Pontryagin duals. This requires, however, the use of the categories $\mathcal{H}_C^n(\mathcal{F}_C(R))$ and $\mathcal{H}_C^n(\mathcal{F}_C^{\mathrm{cot}}(R))$ as a bridge; see Definition 3.12.

Lemma 6.1 *Let C be a semidualizing R-module, and let X be an R-complex. If X is $\mathrm{Hom}_R(-, \mathcal{F}_C^{cot})$-exact, then it is $- \otimes_R \mathcal{I}_C$-exact.*

Proof Let $N \in \mathcal{I}_C(R)$. From Lemmas 4.1(d) and 4.3 we know that the Pontryagin dual N^* is in $\mathcal{F}_C^{\mathrm{cot}}(R)$. Hence, the following complex is exact by assumption

$$\mathrm{Hom}_R(X, N^*) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(X \otimes_R N, \mathbb{Q}/\mathbb{Z}).$$

As $\mathbb{Q}/\mathbb{Z}$ is faithfully injective over $\mathbb{Z}$, we conclude that $X \otimes_R N$ is exact, and so X is $- \otimes_R \mathcal{I}_C$-exact. □

Note that the hypotheses of the next lemma are satisfied whenever $\mathcal{X} \subseteq \mathcal{GF}_C(R)$ by Fact 3.9 and Lemma 5.1.

Lemma 6.2 *Let C be a semidualizing R-module and $\mathcal{X}$ a subcategory of $\mathcal{M}(R)$.*

(a) *If $\mathrm{Tor}^R_{\geqslant 1}(\mathcal{X}, \mathcal{I}_C) = 0$, then $\mathrm{Tor}^R_{\geqslant 1}(\mathcal{H}_C^n(\mathcal{X}), \mathcal{I}_C) = 0$ for each $n \geqslant 1$.*
(b) *If $\mathcal{X} \perp \mathcal{F}_C^{cot}(R)$, then $\mathcal{H}_C^n(\mathcal{X}) \perp \mathcal{F}_C^{cot}(R)$ for each $n \geqslant 1$.*

Proof By induction on n, it suffices to prove the result for $n = 1$. We prove part (a). The proof of part (b) is similar. Let $M \in \mathcal{H}_C(\mathcal{X})$ with $\mathcal{P}_C\mathcal{F}_C^{\mathrm{cot}}$-complete $\mathcal{X}$-resolution X. The complex X is $- \otimes_R \mathcal{I}_C$-exact by Lemma 6.1. Since we have assumed that $\mathrm{Tor}^R_{\geqslant 1}(\mathcal{X}, \mathcal{I}_C) = 0$, the desired conclusion follows from Lemma 2.9(c) because $M \cong \mathrm{Ker}(\partial^X_{-1})$. □

The converse of the next result is in Proposition 6.5.

Lemma 6.3 *If C is a semidualizing R-module and $M \in \mathcal{H}_C(\mathcal{F}_C(R))$, then $M^* \in \mathcal{G}(\mathcal{I}_C(R))$.*

Proof Let X be a $\mathcal{P}_C\mathcal{F}_C^{\mathrm{cot}}$-complete $\mathcal{F}_C$-resolution of M. Lemma 4.1(b) implies that the complex $X^* = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ is an exact complex in $\mathcal{I}_C(R)$. Furthermore $M^* \cong \mathrm{Coker}(\partial_1^{X^*})$. Thus, it suffices to show that X^* is $\mathrm{Hom}_R(\mathcal{I}_C, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{I}_C)$-exact. Let I be an injective R-module.

The second isomorphism in the next sequence is Hom-evaluation [7, Prop. 2.1(ii)]

$$C \otimes_R X^* \cong C \otimes_R \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(C, X), \mathbb{Q}/\mathbb{Z}).$$

Since $\mathrm{Hom}_R(C, X)$ is exact by assumption, we conclude that $C \otimes_R X^* \cong X^* \otimes_R C$ is also exact. It follows that the following complexes are also exact

$$\mathrm{Hom}_R(X^* \otimes_R C, I) \cong \mathrm{Hom}_R(X^*, \mathrm{Hom}_R(C, I))$$

where the isomorphism is Hom-tensor adjointness. Thus X^* is $\mathrm{Hom}_R(-, \mathcal{I}_C)$-exact.

Lemma 6.1 implies that the complex $\mathrm{Hom}_R(C, I) \otimes_R X$ is exact. Hence, the following complexes are also exact

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(C, I) \otimes_R X, \mathbb{Q}/\mathbb{Z}) &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(C, I), \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(C, I), X^*)\end{aligned}$$

and so X^* is $\mathrm{Hom}_R(\mathcal{I}_C, -)$-exact. □

The next result is a version of [23, (5.2)] for $\mathcal{H}_C(\mathcal{F}_C(R))$.

Proposition 6.4 *If C is a semidualizing R-module, then there is an equality $\mathcal{H}_C(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$.*

Proof For the containment $\mathcal{H}_C(\mathcal{F}_C(R)) \subseteq \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$, let $M \in \mathcal{H}_C(\mathcal{F}_C(R))$, and let X be a $\mathcal{P}_C\mathcal{F}_C^{\mathrm{cot}}$-complete $\mathcal{F}_C$-resolution of M. Lemma 6.1 implies that X is $- \otimes_R \mathcal{I}_C$-exact, and so the sequence

$$0 \to M \to X_{-1} \to X_{-2} \to \cdots$$

satisfies condition 3.9(1). Fact 3.9 implies $\mathrm{Tor}^R_{\geqslant 1}(\mathcal{F}_C, \mathcal{I}_C) = 0$ and so Lemma 6.2(a) provides $\mathrm{Tor}^R_{\geqslant 1}(M, \mathcal{I}_C) = 0$. From Fact 3.9 we conclude $M \in \mathcal{GF}_C(R)$. Also, Lemma 6.3 guarantees that $M^* \in \mathcal{G}(\mathcal{I}_C(R))$, and so $M^* \in \mathcal{A}_C(R)$ by Fact 3.11. Thus, Fact 3.7 implies $M \in \mathcal{B}_C(R)$.

For the reverse containment, let $M \in \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$, and let Y be a complete $\mathcal{FF}_C$-resolution of M. In particular, the complex

$$0 \to M \to Y_{-1} \to Y_{-2} \to \cdots \qquad (\dagger)$$

is an augmented $\mathcal{F}_C$-coresolution of M and is $- \otimes_R \mathcal{I}_C$-exact. We claim that this complex is also $\mathrm{Hom}_R(C, -)$-exact and $\mathrm{Hom}_R(-, \mathcal{F}_C^{\mathrm{cot}})$-exact. For each $i \in \mathbb{Z}$ set $M_i = \mathrm{Coker}(\partial_i^Y)$. This yields an isomorphism $M \cong M_1$. By assumption, we have $M, Y_i \in \mathcal{B}_C(R)$ for each $i < 0$, and so $C \perp M$ and $C \perp Y_i$. Thus, Lemma 2.8(b) implies that the complex (†) is $\mathrm{Hom}_R(C, -)$-exact. From Lemma 5.4 we conclude $M_i \in \mathcal{GF}_C(R)$ for each i, and so $M_i \perp \mathcal{F}_C^{\mathrm{cot}}(R)$ by Lemma 5.1. Lemma 4.3 implies $Y_i \perp \mathcal{F}_C^{\mathrm{cot}}(R)$ for each $i < 0$, and so Lemma 2.9(a) guarantees that Eq. † is also $\mathrm{Hom}_R(-, \mathcal{F}_C^{\mathrm{cot}})$-exact.

Because $M \in \mathcal{B}_C(R)$, Fact 3.7 provides an augmented $\mathcal{P}_C$-proper $\mathcal{P}_C$-resolution

$$\cdots \xrightarrow{\partial_2^Z} Z_1 \xrightarrow{\partial_1^Z} Z_0 \to M \to 0. \qquad (\ddagger)$$

Since each $Z_i \in \mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$, we have $Z_i \perp \mathcal{F}_C^{\mathrm{cot}}(R)$ by Lemma 4.3. Since $M \perp \mathcal{F}_C^{\mathrm{cot}}(R)$, we see from Lemma 2.9(a) that Eq. ‡ is also $\mathrm{Hom}_R(-, \mathcal{F}_C^{\mathrm{cot}})$-exact.

It follows that the complex obtained by splicing the sequences (†) and (‡) is a $\mathcal{P}_C\mathcal{F}_C^{\mathrm{cot}}$-complete $\mathcal{F}_C$-resolution of M. Thus $M \in \mathcal{H}_C(\mathcal{F}_C(R))$, as desired. □

Our next result contains the converse to Lemma 6.3.

Proposition 6.5 *Let C be a semidualizing R-module and M an R-module. Then $M \in \mathcal{H}_C(\mathcal{F}_C(R))$ if and only if $M^* \in \mathcal{G}(\mathcal{I}_C(R))$.*

Proof One implication is in Lemma 6.3. For the converse, assume that M^* is in $\mathcal{G}(\mathcal{I}_C(R)) = \mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$; see Fact 3.11. Fact 3.7 and Lemma 5.2 combine with Proposition 6.4 to yield $M \in \mathcal{B}_C(R) \cap \mathcal{GF}_C(R) = \mathcal{H}_C(\mathcal{F}_C(R))$. □

The next three lemmata are for use in Theorem 6.9.

Lemma 6.6 *If C is a semidualizing R-module, then* $\mathcal{H}_C^2(\mathcal{F}_C(R)) \subseteq \mathcal{B}_C(R)$.

Proof Let $M \in \mathcal{H}_C^2(\mathcal{F}_C(R))$ and let X be a $\mathcal{P}_C\mathcal{F}_C^{\mathrm{cot}}$-complete $\mathcal{H}_C(\mathcal{F}_C)$-resolution of M. In particular, the complex $\mathrm{Hom}_R(C, X)$ is exact. Each module X_i is in $\mathcal{H}_C(\mathcal{F}_C(R)) \subseteq \mathcal{B}_C(R)$ by Proposition 6.4, and so $\mathrm{Ext}_R^{\geqslant 1}(C, X_i) = 0$ for each i. Thus, Lemma 2.8(b) implies that $\mathrm{Ext}_R^{\geqslant 1}(C, M) = 0$. Also, since $M \cong \mathrm{Ker}(\partial_{-1}^X)$, the left-exactness of $\mathrm{Hom}_R(C, -)$ implies that $\mathrm{Hom}_R(C, M) \cong \mathrm{Ker}(\partial_{-1}^{\mathrm{Hom}_R(C,X)})$.

The natural evaluation map $C \otimes_R \mathrm{Hom}_R(C, X_i) \to X_i$ is an isomorphism for each i because $X_i \in \mathcal{B}_C(R)$, and so we have $C \otimes_R \mathrm{Hom}_R(C, X) \cong X$. In particular, the complex $\mathrm{Hom}_R(C, X)$ is $- \otimes_R C$-exact. As $\mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, X_i)) = 0$ for each i, Lemma 2.9(c) implies that $\mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, M)) = 0$.

Finally, each row in the following diagram is exact

$$\begin{array}{ccccccc}
C \otimes_R \mathrm{Hom}_R(C, X_1) & \longrightarrow & C \otimes_R \mathrm{Hom}_R(C, X_0) & \longrightarrow & C \otimes_R \mathrm{Hom}_R(C, M) & \longrightarrow & 0 \\
\cong \downarrow & & \cong \downarrow & & \downarrow & & \\
X_1 & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and the vertical arrows are the natural evaluation maps. A diagram chase shows that the rightmost vertical arrow is an isomorphism, and so $M \in \mathcal{B}_C(R)$. □

Lemma 6.7 *If C is a semidualizing R-module, then* $\mathcal{F}_C^{cot}(R)$ *is an injective cogenerator for* $\mathcal{H}_C(\mathcal{F}_C(R))$.

Proof The containment in the following sequence is from Facts 3.7 and 3.9

$$\mathcal{F}_C^{\mathrm{cot}}(R) \subseteq \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) = \mathcal{H}_C(\mathcal{F}_C(R))$$

and the equality is from Proposition 6.4. Lemma 5.1 implies $\mathcal{GF}_C(R) \perp \mathcal{F}_C^{\mathrm{cot}}(R)$. Thus, the conditions $\mathcal{H}_C(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \subseteq \mathcal{GF}_C(R)$ imply that we have $\mathcal{H}_C(\mathcal{F}_C(R)) \perp \mathcal{F}_C^{\mathrm{cot}}(R)$.

Let $M \in \mathcal{H}_C(\mathcal{F}_C(R)) \subseteq \mathcal{GF}_C(R)$. Since $\mathcal{F}_C^{\mathrm{cot}}(R)$ is an injective cogenerator for $\mathcal{GF}_C(R)$ by Proposition 5.7, there is an exact sequence

$$0 \to M \to X \to M' \to 0$$

with $X \in \mathcal{F}_C^{\mathrm{cot}}(R)$ and $M' \in \mathcal{GF}_C(R)$. Since M and X are in $\mathcal{B}_C(R)$, Fact 3.7 implies that $M' \in \mathcal{B}_C(R)$. That is $M' \in \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) = \mathcal{H}_C(\mathcal{F}_C(R))$. This establishes the desired conclusion. □

Lemma 6.8 *If C is a semidualizing R-module, then* $\mathcal{H}_C^2(\mathcal{F}_C(R)) \subseteq \mathrm{cores}\,\widetilde{\mathcal{F}_C^{cot}}(R)$.

Proof Lemma 6.7 says that $\mathcal{F}_C^{\text{cot}}(R)$ is an injective cogenerator for $\mathcal{H}_C(\mathcal{F}_C(R))$. By Lemma 6.2(b) we know that $\mathcal{H}_C^2(\mathcal{F}_C(R)) \perp \mathcal{F}_C^{\text{cot}}(R)$. Let $M \in \mathcal{H}_C^2(\mathcal{F}_C(R))$ and let X be a $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$-complete $\mathcal{H}_C(\mathcal{F}_C)$-resolution of M. By definition, the complex

$$0 \to M \to X_{-1} \to X_{-2} \to \cdots$$

is an augmented $\mathcal{H}_C(\mathcal{F}_C)$-coresolution that is $\mathcal{F}_C$-proper and therefore $\mathcal{F}_C^{\text{cot}}$-proper. Hence, Lemma 2.10 implies $M \in \text{cores}\,\widetilde{\mathcal{F}_C^{\text{cot}}(R)}$. □

Theorem II *For each semidualizing R-module C and each integer $n \geqslant 1$, there is an equality $\mathcal{H}_C^n(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$.*

Proof We first verify the equality $\mathcal{H}_C^2(\mathcal{F}_C(R)) = \mathcal{H}_C(\mathcal{F}_C(R))$. Remark 3.13 implies $\mathcal{H}_C^2(\mathcal{F}_C(R)) \supseteq \mathcal{H}_C(\mathcal{F}_C(R))$. For the reverse containment, let $M \in \mathcal{H}_C^2(\mathcal{F}_C(R))$. Lemma 4.3 implies $\mathcal{F}_C(R) \perp \mathcal{F}_C^{\text{cot}}(R)$, and so $M \perp \mathcal{F}_C^{\text{cot}}(R)$ by Lemma 6.2(b). From Lemma 6.6 we have $M \in \mathcal{B}_C(R)$, and so Fact 3.7 provides an augmented $\mathcal{P}_C$-proper $\mathcal{P}_C$-resolution

$$\cdots \xrightarrow{\partial_2^Z} Z_1 \xrightarrow{\partial_1^Z} Z_0 \to M \to 0. \tag{‡}$$

Each $Z_i \in \mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$, so we have $Z_i \perp \mathcal{F}_C^{\text{cot}}(R)$ by Lemma 4.3. We conclude from Lemma 2.9(a) that Eq. ‡ is $\text{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$-exact.

Lemma 6.8 yields a $\mathcal{F}_C^{\text{cot}}$-proper augmented $\mathcal{F}_C^{\text{cot}}$-coresolution

$$0 \to M \to Y_{-1} \to Y_{-2} \to \cdots . \tag{†}$$

Since each $Y_i \in \mathcal{F}_C^{\text{cot}}(R) \subseteq \mathcal{B}_C(R)$ by Fact 3.7, we have $C \perp Y_i$ for each $i < 0$, and similarly $C \perp M$. Thus, Lemma 2.8(b) implies that Eq. † is $\text{Hom}_R(C, -)$-exact. It follows that the complex obtained by splicing the sequences (‡) and (†) is a $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$-complete $\mathcal{F}_C$-resolution of M. Thus, we have $M \in \mathcal{H}_C(\mathcal{F}_C(R))$.

To complete the proof, use the previous two paragraphs and argue by induction on n to verify the first equality in the next sequence

$$\mathcal{H}_C^n(\mathcal{F}_C(R)) = \mathcal{H}_C(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R).$$

The second equality is from Proposition 6.4. □

Our next result contains Theorem II(a) from the introduction.

Corollary 6.10 *If C is a semidualizing R-module, then $\mathcal{G}^n(\mathcal{GF}_C(R) \cap \mathcal{B}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ for each $n \geqslant 1$.*

Proof In the next sequence, the containments are from Fact 3.11 and Remark 3.13

$$\begin{aligned}\mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \subseteq \mathcal{G}^n(\mathcal{GF}_C(R) \cap \mathcal{B}_C(R)) &= \mathcal{G}^n(\mathcal{H}_C(\mathcal{F}_C(R)))\\ &\subseteq \mathcal{H}_C^n(\mathcal{H}_C(\mathcal{F}_C(R))) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)\end{aligned}$$

and the equalities are by Proposition 6.4 and Theorem 6.9. □

Remark 6.11 In light of Corollary 6.10, it is natural to ask whether we have $\mathcal{G}(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ for each semidualizing R-module C. While Remark 3.13 and Proposition 6.4 imply that $\mathcal{G}(\mathcal{F}_C(R)) \subseteq \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$, we do not know whether the reverse containment holds.

We now turn our attention to $\mathcal{H}^n_C(\mathcal{F}^{\text{cot}}_C(R))$ and $\mathcal{G}^n(\mathcal{F}^{\text{cot}}_C(R))$.

Proposition 6.12 *Let C be a semidualizing R-module and let $n \geqslant 1$.*

(a) *We have $\mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp \subseteq \mathcal{H}^n_C(\mathcal{F}^{cot}_C(R)) \subseteq \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$.*
(b) *If $\dim(R) < \infty$, then $\mathcal{F}_C(R) \perp \mathcal{H}^n_C(\mathcal{F}^{cot}_C(R))$.*
(c) *If $\dim(R) < \infty$, then $\mathcal{H}^n_C(\mathcal{F}^{cot}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$.*

Proof (a) For the first containment, let $M \in \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$. Since $M \in \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^\perp$, Lemma 4.5(c) yields an augmented $\mathcal{F}^{\text{cot}}_C$-resolution

$$\cdots \to Z_1 \to Z_0 \to M \to 0$$

that is $\operatorname{Hom}_R(C, -)$-exact; the argument of Proposition 6.4 shows that this resolution is $\operatorname{Hom}_R(-, \mathcal{F}^{\text{cot}}_C)$-exact. Because M is in $\mathcal{GF}_C(R)$, Proposition 5.7 provides an augmented $\mathcal{F}^{\text{cot}}_C$-coresolution

$$0 \to M \to Y_{-1} \to Y_{-2} \to \cdots$$

that is $\operatorname{Hom}_R(-, \mathcal{F}^{\text{cot}}_C)$-exact. Since $M \in \mathcal{B}_C(R)$, the proof of Proposition 6.4 shows that this coresolution is also $\operatorname{Hom}_R(C, -)$-exact. Splicing these resolutions yields a $\mathcal{P}_C\mathcal{F}^{\text{cot}}_C$-complete $\mathcal{F}^{\text{cot}}_C$-resolution of M, and so $M \in \mathcal{H}_C(\mathcal{F}^{\text{cot}}_C(R)) \subseteq \mathcal{H}^n_C(\mathcal{F}^{\text{cot}}_C(R))$.

The second containment follows from the next sequence

$$\mathcal{H}^n_C(\mathcal{F}^{\text{cot}}_C(R)) \subseteq \mathcal{H}^n_C(\mathcal{F}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$$

wherein the containment is by definition, and the equality is by Theorem 6.9.

(b) Assume $d = \dim(R) < \infty$. A result of Gruson and Raynaud [20, Seconde Partie, Thm. (3.2.6)] and Jensen [19, Prop. 6] implies $\operatorname{pd}_R(F) \leqslant d < \infty$ for each flat R-module F.

We prove the result for all $n \geqslant 0$ by induction on n. The base case $n = 0$ follows from Lemma 4.3. Assume $n \geqslant 1$ and that $\mathcal{F}_C(R) \perp \mathcal{H}^{n-1}_C(\mathcal{F}^{\text{cot}}_C(R))$. Let $M \in \mathcal{H}^n_C(\mathcal{F}^{\text{cot}}_C(R))$, and let X be a $\mathcal{P}_C\mathcal{F}^{\text{cot}}_C$-complete $\mathcal{H}^{n-1}_C(\mathcal{F}^{\text{cot}}_C)$-resolution of M. For each i set $M_i = \operatorname{Im}(\partial^X_i)$. This yields an isomorphism $M \cong M_0$ and, for each i, an exact sequence

$$0 \to M_{i+1} \to X_i \to M_i \to 0.$$

Note that $M_i, X_i \in \mathcal{B}_C(R)$ by part (a). Let $F \otimes_R C \in \mathcal{F}_C(R)$ and let $t \geqslant 1$. Since $\mathcal{F}_C(R) \perp X_i$ for each i, a standard dimension-shifting argument yields the first isomorphism in the next sequence

$$\operatorname{Ext}^t_R(F \otimes_R C, M) \cong \operatorname{Ext}^{t+d}_R(F \otimes_R C, M_d) \cong \operatorname{Ext}^{t+d}_R(F, \operatorname{Hom}_R(C, M_d)) = 0.$$

The second isomorphism is a form of Hom-tensor adjointness using the fact that F is flat with the Bass class condition $\operatorname{Ext}^{\geqslant 1}_R(C, M_d) = 0$. The vanishing follows from the inequality $\operatorname{pd}_R(F) \leqslant d$.

(c) This follows from parts (a) and (b). □

Lemma 6.13 *Let C be a semidualizing R-module and assume* $\dim(R) < \infty$. *If* $M \in \mathcal{F}_C(R)$, *then* $\mathcal{F}_C^{cot}\text{-}\operatorname{id}_R(M) \leqslant \dim(R) < \infty$.

Proof Let F be a flat R-module such that $M \cong F \otimes_R C$. Since $d = \dim(R)$ is finite, the flat module F has an $\mathcal{F}^{\text{cot}}$-coresolution X such that $X_i = 0$ for all $i < -d$; see [9, (8.5.12)]. Since $M \in \mathcal{A}_C(R)$ and each $X_i \in \mathcal{A}_C(R)$, it follows readily that the complex $X \otimes_R F$ is an $\mathcal{F}_C^{\text{cot}}$-coresolution of M of length at most d, as desired. □

Our final result contains Theorem II(b) from the introduction.

Theorem II *Let C be a semidualizing R-module and assume* $\dim(R) < \infty$. *Then* $\mathcal{G}^n(\mathcal{F}_C^{cot}(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^{\perp}$ *for each $n \geqslant 1$, and $\mathcal{F}_C^{cot}(R)$ is an injective cogenerator and a projective generator for $\mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^{\perp}$.*

Proof We first show $\mathcal{G}(\mathcal{F}_C^{\text{cot}}(R)) \supseteq \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$. Let $M \in \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$ and let X be a $\mathcal{P}_C\mathcal{F}_C^{\text{cot}}$-complete $\mathcal{F}_C^{\text{cot}}$-resolution of M. To show that M is in $\mathcal{G}(\mathcal{F}_C^{\text{cot}}(R))$, it suffices to show that X is $\operatorname{Hom}_R(\mathcal{F}_C^{\text{cot}}, -)$-exact, since it is $\operatorname{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$-exact by definition. For each i, set $M_i = \operatorname{Im}(\partial_i^X) \in \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$. Lemma 4.3 and Proposition 6.12(b) imply $\mathcal{F}_C(R) \perp X_i$ and $\mathcal{F}_C(R) \perp M_i$ for all i. Hence, Lemma 2.8(b) implies that X is $\operatorname{Hom}_R(\mathcal{F}_C, -)$-exact, and so X is $\operatorname{Hom}_R(\mathcal{F}_C^{\text{cot}}, -)$-exact.

We next show $\mathcal{G}(\mathcal{F}_C^{\text{cot}}(R)) \subseteq \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$. Let $N \in \mathcal{G}(\mathcal{F}_C^{\text{cot}}(R))$ and let Y be a complete $\mathcal{F}_C^{\text{cot}}$-resolution of N. We will show that Y is $\operatorname{Hom}_R(\mathcal{F}_C, -)$-exact; the containment $\mathcal{P}_C(R) \subseteq \mathcal{F}_C(R)$ will then imply that Y is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact. Since Y is $\operatorname{Hom}_R(-, \mathcal{F}_C^{\text{cot}})$-exact by definition, we will then conclude that N is in $\mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R))$. We have $\mathcal{F}_C(R) \perp Y_i$ for each i by Lemma 4.3, and so $\mathcal{F}_C^{\text{cot}}(R) \perp Y_i$. Since Y is $\operatorname{Hom}_R(\mathcal{F}_C^{\text{cot}}, -)$-exact, Lemma 2.9(b) implies $\mathcal{F}_C^{\text{cot}}(R) \perp M$. From Lemma 2.8 we conclude that cores $\widehat{\mathcal{F}_C^{\text{cot}}(R)} \perp M$. Since $\dim(R) < \infty$, Lemma 6.13 implies that $\mathcal{F}_C(R) \subseteq$ cores $\widehat{\mathcal{F}_C^{\text{cot}}(R)}$ and so $\mathcal{F}_C(R) \perp M$. With the condition $\mathcal{F}_C(R) \perp Y_i$ from above, this implies that Y is $\operatorname{Hom}_R(\mathcal{F}_C, -)$-exact by Lemma 2.8(b).

The above paragraphs yield the second equality in the next sequence

$$\mathcal{G}^n(\mathcal{F}_C^{\text{cot}}(R)) = \mathcal{G}(\mathcal{F}_C^{\text{cot}}(R)) = \mathcal{H}_C(\mathcal{F}_C^{\text{cot}}(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R) \cap \mathcal{F}_C(R)^{\perp}.$$

The first equality is from [23, (4.10)] since Lemma 4.3 implies $\mathcal{F}_C^{\text{cot}}(R) \perp \mathcal{F}_C^{\text{cot}}(R)$, and the third equality is from Proposition 6.12(c). The final conclusion follows from [23, (4.7)]. □

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