J. London Math. Soc. (2) 96 (2017) 271–292
doi:10.1112/jlms.12055

Geometric aspects of representation theory for DG algebras: answering a question of Vasconcelos

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Dedicated to Wolmer V. Vasconcelos

Abstract

We apply geometric techniques from representation theory to the study of homologically finite differential graded (DG) modules M over a finite dimensional, positively graded, commutative DG algebra U. In particular, in this setting we prove a version of a theorem of Voigt by exhibiting an isomorphism between the Yoneda Ext group $\mathrm{YExt}^1_U(M,M)$ and a quotient of tangent spaces coming from an algebraic group action on an algebraic variety. As an application, we answer a question of Vasconcelos from 1974 by showing that a local ring has only finitely many semidualizing complexes up to shift-isomorphism in the derived category $\mathcal{D}(R)$.

Contents

1. *Introduction*

Convention. In this paper, R is a commutative noetherian ring with identity.

The main result of this paper is concerned with a question of Vasconcelos about *semidualizing R-modules*, that is, finitely generated R-modules C such that $\mathrm{Hom}_R(C,C) \cong R$ and $\mathrm{Ext}^i_R(C,C) = 0$ for $i \geqslant 1$. These modules were introduced by Foxby [**20**] as 'PG modules of rank 1', providing a common generalization of Grothendieck's canonical modules over Cohen–Macaulay rings [**30**] and the projective modules of rank 1. (The P in PG is for 'projective' and the G is for 'Gorenstein', referring to Sharp's Gorenstein modules [**37**]. However, the definition makes no assumption about the projective or injective dimension of C.) Foxby's objective was to provide a single framework that explained useful functorial properties exhibited by these two seemingly different classes of modules.

Vasconcelos [**39**] discovered these objects independently (calling them 'spherical modules') in his investigation of divisors. Later Golod [**26**] singled them out (also independently, but calling them 'suitable modules'[†]) to give a general duality for Auslander and Bridger's G-dimension [**3**]. Wakamatsu [**43**] similarly identified them in as a useful generalization of tilting modules

Received 21 April 2016; revised 19 December 2016; published online 9 July 2017.

2010 *Mathematics Subject Classification* 13D02, 13D09, 13E10, 14L30, 16G30 (primary).

Sather-Wagstaff was supported in part by North Dakota EPSCoR, National Science Foundation Grant EPS-0814442, and NSA grant H98230-13-1-0215.

[†]According to private communication with Foxby, the Russian term originally used by Golod has several translations, including 'comfortable modules'.

in representation theory, calling them, appropriately, 'generalized tilting modules'. The point in much of this work is that these generalizations give enough additional flexibility to allow one to verify properties for them that specialize to interesting results about their progenitors. For instance, Sather-Wagstaff [**36**] was able to use them to get some leverage on a question of Huneke about Betti numbers of canonical modules, that is, Bass numbers of rings.

Avramov and Foxby [**6**] used a generalized form of Golod's construction in their study of G-dimensions of local ring homomorphisms $R \to S$. In particular, they establish a sweeping extension of the classical result characterizing the Gorenstein property of S in terms of the Gorensteinness of R and of the closed fibre of the homomorphism when S is flat over R. In this work, they posed a question that has dogged many of us in the area: if two local ring homomorphisms $R \to S \to T$ have finite G-dimension, must the same be true of the composite map $R \to T$? This question, with its relation to semidualizing modules (and more generally, semidualizing complexes; see the appendix for background information on these) is one of our main reasons for investigating these objects, in addition to the connection with Bass numbers.

In spite of the utility of the semidualizing modules, we know surprisingly little about them. For instance, we are only now in a position to answer the following aforementioned question of Vasconcelos.

Question 1.1 [**39**, p. 97]. If R is local and Cohen–Macaulay, must the set of isomorphism classes of semidualizing R-modules be finite?

The main result of this paper, stated next, provides a complete answer to Vasconcelos' question. Note that it does not assume that R is Cohen–Macaulay, and deals with the more general semidualizing complexes.

Theorem A. *If R is a local ring, then the set of shift-isomorphism classes of semidualizing R-complexes in the derived category $\mathcal{D}(R)$ is finite.*

Section 4 is devoted to the proof of this result (contained in 4.2) and some consequences, including a version of this result for semi-local rings (in Theorem 4.6).

Our proof is inspired by Christensen and Sather-Wagstaff's [**15**] treatment of the special case where R is Cohen–Macaulay and contains a field, so we summarize their proof here. First, they reduce to the case where R is complete with algebraically closed residue field F. They replace R with the finite dimensional F-algebra $R/(\mathbf{x})$, where $\mathbf{x}$ is a maximal R-sequence. The desired result then follows from a theorem of Happel [**28**]: in this context, there are only finitely many isomorphism classes of R-modules C of a given length r such that $\operatorname{Ext}^1_R(C,C) = 0$.

Happel's result is proved using geometric techniques. One parametrizes all R-modules of length r by an algebraic variety Mod^R_r that is acted on by the general linear group GL^F_r. The isomorphism class of a module M is the orbit $\operatorname{GL}^F_r \cdot M$, and the tangent space $\mathsf{T}^{\operatorname{GL}^F_r \cdot M}_M$ to the orbit $\operatorname{GL}^F_r \cdot M$ at M is naturally identified with a subspace of the tangent space $\mathsf{T}^{\operatorname{Mod}^R_r}_M$. A theorem of Voigt [**42**] (see also Gabriel [**24**] and Theorem 2.8 below) provides an isomorphism between $\operatorname{Ext}^1_R(M,M)$ and the quotient $\mathsf{T}^{\operatorname{Mod}^R_r}_M / \mathsf{T}^{\operatorname{GL}^F_r \cdot M}_M$. If $\operatorname{Ext}^1_R(M,M) = 0$, then the orbit $\operatorname{GL}^F_r \cdot M$ is open in Mod^R_r. As Mod^R_r is quasi-compact, it can only have finitely many open orbits, and Happel's result follows.

The idea behind our proof is the same as that of [**15**], with important differences. First, without the Cohen–Macaulay assumption, given a maximal R-regular sequence $\mathbf{x}$, the quotient $R/(\mathbf{x})$ is not artinian; and if R does not contain a field, then there is no reason to expect that the quotient will either. One cannot remedy this problem by replacing the regular sequence with a system of parameters $\mathbf{y}$ (or in fact any sequence generating an ideal of finite colength) because in general one loses too much homological information in the passage from R to $R/(\mathbf{y})$.

To deal with this, we employ a technique pioneered by Avramov: instead of replacing R with $R/(\mathbf{x})$, we use the Koszul complex K on a minimal generating sequence for the maximal ideal of R. More specifically, we replace R with a related finite dimensional *DG algebra* over an algebraically closed field F. (See the appendix for background information on DG algebras and DG modules.)

To prove versions of the results of Happel and Voigt, in Section 3 we develop certain geometric aspects of representation theory for DG algebras. In short, we parametrize all finite dimensional DG U-modules M with fixed underlying graded F-vector space W by an algebraic variety $\mathrm{Mod}^U(W)$. A product $\mathrm{GL}(W)_0$ of general linear groups acts on this variety so that the isomorphism class of M is precisely the orbit $\mathrm{GL}(W)_0 \cdot M$. Our version of Voigt's result for this context is the following; its proof is in 3.10.

THEOREM B. *Let U be a finite dimensional DG algebra over an algebraically closed field F, and let W be a finite dimensional graded F-vector space. Given an element $M \in \mathrm{Mod}^U(W)$, the quotient of tangent spaces $T_M^{\mathrm{Mod}^U(W)} / T_M^{\mathrm{GL}(W)_0 \cdot M}$ is isomorphic to the Yoneda Ext group $\mathrm{YExt}^1_U(M,M)$ from Definition A.7*[†].

As a consequence, if $\mathrm{YExt}^1_U(M,M) = 0$, then the orbit $\mathrm{GL}(W)_0 \cdot M$ is open in $\mathrm{Mod}^U(W)$, and our version of Happel's result follows; see Proposition 4.1.

Our first proof of Theorem B mimicked the proof of Voigt's original result. In contrast, the proof currently given obtains our result as a corollary of Theorem 2.22, which is itself a corollary of Voigt's original result and is the subject of Section 2.

2. *Graded version of Voigt's theorem*

In this section we derive a graded version of Voigt's theorem for use in our proof of Theorem B in the next section. We state Voigt's original result in Theorem 2.8 below, after developing the necessary notation. Our graded version is a corollary of this; see Theorem 2.22. The ideas here are from [**2**, **11**, **24**, **28**].

NOTATION 2.1. Let F be an algebraically closed field, and set $G_{\mathrm{m}} := F^{\times}$, the multiplicative group over F.

Let A be a finite dimensional associative graded F-algebra such that $A_i = 0$ for $|i| > q$. (Note that we do not assume multiplicative commutativity for A.) Let $\dim_F(A_i) = c_i$ for $i = -q, \dots, q$. Let $W := \bigoplus_{i=0}^s W_i$ be a finite dimensional graded F-vector space with $r_i := \dim_F(W_i)$ for $i = 0, \dots, s$. We use the convention $W_j = 0$ for all $j \notin \{0, \dots, s\}$.

Let $\mathrm{End}_F(W)$ denote the set of F-linear endomorphisms of W:

$$\mathrm{End}_F(W) = \prod_{i=0}^{s} \prod_{j=-s}^{s} \mathrm{Hom}_F(W_i, W_{i+j}). \tag{2.1.1}$$

For each $j \in \mathbb{Z}$, let $\mathrm{End}_F(W)_j$ denote the set of F-linear endomorphisms of W that are homogeneous of degree j:

$$\mathrm{End}_F(W)_j = \mathrm{Hom}_F(W,W)_j = \prod_{i=0}^{s} \mathrm{Hom}_F(W_i, W_{i+j}).$$

[†]See Definition A.9 for the different but equally important $\mathrm{Ext}^1_U(M,M)$.

Representations of A-module structures

A (left) A-module structure on W consists of a scalar multiplication

$$\mu \in \operatorname{Hom}_F(A \otimes_F W, W) \cong \prod_{i=0}^{q} \prod_{j=0}^{s} \prod_{k=-q-s}^{q+s} \operatorname{Hom}_F(A_i \otimes_F W_j, W_{i+j+k}) \tag{2.1.2}$$

that is associative and unital. A graded A-module structure on W consists of a scalar multiplication

$$\mu \in \operatorname{Hom}_F(A \otimes_F W, W)_0 \cong \prod_{i=0}^{q} \prod_{j=0}^{s} \operatorname{Hom}_F(A_i \otimes_F W_j, W_{i+j})$$

that is associative and unital.

NOTATION 2.2. Let $\operatorname{Mod}^A(W)$ denote the set of all maps μ making W into an A-module. Sometimes we identify a map $\mu \in \operatorname{Mod}^A(W)$ with the corresponding A-module $M = (W, \mu)$. Let $\operatorname{Mod}^A_{\mathrm{gr}}(W) \subseteq \operatorname{Mod}^A(W)$ denote the set of all maps μ making W into a graded A-module.

REMARK 2.3. The multiplicative group G_{m} acts on W; on homogeneous elements w, this action has the form $\alpha \cdot w = \alpha^{|w|} w$ where the right-hand expression uses the scalar multiplication of F on W. One defines an action of G_{m} on A using the same formula. The actions of G_{m} on W and A allow G_{m} to act on $\operatorname{Mod}^A(W)$ by the formula $\alpha\mu := \alpha \circ \mu \circ (\alpha^{-1} \otimes_F \alpha^{-1})$. Because F is infinite, one checks that μ respects the grading on W if and only if μ is fixed by this action, that is, $\operatorname{Mod}^A_{\mathrm{gr}}(W)$ is the set of fixed points for this action: $\operatorname{Mod}^A_{\mathrm{gr}}(W) = \operatorname{Mod}^A(W)^{G_{\mathrm{m}}}$.

We next describe geometric structures on the sets $\operatorname{Mod}^A(W)$ and $\operatorname{Mod}^A_{\mathrm{gr}}(W)$.

REMARK 2.4. The F-vector space $\operatorname{Hom}_F(A \otimes_F W, W)$ from (2.1.2) has dimension $b := (\sum_i c_i)(\sum_j r_j)^2$, so a map μ corresponds to an element of the affine space $\mathbb{A}^b_F$. The condition that μ be associative and unital is equivalent to the entries of the matrices representing μ satisfying certain fixed polynomials over F. Thus, the set $\operatorname{Mod}^A(W)$ is a Zariski-closed subset of $\mathbb{A}^b_F$.

In the decomposition (2.1.2), the maps $\mu \in \operatorname{Hom}_F(A \otimes_F W, W)$ that are homogeneous of degree 0 are exactly the ones where the matrices from $\operatorname{Hom}_F(A_i \otimes_F W_j, W_k)$ with $k \neq i + j$ are all 0. In other words, the space $\operatorname{Hom}_F(A \otimes_F W, W)_0$ is a linear subspace of $\operatorname{Hom}_F(A \otimes_F W, W)$ of dimension $b' := \sum_t \sum_i c_i r_{t-i} r_t$. We identify the affine space $\mathbb{A}^{b'}_F$ with the corresponding linear subvariety of $\mathbb{A}^b_F$; the defining conditions for $\operatorname{Mod}^A_{\mathrm{gr}}(W)$ and $\operatorname{Mod}^A(W)$, namely the associative and unital conditions, then say that $\operatorname{Mod}^A_{\mathrm{gr}}(W) = \mathbb{A}^{b'}_F \cap \operatorname{Mod}^A(W)$.

NOTATION 2.5. Let $\operatorname{GL}(W)$ denote the set of F-linear automorphisms of W, that is, the invertible elements of $\operatorname{End}_F(W)$. Let $\operatorname{GL}(W)_0$ denote the set of F-linear automorphisms of W that are homogeneous of degree 0, that is, the invertible elements of $\operatorname{End}_F(W)_0$; so $\operatorname{GL}(W)_0 = \operatorname{GL}(W) \cap \operatorname{End}_F(W)_0$.

REMARK 2.6. An element $\alpha \in \operatorname{GL}(W)$ is an element of $\operatorname{End}_F(W)$ with a multiplicative inverse. The vector space $\operatorname{End}_F(W)$ has dimension $\sigma := (\sum_i r_i)^2$, so the map α corresponds to an element of the affine space $\mathbb{A}^\sigma_F$. The invertibility of α is an open condition, given by the non-vanishing of the determinant. Thus, the group $\operatorname{GL}(W)$ is a Zariski-open subset of $\mathbb{A}^\sigma_F$; in particular, $\operatorname{GL}(W)$ is non-singular of dimension σ.

Note that the maps defining the operations in $\operatorname{GL}(W)$ are regular, that is, they are determined by rational functions on the entries of the matrix with non-vanishing denominators on $\mathrm{GL}(W)$. For multiplication, this is from the definition of matrix multiplication in terms of quadratic forms on the entries. For inversion, this follows from the formula $\alpha^{-1} = \det(\alpha)^{-1}\operatorname{adj}(\alpha)$. Thus, $\mathrm{GL}(W)$ is an algebraic group.

In the decomposition (2.1.1), the maps $\alpha \in \mathrm{End}_F(W)$ that are homogeneous of degree 0 are exactly the ones with a block-diagonal form, that is, where the matrices from $\mathrm{Hom}_F(W_i, W_j)$ with $j \neq i$ are all 0. In other words, the space $\mathrm{End}_F(W)_0$ is a linear subspace of $\mathrm{End}_F(W)$ of dimension $e := \sum_i r_i^2$. We identify the affine space $\mathbb{A}_F^e$ with the corresponding linear subvariety of $\mathbb{A}_F^\sigma$. The defining conditions for $\mathrm{GL}(W)_0$ and $\mathrm{GL}(W)$, namely the invertibility conditions, then say that $\mathrm{GL}(W)_0 = \mathbb{A}_F^e \cap \mathrm{GL}(W)$. In particular, $\mathrm{GL}(W)_0$ is non-singular of dimension e. Moreover, as with $\mathrm{GL}(W)$, the subset $\mathrm{GL}(W)_0 \subseteq \mathrm{GL}(W)$ is an algebraic subgroup. (Alternately, one sees part of this by viewing $\mathrm{GL}(W)_0$ as the product $\mathrm{GL}(W_0) \times \cdots \times \mathrm{GL}(W_s)$, since each $\mathrm{GL}(W_i)$ is a non-singular algebraic group.)

Note that the action of G_m on W from Remark 2.3 allows us to identify G_m naturally with a subgroup of $\mathrm{GL}(W)_0$, hence of $\mathrm{GL}(W)$. Under this identification, because F is infinite, one checks that $\mathrm{GL}(W)_0$ is the centralizer of G_m in $\mathrm{GL}(W)$.

General linear group action on $\mathrm{Mod}^A(W)$

Next, we describe an action of the group $\mathrm{GL}(W)$ from Notation 2.5 on $\mathrm{Mod}^A(W)$ by conjugation.

NOTATION 2.7. Let $\alpha \in \mathrm{GL}(W)$. For every $\mu \in \mathrm{Mod}^A(W)$, we define

$$\alpha \cdot \mu := \alpha \circ \mu \circ (A \otimes_F \alpha^{-1}). \tag{2.7.1}$$

Let $\alpha \in \mathrm{GL}(W)$ and $\mu \in \mathrm{Mod}^A(W)$. It is straightforward to show that (2.7.1) describes an A-module structure for W, so $\alpha \cdot \mu \in \mathrm{Mod}^A(W)$. From the definition of $\alpha \cdot \mu$, it follows readily that this describes a $\mathrm{GL}(W)$-action on $\mathrm{Mod}^A(W)$. In addition, the maps defining this action are regular, essentially by the second paragraph of Remark 2.6. In the language of [**11**], this says that we have an algebraic action of $\mathrm{GL}(W)$ on $\mathrm{Mod}^A(W)$. Furthermore, this restricts to an algebraic action of $\mathrm{GL}(W)_0$ on $\mathrm{Mod}^A_{\mathrm{gr}}(W)$: if $\alpha \in \mathrm{GL}(W)_0 \subseteq \mathrm{GL}(W)$ and $\mu \in \mathrm{Mod}^A_{\mathrm{gr}}(W) \subseteq \mathrm{Mod}^A(W)$, then $\alpha \cdot \mu \in \mathrm{Mod}^A_{\mathrm{gr}}(W)$.

We now have the notation needed to state Voigt's original theorem. Note that we will apply it to the algebra A and the vector space W with the gradings forgotten.

THEOREM 2.8 (Voigt [**42**]). *Let B be a finite dimensional associative algebra over an algebraically closed field F, and let X be a finite dimensional F-vector space. Given an element $N \in \mathrm{Mod}^B(X)$, there is a natural identification of the tangent space $T_N^{\mathrm{GL}(X)\cdot N}$ as a subspace of $T_N^{\mathrm{Mod}^B(X)}$, and under this identification the quotient of tangent spaces $T_N^{\mathrm{Mod}^B(X)} / T_N^{\mathrm{GL}(X)\cdot N}$ is isomorphic to the Ext group $\mathrm{Ext}^1_B(N,N)$.*[†]

Next, we describe the orbits of the action of $\mathrm{GL}(W)$ on $\mathrm{Mod}^A(W)$. Let $\alpha \in \mathrm{GL}(W)$ and $\mu \in \mathrm{Mod}^A(W)$, and set $\widetilde{\mu} := \alpha \cdot \mu$. It is straightforward to show that the map α gives an A-module isomorphism $\alpha\colon (W,\mu) \xrightarrow{\cong} (W,\widetilde{\mu})$. Furthermore, if $\alpha \in \mathrm{GL}(W)_0$ and $\mu \in \mathrm{Mod}^A_{\mathrm{gr}}(W)$, then the isomorphism $\alpha\colon (W,\mu) \xrightarrow{\cong} (W,\widetilde{\mu})$ is homogeneous.

Conversely, given another element $\mu' \in \mathrm{Mod}^A(W)$, if there is an A-module isomorphism $\beta\colon (W,\mu) \xrightarrow{\cong} (W,\mu')$, then $\beta \in \mathrm{GL}(W)$ and $\mu' = \beta \cdot \mu$. Furthermore, if $\mu, \mu' \in \mathrm{Mod}^A_{\mathrm{gr}}(W)$ and

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[†]In contrast with the DG-setting of Theorem B, there is no need for the notation YExt in the non-DG-setting.

if β is a homogeneous isomorphism, then $\beta \in \mathrm{GL}(W)_0$ and $\mu' = \beta \cdot \mu$. Thus, we have the following.

PROPOSITION 2.9. *The orbits in* $\mathrm{Mod}^A(W)$ *under the action of* $\mathrm{GL}(W)$ *are in bijection with the isomorphism classes of* A*-module structures on* W*. The orbits in* $\mathrm{Mod}^A_{\mathrm{gr}}(W)$ *under the action of* $\mathrm{GL}(W)_0$ *are in bijection with the isomorphism classes of graded* A*-module structures on* W*.*

Tangent spaces

NOTATION 2.10. The Zariski tangent space of $\mathrm{Mod}^A(W)$ at an element $M = \mu \in \mathrm{Mod}^A(W)$ is denoted $\mathsf{T}_M^{\mathrm{Mod}^A(W)}$. Similarly, for any $\alpha \in \mathrm{GL}(W)$, we have tangent spaces $\mathsf{T}_\alpha^{\mathrm{GL}(W)}$ and $\mathsf{T}_{\alpha\cdot M}^{\mathrm{GL}(W)\cdot M}$, and similarly for the homogeneous cases.

Set $F[\epsilon] := F\epsilon \oplus F$, where $\epsilon^2 = 0$ and $|\epsilon| = 0$. We set

$$A[\epsilon] := F[\epsilon] \otimes_F A \cong A\epsilon \oplus A \cong A \oplus A$$

$$W[\epsilon] := F[\epsilon] \otimes_F W \cong W\epsilon \oplus W \cong W \oplus W.$$

We write elements of $A[\epsilon]$ as $u\epsilon + v = (\epsilon \otimes u) + (1 \otimes v)$, and similarly for $W[\epsilon]$.

REMARK 2.11. The vector space $A[\epsilon]$ is a graded $F[\epsilon]$-algebra using the multiplication $(\zeta \otimes u)(\zeta' \otimes u') := (\zeta\zeta') \otimes (uu')$. In other words, since $\epsilon^2 = 0$, this works out to be

$$(u\epsilon + v)(u'\epsilon + v') = (uv' + vu')\epsilon + vv'.$$

In particular, $\epsilon u = u\epsilon$ for all $u \in A$. Similarly, $W[\epsilon]$ is a graded $F[\epsilon]$-module.

REMARK 2.12. As above, an $A[\epsilon]$-module structure on $W[\epsilon]$ consists of a scalar multiplication from $\mathrm{Hom}_{F[\epsilon]}(A[\epsilon] \otimes_{F[\epsilon]} W[\epsilon], W[\epsilon])$, and a graded module structure comes from $\mathrm{Hom}_{F[\epsilon]}(A[\epsilon] \otimes_{F[\epsilon]} W[\epsilon], W[\epsilon])_0$.

For example, given a (graded) A-module structure $M = \mu$ on W, we have a trivial (graded) $A[\epsilon]$-module structure $M[\epsilon] = F[\epsilon] \otimes_F M$ on $W[\epsilon]$. In terms of the tensor descriptions of $A[\epsilon]$ and $M[\epsilon]$, this is given as $(\xi \otimes a)(\zeta \otimes m) = (\xi\zeta) \otimes (am)$. Writing elements of $A[\epsilon]$ as $u\epsilon + v$ and similarly for $M[\epsilon]$, this reads $(u\epsilon + v)(m\epsilon + n) = (un + vm)\epsilon + vn$.

NOTATION 2.13. Let $\mathrm{Mod}^{A[\epsilon]}(W[\epsilon])$ denote the set of all maps μ making $W[\epsilon]$ into an $A[\epsilon]$-module, and let $\mathrm{Mod}^{A[\epsilon]}_{\mathrm{gr}}(W[\epsilon]) \subseteq \mathrm{Mod}^{A[\epsilon]}(W[\epsilon])$ denote the subset of all maps μ making $W[\epsilon]$ into a graded $A[\epsilon]$-module.

REMARK 2.14. The natural ring epimorphism $F[\epsilon] \to F$ induces well-defined maps $\mathrm{Mod}^{A[\epsilon]}(W[\epsilon]) \xrightarrow{\kappa} \mathrm{Mod}^A(W)$ and $\mathrm{Mod}^{A[\epsilon]}_{\mathrm{gr}}(W[\epsilon]) \xrightarrow{\kappa'} \mathrm{Mod}^A_{\mathrm{gr}}(W)$ obtained by reducing modulo ϵ. The point is that any unital, associative (graded) scalar multiplication of $A[\epsilon]$ on $W[\epsilon]$ reduces modulo ϵ to a unital, associative (graded) scalar multiplication of A on W.

The tangent space $\mathsf{T}_M^{\mathrm{Mod}^A(W)}$ of $\mathrm{Mod}^A(W)$ at an element $M \in \mathrm{Mod}^A(W)$ is the fibre of κ over M

$$\mathsf{T}_M^{\mathrm{Mod}^A(W)} = \{N \in \mathrm{Mod}^{A[\epsilon]}(W[\epsilon]) \mid \kappa(N) = M\}.$$

See, for example, [17, VI.1.3]. In other words, the tangent space $\mathsf{T}_M^{\mathrm{Mod}^A(W)}$ is the set of elements of $\mathrm{Mod}^{A[\epsilon]}(W[\epsilon])$ that reduce to M modulo ϵ. A similar conclusion holds for $\mathsf{T}_M^{\mathrm{Mod}^A_{\mathrm{gr}}(W)}$.

Lemma 2.15. *Let $M \in \mathrm{Mod}^A_{gr}(W)$. Then $T_M^{\mathrm{Mod}^A_{gr}(W)}$ is naturally a subspace of $T_M^{\mathrm{Mod}^A(W)}$. Moreover, G_m acts on $T_M^{\mathrm{Mod}^A(W)}$ by linear maps (that is, by automorphisms) so that $T_M^{\mathrm{Mod}^A_{gr}(W)} = (T_M^{\mathrm{Mod}^A(W)})^{G_m}$.*

Proof. Remark 2.3 shows how to define an action of G_{m} on $\mathrm{Mod}^{A[\epsilon]}(W[\epsilon])$. It is straightforward to show that the subspace $\mathsf{T}_M^{\mathrm{Mod}^A(W)} \subseteq \mathrm{Mod}^{A[\epsilon]}(W[\epsilon])$ is invariant under this action. One checks readily that G_{m} acts on $\mathsf{T}_M^{\mathrm{Mod}^A(W)}$ by linear maps.

The invariance of $\mathsf{T}_M^{\mathrm{Mod}^A(W)}$ yields the last equality in the next display

$$\begin{aligned} \mathsf{T}_M^{\mathrm{Mod}^A_{\mathrm{gr}}(W)} &= \{N \in \mathrm{Mod}^{A[\epsilon]}_{\mathrm{gr}}(W[\epsilon]) \mid \kappa'(N) = M\} \\ &= \mathsf{T}_M^{\mathrm{Mod}^A(W)} \cap \mathrm{Mod}^{A[\epsilon]}_{\mathrm{gr}}(W[\epsilon]) \\ &= \mathsf{T}_M^{\mathrm{Mod}^A(W)} \cap (\mathrm{Mod}^{A[\epsilon]}(W[\epsilon]))^{G_{\mathrm{m}}} \\ &= \left(\mathsf{T}_M^{\mathrm{Mod}^A(W)}\right)^{G_{\mathrm{m}}}. \end{aligned}$$

The third equality here is from Remark 2.3, applied to the algebra $A[\epsilon]$. The other equalities are from Remark 2.14. □

Notation 2.16. As above, let $\mathrm{GL}(W[\epsilon])$ denote the set of $F[\epsilon]$-linear automorphisms of $W[\epsilon]$, that is, the invertible elements of $\mathrm{End}_{F[\epsilon]}(W[\epsilon])$, and let $\mathrm{GL}(W[\epsilon])_0$ denote the set of $F[\epsilon]$-linear automorphisms of $W[\epsilon]$ that are homogeneous of degree 0, that is, the invertible elements of $\mathrm{End}_{F[\epsilon]}(W[\epsilon])_0$.

For each $\alpha \in \mathrm{GL}(W[\epsilon])$ and $\mu \in \mathrm{Mod}^{A[\epsilon]}(W[\epsilon])$, we define

$$\alpha \cdot \mu := \alpha \circ \mu \circ (A[\epsilon] \otimes_{F[\epsilon]} \alpha^{-1}).$$

It follows readily that this describes a $\mathrm{GL}(W[\epsilon])$-action on $\mathrm{Mod}^{A[\epsilon]}(W[\epsilon])$. The same formula describes a $\mathrm{GL}(W[\epsilon])_0$-action on $\mathrm{Mod}^{A[\epsilon]}_{\mathrm{gr}}(W[\epsilon])$.

The next result is proved like Lemma 2.15. The subsequent result, however, requires slightly more effort.

Lemma 2.17. *The tangent space $T_{\mathrm{id}_W}^{\mathrm{GL}(W)_0}$ is naturally a subspace of $T_{\mathrm{id}_W}^{\mathrm{GL}(W)}$. Moreover, G_m acts on $T_{\mathrm{id}_W}^{\mathrm{GL}(W)}$ by linear maps so that $T_{\mathrm{id}_W}^{\mathrm{GL}(W)_0} = (T_{\mathrm{id}_W}^{\mathrm{GL}(W)})^{G_m}$.*

Lemma 2.18. *Let $M \in \mathrm{Mod}^A_{gr}(W)$. Then the tangent space $T_M^{\mathrm{GL}(W)_0 \cdot M}$ is naturally a subspace of $T_M^{\mathrm{GL}(W)\cdot M}$. Moreover, G_m acts on $T_M^{\mathrm{GL}(W)\cdot M}$ by linear maps so that $T_M^{\mathrm{GL}(W)_0\cdot M} = (T_M^{\mathrm{GL}(W)\cdot M})^{G_m}$.*

Proof. Consider the map $\mathrm{GL}(W) \xrightarrow{\cdot M} \mathrm{Mod}^A(W)$ coming from the action of $\mathrm{GL}(W)$ on $\mathrm{Mod}^A(W)$, and the induced linear transformation $\mathsf{T}_{\mathrm{id}_W}^{\mathrm{GL}(W)} \xrightarrow{D_M} \mathsf{T}_M^{\mathrm{Mod}^A(W)}$ of tangent spaces. We describe the map D_M, using the following diagram as a guide.

$$\begin{array}{ccc} \mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W} & \xrightarrow{D_M} & \mathsf{T}^{\mathrm{Mod}^A(W)}_M \\ \big\downarrow \subseteq & & \big\downarrow \subseteq \\ \mathrm{GL}(W[\epsilon]) & \xrightarrow{\cdot M[\epsilon]} & \mathrm{Mod}^{A[\epsilon]}(W[\epsilon]) \end{array}$$

Identify $\mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W}$, as above, with the set of elements of $\mathrm{GL}(W[\epsilon])$ that reduce to id_W modulo ϵ. The action of $\mathrm{GL}(W[\epsilon])$ on $\mathrm{Mod}^{A[\epsilon]}(W[\epsilon])$ yields a map

$$\mathrm{GL}(W[\epsilon]) \xrightarrow{\cdot M[\epsilon]} \mathrm{Mod}^{A[\epsilon]}(W[\epsilon]).$$

The restriction of this map to $\mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W}$ lands in $\mathsf{T}^{\mathrm{Mod}^A(W)}_M$, and the restricted map $\mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W} \to \mathsf{T}^{\mathrm{Mod}^A(W)}_M$ is exactly D_M. In other words, for each $\alpha \in \mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W}$, we have $D_M(\alpha) = \alpha \cdot M[\epsilon] \in \mathsf{T}^{\mathrm{Mod}^A(W)}_M$.

Recall that [11, Theorem 2.7(ii)] shows that $\mathrm{GL}_F(W) \cdot M$ is a locally closed subset of $\mathrm{Mod}^A(W)$. Hence, the tangent space $\mathsf{T}^{\mathrm{GL}(W)\cdot M}_M$ is naturally identified with a subspace of $\mathsf{T}^{\mathrm{Mod}^A(W)}_M$. Part of the proof of Voigt's Theorem 2.8 states that, under this identification, we have

$$D_M(\mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W}) = \mathsf{T}^{\mathrm{GL}(W)\cdot M}_M. \tag{2.18.1}$$

From the above description of the map D_M, it is straightforward to show that D_M is G_{m}-equivariant. It follows that the subspace $\mathsf{T}^{\mathrm{GL}(W)\cdot M}_M$ is invariant under the action of G_{m} on $\mathsf{T}^{\mathrm{Mod}^A(W)}_M$. Since Lemma 2.17 implies that G_{m} acts on $\mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W}$ by linear maps, we also conclude that G_{m} acts on the invariant subspace $\mathsf{T}^{\mathrm{Mod}^A(W)}_M$ by linear maps.

Next, consider the map $\mathrm{GL}(W)_0 \xrightarrow{\cdot M} \mathrm{Mod}^A_{\mathrm{gr}}(W)$ and the induced linear transformation $\mathsf{T}^{\mathrm{GL}(W)_0}_{\mathrm{id}_W} \xrightarrow{D_M} \mathsf{T}^{\mathrm{Mod}^A_{\mathrm{gr}}(W)}_M$. The above argument translates directly to show that $\mathsf{T}^{\mathrm{GL}(W)_0\cdot M}_M$ is naturally identified with a subspace of $\mathsf{T}^{\mathrm{Mod}^A_{\mathrm{gr}}(W)}_M$, and under this identification we have

$$D_M(\mathsf{T}^{\mathrm{GL}(W)_0}_{\mathrm{id}_W}) = \mathsf{T}^{\mathrm{GL}(W)_0\cdot M}_M. \tag{2.18.2}$$

Recall that the group G_{m} is linearly reductive. It follows that the functor $(-)^{G_{\mathrm{m}}}$ is exact on the category with objects equal to the finite dimensional F-vector spaces with G_{m}-actions by linear maps and with morphisms equal to the equivariant linear maps. Lemma 2.17 says that G_{m} acts on $\mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W}$ by linear maps, so the exactness of $(-)^{G_{\mathrm{m}}}$ yields the third equality in the following sequence.

$$\begin{aligned} \mathsf{T}^{\mathrm{GL}(W)_0\cdot M}_M &= D_M\left(\mathsf{T}^{\mathrm{GL}(W)_0}_{\mathrm{id}_W}\right) \\ &= D_M\left(\left(\mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W}\right)^{G_{\mathrm{m}}}\right) \\ &= \left(D_M\left(\mathsf{T}^{\mathrm{GL}(W)}_{\mathrm{id}_W}\right)\right)^{G_{\mathrm{m}}} \\ &= \left(\mathsf{T}^{\mathrm{GL}(W)\cdot M}_M\right)^{G_{\mathrm{m}}} \end{aligned}$$

The first equality here is (2.18.2), the second equality is from Lemma 2.17, and the fourth equality is (2.18.1). □

Extensions and our graded version of Voigt's Theorem

REMARK 2.19. Let $M = \mu \in \mathrm{Mod}^A_{\mathrm{gr}}(W)$. The elements of $\mathrm{Ext}^1_A(M, M)$ are equivalence classes of short exact sequences

$$0 \to M \to X \to M \to 0.$$

Since the underlying vector space for M is the graded vector space W, such sequences are split over F, and they can be expressed in the form

$$\xi = 0 \to (W, \mu) \xrightarrow{i} (W[\epsilon], \nu) \xrightarrow{p} (W, \mu) \to 0 \tag{2.19.1}$$

where $i(w) = w\epsilon$ and $p(w\epsilon + x) = x$.

Within this form, the F-vector space structure on $\mathrm{Ext}^1_A(M, M)$ takes the following form. For an element $\lambda \in F$, we have

$$\lambda\xi = 0 \to (W, \mu) \xrightarrow{i} (W[\epsilon], \nu^\lambda) \xrightarrow{p} (W, \mu) \to 0$$

where $\nu^\lambda(a \otimes (w\epsilon + x)) := \nu(a \otimes (w\epsilon + \lambda x)) - \mu(a \otimes (\lambda x)) + \mu(a \otimes x)$. For a second extension

$$\zeta = 0 \to (W, \mu) \xrightarrow{i} (W[\epsilon], \tau) \xrightarrow{p} (W, \mu) \to 0$$

the Baer sum in $\mathrm{YExt}^1_A(M, M)$ gives

$$\xi + \zeta = 0 \to (W, \mu) \xrightarrow{i} (W[\epsilon], \nu \oplus \tau) \xrightarrow{p} (W, \mu) \to 0$$

where $(\nu \oplus \tau)(a \otimes (w\epsilon + x)) := \nu(a \otimes (w\epsilon + x)) + \tau(a \otimes x) - \mu(a \otimes x)$.

Since M is graded, the multiplicative group G_{m} acts on $\mathrm{Ext}^1_A(M, M)$ as

$$\alpha\xi = 0 \to (W, \mu) \xrightarrow{i} (W[\epsilon], \alpha\nu) \xrightarrow{p} (W, \mu) \to 0$$

where the notation $\alpha\nu$ is from Remark 2.3. One checks readily that this describes a group action and furthermore that G_{m} acts on $\mathrm{Ext}^1_A(M, M)$ by linear maps.

Moreover, the map $\eta\colon \mathsf{T}_M^{\mathrm{Mod}^A(W)} \to \mathrm{Ext}^1_A(M, M)$ from Voigt's Theorem 2.8 sends an element $\nu \in \mathsf{T}_M^{\mathrm{Mod}^A(W)}$ to the extension (2.19.1). Thus, it follows from the definitions that the map η is G_{m}-equivariant.

DEFINITION 2.20. Let $M = \mu \in \mathrm{Mod}^A_{\mathrm{gr}}(W)$. An extension (2.19.1) of M by M is *graded* when the modules in it are graded as A-modules and the maps are homogeneous. Let $\text{gr-}\mathrm{Ext}^1_A(M, M)$ denote the set of equivalence classes (under the usual equivalence relation) of graded extensions of M by M.

LEMMA 2.21. *Let $M = \mu \in \mathrm{Mod}^A_{gr}(W)$. Then $\text{gr-}\mathrm{Ext}^1_A(M, M)$ is naturally a subspace of $\mathrm{Ext}^1_A(M, M)$ so that $\text{gr-}\mathrm{Ext}^1_A(M, M) = \mathrm{Ext}^1_A(M, M)^{G_m}$.*

Proof. In an extension (2.19.1), the module $M = (W, \mu)$ is graded by assumption, as are the maps i and p. Thus, such an extension is graded if and only if the module $(W[\epsilon], \nu)$ is graded. That is, $\xi \in \text{gr-}\mathrm{Ext}^1_A(M, M)$ if and only if ν is graded; by Remark 2.3, ν is graded if and only if it is fixed by the action of G_{m}, and this is so if and only if ξ is fixed by the action of G_{m}. In summary, this shows that $\text{gr-}\mathrm{Ext}^1_A(M, M) = \mathrm{Ext}^1_A(M, M)^{G_{\mathrm{m}}}$. □

We are finally in a position to prove our graded version of Voigt's Theorem.

THEOREM 2.22. *Let A be a finite dimensional graded associative algebra over an algebraically closed field F, and let W be a finite dimensional graded F-vector space. Given an element $M \in \mathrm{Mod}^A_{gr}(W)$, there is a natural identification of the tangent space $T_M^{\mathrm{GL}(W)_0 \cdot M}$*

as a subspace of $T_M^{\mathrm{Mod}^A_{gr}(W)}$*, and under this identification the quotient of tangent spaces* $T_M^{\mathrm{Mod}^A_{gr}(W)} \,/\, T_M^{\mathrm{GL}(W)_0\cdot M}$ *is isomorphic to* $\text{gr-Ext}^1_A(M,M)$.

Proof. Voigt's Theorem 2.8 provides a short exact sequence

$$0 \to \mathsf{T}_M^{\mathrm{GL}(W)\cdot M} \xrightarrow{\subseteq} \mathsf{T}_M^{\mathrm{Mod}^A(W)} \xrightarrow{\eta} \mathrm{Ext}^1_A(M,M) \to 0.$$

The multiplicative group G_{m} acts on these spaces by linear maps, and η is G_{m}-equivariant; as we have noted above, the fact that G_{m} is linearly reductive implies that the induced sequence

$$0 \to \left(\mathsf{T}_M^{\mathrm{GL}(W)\cdot M}\right)^{G_{\mathrm{m}}} \xrightarrow{\subseteq} \left(\mathsf{T}_M^{\mathrm{Mod}^A(W)}\right)^{G_{\mathrm{m}}} \xrightarrow{\eta^{G_{\mathrm{m}}}} \mathrm{Ext}^1_A(M,M)^{G_{\mathrm{m}}} \to 0$$

is exact; see Remark 2.19 and Lemmas 2.15, 2.18, and 2.21. These results also show that this sequence is of the form

$$0 \to \mathsf{T}_M^{\mathrm{GL}(W)_0\cdot M} \xrightarrow{\subseteq} \mathsf{T}_M^{\mathrm{Mod}^A_{\mathrm{gr}}(W)} \to \text{gr-Ext}^1_A(M,M) \to 0$$

hence the desired conclusions. □

3. *DG version of Voigt's theorem*

In this section, we prove Theorem B from the introduction; see 3.10. The ideas here originate in [**2**, **11**, **24**, **28**]. See appendix for background information on DG modules.

NOTATION 3.1. Let F be an algebraically closed field, and U a finite dimensional DG F-algebra such that $U_i = 0$ for $i > q$ and for $i < 0$. Let $\dim_F(U_i) = n_i$ for $i = 0,\dots,q$. Let $W := \bigoplus_{i=0}^{s} W_i$ be as in Notation 2.1.

Representations of DG-module structures

A DG U-module structure on W consists of two pieces of data: a differential

$$\partial \in \mathrm{End}_F(W)_{-1} = \mathrm{Hom}_F(W,W)_{-1} = \prod_{i=0}^{s} \mathrm{Hom}_F(W_i, W_{i-1})$$

and a scalar multiplication

$$\mu \in \mathrm{Hom}_F(U \otimes_F W, W)_0 \cong \prod_{i=0}^{q}\prod_{j=0}^{s} \mathrm{Hom}_F(U_i \otimes_F W_j, W_{i+j}).$$

NOTATION 3.2. Let $\mathrm{Mod}^U(W)$ denote the set of all ordered pairs (∂,μ) making W into a DG U-module. Sometimes we identify an ordered pair $(\partial,\mu) \in \mathrm{Mod}^U(W)$ with the corresponding DG U-module $M = (W,\partial,\mu)$.

We next describe a geometric structure on the set $\mathrm{Mod}^U(W)$.

REMARK 3.3. As in Section 2, a differential ∂ on W corresponds to an element of the affine space $\mathbb{A}^d_F$ where $d := \sum_i r_i r_{i-1}$. The vanishing condition $\partial\partial = 0$ is equivalent to the entries of the matrices representing ∂ satisfying certain fixed homogeneous quadratic polynomial equations over F. Hence, the set of all differentials on W is a Zariski-closed subset of $\mathbb{A}^d_F$. Also, a map μ corresponds to an element of the affine space $\mathbb{A}^{d'}_F$ where $d' := \sum_c \sum_i n_i r_{c-i} r_c$. The condition that μ be an associative, unital, cycle satisfying the Leibniz Rule is equivalent to the entries of the matrices representing ∂ and μ satisfying certain fixed polynomials over F. Thus, the set $\operatorname{Mod}^U(W)$ is a Zariski-closed subset of $\mathbb{A}_F^d \times \mathbb{A}_F^{d'} \cong \mathbb{A}_F^{d+d'}$.

General linear group action on $\operatorname{Mod}^U(W)$

Next, we describe an action of the group $\operatorname{GL}(W)_0$ from Notation 2.5 on $\operatorname{Mod}^U(W)$ by conjugation.

NOTATION 3.4. Let $\alpha \in \operatorname{GL}(W)_0$. For every $(\partial, \mu) \in \operatorname{Mod}^U(W)$, we define

$$\alpha \cdot (\partial, \mu) := (\alpha \circ \partial \circ \alpha^{-1}, \alpha \circ \mu \circ (U \otimes_F \alpha^{-1})). \tag{3.4.1}$$

Let $\alpha \in \operatorname{GL}(W)_0$ and $M = (\partial, \mu) \in \operatorname{Mod}^U(W)$. As in Section 2, this describes an algebraic action of $\operatorname{GL}(W)_0$ on $\operatorname{Mod}^U(W)$. Thus, most of the following result is from [**11**, Proposition 2.7(ii)].

LEMMA 3.5. *Let $M \in \operatorname{Mod}^U(W)$ be given.*

(a) *The stabilizer* (*that is, the isotropy group*)

$$\operatorname{Stab}(M) = \{\alpha \in \operatorname{GL}(W)_0 \mid \alpha \cdot M = M\}$$

is closed in $\operatorname{GL}(W)_0$ *and non-singular.*

(b) *The orbit*

$$\operatorname{GL}(W)_0 \cdot M = \{\alpha \cdot M \mid \alpha \in \operatorname{GL}(W)_0\}$$

is a locally closed, non-singular subvariety† *of* $\operatorname{Mod}^U(W)$. *All connected components of* $\operatorname{GL}(W)_0 \cdot M$ *have dimension* $\dim(\operatorname{GL}(W)_0) - \dim(\operatorname{Stab}(M))$.

Proof. By [**11**, Proposition 2.7], we need only to show that $\operatorname{Stab}(M)$ is non-singular. Since F is algebraically closed, it suffices to show that $\operatorname{Stab}(M)$ is regular. As $\operatorname{GL}(W)_0$ is regular, to show that $\operatorname{Stab}(M) \subseteq \operatorname{GL}(W)_0$ is regular it is enough to show that $\operatorname{Stab}(M)$ is defined by linear equations. To find these linear equations, note that the stabilizer condition $\alpha \cdot M = M$ is equivalent to the conditions $\partial = \alpha \circ \partial \circ \alpha^{-1}$ and $\mu = \alpha \circ \mu \circ (U \otimes_F \alpha^{-1})$, that is, $\partial \circ \alpha = \alpha \circ \partial$ and $\mu \circ (U \otimes_F \alpha) = \alpha \circ \mu$; since the matrices defining ∂ and μ are fixed, these equations are described by a system of linear equations in the variables describing α. Thus, $\operatorname{Stab}(M)$ is non-singular. □

The next result is verified like Proposition 2.9.

PROPOSITION 3.6. *The orbits in* $\operatorname{Mod}^U(W)$ *under the action of* $\operatorname{GL}(W)_0$ *are in bijection with the isomorphism classes of DG* U*-module structures on* W.

Translating from the DG setting to the graded setting

The following notation remains in effect for the rest of section.

NOTATION 3.7. Let δ be an exterior variable of degree -1, so $\delta^2 = 0$, and set $V = U^\natural[\delta]$ subject to the relations $\delta u = \partial^U(u) + (-1)^{|u|} u\delta$ for all $u \in U$. By definition, this yields formulas like the following:

$$u(u'\delta) = uu'\delta$$

†In [**11**], varieties are not necessarily irreducible.

$$(u\delta)u' = u\partial^U(u') + (-1)^{|u'|}uu'\delta$$
$$(u\delta)(u'\delta) = u\partial^U(u')\delta + (-1)^{|u'|}uu'\delta^2 = u\partial^U(u')\delta.$$

Next, we summarize some important connections between U and V. The verifications are straightforward, so we omit them.

FACT 3.8. (a) V is an associative algebra concentrated in degrees $-1, 0, \dots, q$. Note that the relation $\delta u = \partial^U(u) + (-1)^{|u|}u\delta$ shows that V is not in general graded commutative.

(b) Given a graded F-vector space X, the graded left V-modules with underlying F-vector space X are in bijection with the DG U-modules with underlying F-vector space X. Specifically, given a DG U-module M, one defines a graded V-module structure on it via the formula $\delta m = \partial^M(m)$. Conversely, given a graded V-module N, one defines a DG U-module structure on it via the same formula: $\partial^N(n) = \delta n$.

NOTATION 3.9. Given a DG U-module M, we write ${}_VM$ for the corresponding graded V-module. We specify the DG U-module structure on M by writing ${}_UM$. Similarly, given a graded V-module N, we write ${}_UN$ for the corresponding DG U-module, and we specify the graded V-module structure on N by writing ${}_VN$.

Proof of Theorem B 3.10. For each $i \in \mathbb{Z}$, we have a natural monomorphism

$$\text{gr-Hom}_V({}_VM, {}_VM)_i \hookrightarrow \text{Hom}_U({}_UM, {}_UM)_i$$

the image of which is exactly the set of degree-i cycles in $\text{Hom}_U({}_UM, {}_UM)$. The point is that a graded map $f\colon M \to M$ that is V-linear is also U-linear and, moreover, it is V-linear if and only if it is U-linear and it is a cycle in the Hom-complex over U. In particular, the case $i = 0$ implies that the graded V-module homomorphisms ${}_VM \to {}_VM$ are in bijection with the DG U-module morphisms ${}_UM \to {}_UM$. From this we obtain an abelian group isomorphism

$$\text{YExt}^1_U({}_UM, {}_UN) \cong \text{gr-Ext}^1_V({}_VM, {}_VN). \tag{3.10.1}$$

Indeed, given an exact sequence

$$0 \to N \to X \to M \to 0 \tag{3.10.2}$$

of graded F-vector spaces, we have the following:

(1) the sequence (3.10.2) consists of DG U-module morphisms if and only if it consists of V-linear maps;
(2) an F-linear isomorphism between the extension (3.10.2) and the trivial extension $0 \to N \to M \oplus N \to M \to 0$ will consist of DG U-module morphisms if and only if it consists of V-linear maps, that is, the extension (3.10.2) will be trivial in $\text{YExt}^1_U({}_UM, {}_UN)$ if and only if it is trivial in $\text{gr-Ext}^1_V({}_VM, {}_VN)$; and
(3) the Baer sum in the DG setting over U corresponds directly to the Baer sum in the graded setting over V.

The correspondence in Fact 3.8(b) between DG U-modules and graded V-modules induces an isomorphism of varieties $\text{Mod}^U(W) \xrightarrow{\cong} \text{Mod}^V_{\text{gr}}(W)$. It is straightforward to show that this is equivariant with respect to the actions of $\text{GL}(W)_0$. In particular, this induces an isomorphism of orbits $\text{GL}(W)_0 \cdot {}_UM \xrightarrow{\cong} \text{GL}(W)_0 \cdot {}_VM$. The first of these isomorphisms induces an isomorphism of tangent spaces $\mathsf{T}^{\text{Mod}^U(W)}_{{}_UM} \xrightarrow{\cong} \mathsf{T}^{\text{Mod}^V_{\text{gr}}(W)}_{{}_VM}$ which maps the subspace $\mathsf{T}^{\text{GL}(W)_0\cdot {}_UM}_{{}_UM} \subseteq \mathsf{T}^{\text{Mod}^U(W)}_{{}_UM}$ isomorphically onto the subspace $\mathsf{T}^{\text{GL}(W)_0\cdot {}_VM}_{{}_VM} \subseteq \mathsf{T}^{\text{Mod}^V_{\text{gr}}(W)}_{{}_VM}$. Thus, we have the first isomorphism in the next sequence

$$\begin{aligned}\mathsf{T}^{\mathrm{Mod}^U(W)}_{{}_UM} / \mathsf{T}^{\mathrm{GL}(W)_0\cdot {}_UM}_{{}_UM} &\cong \mathsf{T}^{\mathrm{Mod}^V_{\mathrm{gr}}(W)}_{{}_VM} / \mathsf{T}^{\mathrm{GL}(W)_0\cdot {}_VM}_{{}_VM} \\ &\cong \text{gr-Ext}^1_V({}_VM, {}_VM) \\ &\cong \mathrm{YExt}^1_U({}_UM, {}_UM).\end{aligned}$$

The other isomorphisms are from Theorem 2.22 and the display (3.10.1). □

4. *Answering Vasconcelos' question*

In this section we prove Theorem A from the introduction; see 4.2. We also verify a semi-local version of this result in Theorem 4.6. We begin with a consequence of the work from Section 3, motivated by Happel's result [**28**] discussed in the introduction. Recall that s and W are fixed in Notation 3.1; other notation comes from the appendix.

Proposition 4.1. *We work with the notation from Section 3. Let $\mathfrak{S}_W(U)$ denote the set of isomorphism classes in $\mathcal{D}(A)$ of degree-wise finite semi-free semidualizing DG U-modules C such that $s \geqslant \sup(C)$, $C_i = 0$ for all $i < 0$, and $(\tau(C)_{(\leqslant s)})^\natural \cong W$. Then $\mathfrak{S}_W(U)$ is a finite set.*

Proof. Let $[C], [C'] \in \mathfrak{S}_W(U)$, and set $M = \tau(C)_{(\leqslant s)}$ and $M' = \tau(C')_{(\leqslant s)}$. Since we have $s \geqslant \sup(C)$, we conclude that $M \simeq C$ in $\mathcal{D}(U)$, and similarly for M'.

We observe that the semidualizing assumption, in particular the condition

$$\mathrm{Ext}^1_U(M, M) \cong \mathrm{Ext}^1_U(C, C) = 0$$

implies that the orbit $\mathrm{GL}(W)_0 \cdot M$ is open in $\mathrm{Mod}^U(W)$. Indeed, from [**34**, Proposition 4.4] we have $\mathrm{YExt}^1_U(M, M) = 0$, so Theorem B shows that $\mathsf{T}^{\mathrm{Mod}^U(W)}_M = \mathsf{T}^{\mathrm{GL}(W)_0\cdot M}_M$. Lemma 3.5(b) implies that $\mathrm{GL}(W)_0 \cdot M$ is locally closed and non-singular. So, the orbit $\mathrm{GL}(W)_0 \cdot M$ is open by [**11**, Lemma 2.14].

Next, note that if $\mathrm{GL}(W)_0 \cdot M = \mathrm{GL}(W)_0 \cdot M'$, then $[C] = [C']$: indeed, by Proposition 3.6, we have $C \simeq M \cong M' \simeq C'$.

As $\mathrm{Mod}^U(W)$ is quasi-compact, it can only have finitely many open orbits. By what we have already shown, the map $[C] \mapsto \mathrm{GL}(W) \cdot M$ is a well-defined injective map from $\mathfrak{S}_W(U)$ into a finite set of open orbits, so $\mathfrak{S}_W(U)$ is finite, as desired. □

Proof of Theorem A 4.2. A result of Grothendieck [**27**, Proposition (0.10.3.1)] provides a flat local ring homomorphism $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ such that k' is algebraically closed. Composing with the natural map from R' to its $\mathfrak{m}'$-adic completion, we assume that R' is complete. By [**22**, Theorem II(c)], the induced map $\mathfrak{S}(R) \to \mathfrak{S}(R')$ is injective. Thus it suffices to prove the result for R', so assume that R is complete with algebraically closed residue field.

Let $\underline{t} = t_1, \ldots, t_n$ be a minimal generating sequence for $\mathfrak{m}$, and set $K = K^R(\underline{t})$, the Koszul complex. The map $\mathfrak{S}(R) \to \mathfrak{S}(K)$ induced by $C \mapsto K \otimes_R C$ is bijective by [**33**, Corollary 3.10]. Thus, it suffices to show that $\mathfrak{S}(K)$ is finite. Note that for each semidualizing R-complex C, we have $\mathrm{amp}(C) \leqslant \dim(R) - \mathrm{depth}(R)$ by [**13**, (3.4) Corollary]. A standard result about K (see, for example, [**21**, 1.3]) implies that

$$\mathrm{amp}(K \otimes_R C) \leqslant \mathrm{amp}(C) + n \leqslant \dim(R) - \mathrm{depth}(R) + n. \tag{4.2.1}$$

Set $s = \dim(R) - \mathrm{depth}(R) + n$.

From [**5**, (2.8)] there is a finite dimensional DG k-algebra U, as in Notation 3.1 with $F = k$, that is linked to K by a sequence of (quasi)isomorphisms of local DG algebras. By

Lemma A.13(c) there is a bijection between $\mathfrak{S}(K)$ and $\mathfrak{S}(U)$. Thus, it suffices to show that $\mathfrak{S}(U)$ is finite.

Let C' be a semidualizing DG U-module, and let C be a semidualizing R-complex corresponding to C' under the bijection given above. From Lemma A.13(b) and the display (4.2.1), we have $\operatorname{amp}(C') = \operatorname{amp}(K \otimes_R C) \leqslant s$. By applying an appropriate shift we assume without loss of generality that $\inf(C) = 0 = \inf(C')$, so we have $\sup(C') \leqslant s$. Let $L \xrightarrow{\simeq} C'$ be a minimal semi-free resolution of C' over U; see Fact A.10. The conditions $\sup(L) = \sup(C') \leqslant s$ imply that L (and hence C') is quasi-isomorphic to the truncation $\widetilde{L} := \tau(L)_{(\leqslant s)}$. We set $W := \widetilde{L}^{\natural}$ and work with the notation set in Section 3.

We claim that there is an integer $\lambda \geqslant 0$, depending only on R and U, such that $\sum_{i=0}^{s} r_i \leqslant \lambda$. (Recall that r_i and other quantities are fixed in Notation 3.1.) To see this, first note that for $i = 0, \dots, s$ we have $L_i = \bigoplus_{j=0}^{i} U_j^{\beta^U_{i-j}(C')}$; see Fact A.10. Using minimal semi-free resolutions and Lemma A.16, we conclude that

$$\beta^U_j(C') = \beta^R_j(C) \leqslant \mu_R^{j+\operatorname{depth}(R)}(R)$$

for all j. With our notation $n_i = \dim_F(U_i)$ from 3.1, it follows that

$$r_i = \operatorname{rank}_F(\widetilde{L}_i) \leqslant \operatorname{rank}_F(L_i) = \sum_{j=0}^{i} n_{i-j}\beta^U_j(C') \leqslant \sum_{j=0}^{i} n_{i-j}\mu_R^{j+\operatorname{depth}(R)}(R).$$

And we conclude that

$$\sum_{i=0}^{s} r_i \leqslant \sum_{i=0}^{s}\sum_{j=0}^{i} n_{i-j}\mu_R^{j+\operatorname{depth}(R)}(R).$$

Since the numbers in the right-hand side of this inequality only depend on R and U, we have found the desired value for λ.

Because there are only finitely many $(r_0, \dots, r_s) \in \mathbb{N}^{s+1}$ with $\sum_{i=0}^{s} r_i \leqslant \lambda$, there are only finitely many W that occur from this construction, say $W^{(1)}, \dots, W^{(b)}$. Proposition 4.1 implies that $\mathfrak{S}(U) = \mathfrak{S}_{W^{(1)}}(U) \cup \cdots \cup \mathfrak{S}_{W^{(b)}}(U)$ is finite.

Remark 4.3. Note that Theorem A provides a positive answer to Question 1.1 since one has $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$; see Definition A.11.

Now, we move toward our semi-local version of Theorem A.

Definition 4.4. The *non-Gorenstein locus* of R is

$$\operatorname{nGor}(R) := \{\text{maximal ideals } \mathfrak{m} \subset R \mid R_{\mathfrak{m}} \text{ is not Gorenstein}\} \subseteq \text{m-Spec}(R)$$

where m-Spec(R) is the set of maximal ideals of R.

Remark 4.5. For 'nice' rings, for example, rings with a dualizing complex [**38**], the set nGor(R) is closed in m-Spec(R).

Theorem 4.6. *Assume that R satisfies one of the following conditions:*

(1) *R is semi-local, or*
(2) *R is Cohen–Macaulay and* nGor(R) *is finite.*

Then the sets $\overline{\mathfrak{S}_0}(R)$ and $\overline{\mathfrak{S}}(R)$ are finite; see Notation A.19.

Proof. Because of the containment $\overline{\mathfrak{S}_0}(R) \subseteq \overline{\mathfrak{S}}(R)$, it suffices to show that $\overline{\mathfrak{S}}(R)$ is finite. Given a finite subset $X = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \subseteq \text{m-Spec}(R)$, let $f_X : \overline{\mathfrak{S}}(R) \to \prod_{i=1}^n \mathfrak{S}(R_{\mathfrak{m}_i})$ be given by the formula $f_X(\text{DPic}(R) \cdot C) := ([C_{\mathfrak{m}_1}], \ldots, [C_{\mathfrak{m}_n}])$. This is well-defined by Fact A.20.

In each case (1)–(2) we show that there is a finite set X such that f_X is injective. Then Theorem A implies that the set $\prod_{i=1}^n \mathfrak{S}(R_{\mathfrak{m}_i})$ is finite, so $\overline{\mathfrak{S}}(R)$ is also finite.

(1) Assume that R is semi-local, and set $X := \text{m-Spec}(R)$. To show that f_X is injective, let $[B], [C] \in \mathfrak{S}(R)$ be such that $B_{\mathfrak{m}_i} \sim C_{\mathfrak{m}_i}$ for $i = 1, \ldots, n$. According to Fact A.20, to show that $\text{DPic}(R) \cdot C = \text{DPic}(R) \cdot B$, it suffices to show that $\text{Ext}^j_R(B, C) = 0$ for $j \gg 0$. Since we know that $\text{Ext}^j_{R_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}, B_{\mathfrak{m}_i}) = 0$ for all $j \geqslant 1$, we conclude that there are integers $j_1, \ldots, j_n$ such that for $i = 1, \ldots, n$ we have $\text{Ext}^j_{R_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}, C_{\mathfrak{m}_i}) = 0$ for all $j \geqslant j_i$. Since B is homologically finite, we have $0 = \text{Ext}^j_{R_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}, C_{\mathfrak{m}_i}) \cong \text{Ext}^j_R(B, C)_{\mathfrak{m}_i}$ for all $j \geqslant \max_i j_i$. Since vanishing is a local property, it follows that $\text{Ext}^j_R(B, C) = 0$ for $j \gg 0$, as desired.

(2) Now, assume that R is Cohen–Macaulay and $\text{nGor}(R)$ is finite, and set $X := \text{nGor}(R)$. To show that f_X is injective, let $[B], [C] \in \mathfrak{S}(R)$ be such that $B_{\mathfrak{m}_i} \sim C_{\mathfrak{m}_i}$ for $i = 1, \ldots, n$. Lemma A.21 provides semidualizing modules B' and C' such that $\text{DPic}(R) \cdot B = \text{DPic}(R) \cdot B'$ and $\text{DPic}(R) \cdot C = \text{DPic}(R) \cdot C'$, so we assume without loss of generality that B and C are modules.

We claim that $B_\mathfrak{m} \sim C_\mathfrak{m}$ for all maximal ideals $\mathfrak{m} \subset R$ and $\text{Ext}^i_R(B, C) = 0$ for all $i \geqslant 1$. (Then the desired conclusion then follows from Fact A.20.) As B is a finitely generated R-module, it suffices to show that $B_\mathfrak{m} \sim C_\mathfrak{m}$ and $\text{Ext}^i_{R_\mathfrak{m}}(B_\mathfrak{m}, C_\mathfrak{m}) = 0$ for all $i \geqslant 1$ and for all maximal ideals $\mathfrak{m} \subset R$.

Case 1: $\mathfrak{m} \in \text{nGor}(R)$. In this case, we have $B_\mathfrak{m} \sim C_\mathfrak{m}$, by assumption. Since B and C are both modules, this implies that $B_\mathfrak{m} \cong C_\mathfrak{m}$, so the fact that $B_\mathfrak{m}$ is semidualizing over $R_\mathfrak{m}$ implies that $\text{Ext}^i_{R_\mathfrak{m}}(B_\mathfrak{m}, C_\mathfrak{m}) \cong \text{Ext}^i_{R_\mathfrak{m}}(B_\mathfrak{m}, B_\mathfrak{m}) = 0$.

Case 2: $\mathfrak{m} \notin \text{nGor}(R)$. In this case, the ring $R_\mathfrak{m}$ is Gorenstein, so we have $B_\mathfrak{m} \cong R_\mathfrak{m} \cong C_\mathfrak{m}$ by [**13**, (8.6) Corollary], and the desired vanishing follows. □

Appendix. *DG modules*

We assume that the reader is familiar with the category of R-complexes. For clarity, though, we include some notation.

Definition A.1. We index chain complexes of R-modules ('R-complexes' for short) homologically:

$$M = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots .$$

The degree of an element $m \in M$ is denoted $|m|$. The *tensor product* of two R-complexes M, N is denoted $M \otimes_R N$, and the *Hom complex* is denoted $\text{Hom}_R(M, N)$. A *chain map* $M \to N$ is a cycle in $\text{Hom}_R(M, N)_0$.

Next we discuss DG algebras, which are treated in, for example, [**4**, **7**, **8**, **10**, **19**, **31**]. We follow the notation and terminology from [**7**, **10**]; given the slight differences in the literature, though, we include a summary next.

Definition A.2. A *positively graded commutative differential graded R-algebra* (*DG R-algebra* for short) is an R-complex A equipped with a chain map $\mu^A : A \otimes_R A \to A$ such that the product $ab := \mu^A(a \otimes b)$ is associative, unital, and graded commutative, and such that $A_i = 0$

for $i < 0$. The map μ^A is the *product* on A. Given a DG R-algebra A, the *underlying algebra* is the graded commutative R-algebra $A^\natural = \oplus_{i=0}^{\infty} A_i$. When R is a field and $\mathrm{rank}_R(A^\natural) < \infty$, then A is *finite dimensional* over R. We say that A is *homologically degree-wise noetherian* if $\mathrm{H}_0(A)$ is noetherian and the $\mathrm{H}_0(A)$-module $\mathrm{H}_i(A)$ is finitely generated for all $i \geqslant 0$.

A *morphism* of DG R-algebras is a chain map $f\colon A \to B$ between DG R-algebras respecting products and multiplicative identities. A *quasi-isomorphism* of DG R-algebras is a morphism that is a quasi-isomorphism.

Assume that $(R, \mathfrak{m})$ is local. We say that A is *local* if it is homologically degree-wise noetherian and the ring $\mathrm{H}_0(A)$ is a local R-algebra, so, $\mathrm{H}_0(A)$ is a local ring with maximal ideal $\mathfrak{m}_{\mathrm{H}_0(A)}$ containing $\mathfrak{m}\,\mathrm{H}_0(A)$. In this case, the composition $A \to \mathrm{H}_0(A) \to \mathrm{H}_0(A)/\mathfrak{m}_{\mathrm{H}_0(A)}$ is a surjective morphism of DG R-algebras with kernel of the form

$$\mathfrak{m}_A = \cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} \mathfrak{m}_0 \to 0$$

where $\mathfrak{m}_0$ is the preimage of $\mathfrak{m}_{\mathrm{H}_0(A)}$ in A_0. The ideal $\mathfrak{m}_0$ is maximal and we have $\mathfrak{m}A_0 \subseteq \mathfrak{m}_0$. The DG ideal $\mathfrak{m}_A$ is the *augmentation ideal* of A.

For this paper, an important example is the next one.

Example A.3. Given a sequence $\mathbf{a} = a_1, \ldots, a_n \in R$, the Koszul complex $K = K^R(\mathbf{a})$ is a DG R-algebra with product given by the exterior (that is, wedge) product. If $(R, \mathfrak{m})$ is local and $\mathbf{a} \in \mathfrak{m}$, then K is a local DG R-algebra with augmentation ideal $\mathfrak{m}_K = (0 \to R \to \cdots \to R^n \to \mathfrak{m} \to 0)$.

In the passage to DG algebras, we focus on DG modules, described next.

Definition A.4. Let A be a DG R-algebra. A *DG A-module* is an R-complex M with a chain map $\mu^M\colon A \otimes_R M \to M$ such that the rule $am := \mu^M(a \otimes m)$ is associative and unital. The map μ^M is the *scalar multiplication* on M. The *underlying $A^\natural$-module* associated to M is the $A^\natural$-module $M^\natural = \oplus_{i\in\mathbb{Z}} M_i$.

Given DG A-modules M, N, the DG A-modules $M \otimes_A N$ and $\mathrm{Hom}_A(M,N)$ are defined in [**8**, Section 1]. A *morphism* of DG A-modules is a cycle in $\mathrm{Hom}_A(M,N)_0$. Isomorphisms in the category of DG A-modules are identified by the symbol $\cong$. Quasi-isomorphisms of DG A-modules are identified by the symbol $\simeq$.

Example A.5. Consider the ring R as a DG R-algebra. A DG R-module is just an R-complex, and a morphism of DG R-modules is simply a chain map.

Definition A.6. Let A be a DG R-algebra, let i be an integer, and let M be a DG A-module. The ith *suspension* of M is the DG A-module $\Sigma^i M$ defined by $(\Sigma^i M)_n := M_{n-i}$ and $\partial_n^{\Sigma^i M} := (-1)^i \partial_{n-i}^M$. The scalar multiplication on $\Sigma^i M$ is defined as $\mu^{\Sigma^i M}(a \otimes m) := (-1)^{i|a|}\mu^M(a \otimes m)$.

Definition A.7. Let A be a DG R-algebra. The category of DG A-modules described above is an abelian category; see, for example, [**31**, Introduction]. So, given DG A-modules M and N, the *Yoneda Ext group* $\mathrm{YExt}^1_A(M,N)$, defined as the set of equivalence classes of exact sequences $0 \to N \to X \to M \to 0$ of DG A-modules, is a well-defined abelian group under the Baer sum; see, for example, [**44**, (3.4.6)]. An explicit description of the Baer sum in this setting is given in the proof of [**34**, Theorem 3.5].

In the remainder of this appendix, we provide some background on semidualizing DG-modules, for example, semidualizing complexes. We assume that the reader is familiar with the derived category $\mathcal{D}(R)$. References for this include [12, 14, 25, 29, 40, 41]. For clarity, we include some definitions and notation.

DEFINITION A.8. Let A be a DG R-algebra. The *infimum*, *supremum*, and *amplitude* of a DG A-module M are

$$\inf(M) := \inf\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}$$
$$\sup(M) := \sup\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}$$
$$\mathrm{amp}(M) := \sup(M) - \inf(M).$$

The DG A-module M is *bounded below* if $M_n = 0$ for all $n \ll 0$; it is *degree-wise finite* if M_i is finitely generated over A_0 for each i; it is *homologically bounded* if $\mathrm{H}_i(M) = 0$ for $|i| \gg 0$; it is *homologically degree-wise finite* if each $\mathrm{H}_0(A)$-module $\mathrm{H}_n(M)$ is finitely generated; and it is *homologically finite* if it is homologically both bounded and degree-wise finite.

A DG A-module Q is *semi-projective* if $\mathrm{Hom}_A(Q, -)$ respects surjective quasi-isomorphisms, that is, if $Q^\natural$ is a projective graded $R^\natural$-module and $\mathrm{Hom}_A(Q, -)$ respects quasi-isomorphisms. A DG A-module L is *semi-free* if the graded $A^\natural$-module $L^\natural$ has a graded basis X which decomposes as a disjoint union $X = \sqcup_{i=0}^{\infty} X^i$ such that $\partial^L(X^i)$ is contained in the DG submodule RX^{i-1} for each $i \geqslant 0$, where $X^{-1} := \emptyset$. A *semi-free* (*respectively, semi-projective*) *resolution* of M is a quasi-isomorphism $L \xrightarrow{\simeq} M$ of DG A-modules such that L is semi-free (respectively, semi-projective). If R and A are local, then a *minimal semi-free resolution* of M is a semi-free resolution $L \xrightarrow{\simeq} M$ such that each graded basis of $L^\natural$ is finite in each degree and the differential on $(A/\mathfrak{m}_A) \otimes_A L$ is 0.

DEFINITION A.9. Let A be a DG R-algebra. The derived category $\mathcal{D}(A)$ is formed from the category of DG A-modules by formally inverting the quasi-isomorphisms; see [31]. Isomorphisms in $\mathcal{D}(A)$ are identified by the symbol $\simeq$, and isomorphisms up to shift in $\mathcal{D}(A)$ are identified by $\sim$.

The derived functors $M \otimes_A^{\mathbf{L}} N$ and $\mathbf{R}\mathrm{Hom}_A(M, N)$ are given, for example, via a semi-projective resolution $P \xrightarrow{\simeq} M$, as $M \otimes_A^{\mathbf{L}} N \simeq P \otimes_A N$ and $\mathbf{R}\mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_A(P, N)$. For each $i \in \mathbb{Z}$, set $\mathrm{Tor}_i^A(M, N) := \mathrm{H}_i(M \otimes_A^{\mathbf{L}} N)$ and $\mathrm{Ext}_A^i(M, N) := \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_A(M, N))$.†

FACT A.10. Let A be a DG R-algebra. If L is a bounded below DG A-module such that $L^\natural$ is a graded free $A^\natural$-module, then L is semi-free. If M is a DG A-module such that $j = \inf(M) > -\infty$, then M has a semi-free (hence, semi-projective) resolution $L \xrightarrow{\simeq} M$ such that $L_i = 0$ for all $i < j$. If A is homologically degree-wise noetherian and M is homologically degree-wise finite, then L can be chosen so that $L^\natural \cong \oplus_{i=j}^{\infty} \Sigma^i (A^\natural)^{\beta_i}$ for some integers β_i. If, in addition, R and A are local, then L can be chosen to be minimal by [18, Lemma A.3.iii].

In the passage from R to U in our proof of Theorem A, we use Christensen and Sather-Wagstaff's notion of semidualizing DG U-modules from [16], defined next.

DEFINITION A.11. Let A be a DG R-algebra, and let M be a DG A-module. The *homothety morphism* $X_M^A \colon A \to \mathrm{Hom}_A(M, M)$ is given by $X_M^A(a)(m) = am$. This induces a *homothety morphism* $\chi_M^A \colon A \to \mathbf{R}\mathrm{Hom}_A(M, M)$ in $\mathcal{D}(A)$.

Assume that A is homologically degree-wise noetherian. Then M is a *semidualizing* DG A-module if M is homologically finite and the homothety morphism $\chi_M^A \colon A \to \mathbf{R}\mathrm{Hom}_A(M, M)$

†See [34] for differences and similarities between, for example, $\mathrm{Ext}_A^1(M, N)$ and $\mathrm{YExt}_A^1(M, N)$.

is an isomorphism in $\mathcal{D}(A)$.[†] Let $\mathfrak{S}(A)$ denote the set of shift-isomorphism classes in $\mathcal{D}(A)$ of semidualizing DG A-modules, that is, the set of equivalence classes of semidualizing DG A-modules under the relation $\sim$ from Definition A.9. Over the ring R, semidualizing DG R-modules are called *semidualizing complexes;* in this setting, we let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes of semidualizing R-modules, which is naturally a subset of $\mathfrak{S}(R)$.

The following base-change construction is used in the passage from R to U in our proof of Theorem A.

Remark A.12. Let $A \to B$ be a morphism of DG R-algebras, and let M and N be DG A-modules. The 'base changed' complex $B \otimes_A M$ has the structure of a DG B-module by the action $b(b' \otimes m) := (bb') \otimes m$. This structure is compatible with the DG A-module structure on $B \otimes_A M$ via restriction of scalars. Furthermore, this induces a well-defined operation $\mathcal{D}(A) \to \mathcal{D}(B)$ given by $M \mapsto B \otimes^{\mathbf{L}}_A M$.

Given an element $f \in \operatorname{Hom}_A(M,N)_i$, define $B \otimes_A f \in \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)_i$ by the formula $(B \otimes_A f)(b \otimes m) := (-1)^{i|b|} b \otimes f(m)$. This yields a morphism of DG A-modules $\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)$ given by $f \mapsto B \otimes_A f$ which, in turn, provides a morphism $\mathbf{R}\operatorname{Hom}_A(M,N) \to \mathbf{R}\operatorname{Hom}_B(B \otimes^{\mathbf{L}}_A M, B \otimes^{\mathbf{L}}_A N)$ in $\mathcal{D}(A)$.

The next lemma is essentially from [**32**] and [**35**].

Lemma A.13. *Let $\varphi\colon A \to B$ be a quasi-isomorphism of DG R-algebras.*

(a) *The base change functor $B \otimes^{\mathbf{L}}_A -$ induces an equivalence of derived categories $\mathcal{D}(A) \to \mathcal{D}(B)$ whose quasi-inverse is given by restriction of scalars.*
(b) *For each DG A-module $X \in \mathcal{D}(A)$, one has $X \simeq B \otimes^{\mathbf{L}}_A X$ in $\mathcal{D}(A)$.*
(c) *The equivalence from part* (a) *induces a bijection from $\mathfrak{S}(A)$ to $\mathfrak{S}(B)$.*

Proof. (a) See, for example, [**32**, 7.6 Example].
(b) The equivalence from part (a) implies that the natural morphism $X \to B \otimes^{\mathbf{L}}_A X$ is an isomorphism in $\mathcal{D}(A)$.
(c) Let X be a DG A-module. We show that X is a semidualizing DG A-module if and only if $B \otimes^{\mathbf{L}}_A X$ is a semidualizing DG B-module. As the maps $X \to B \otimes^{\mathbf{L}}_A X$ and $A \to B$ are (quasi)isomorphisms, it follows that X is homologially finite over A if and only if $B \otimes^{\mathbf{L}}_A X$ is homologially finite over B. It remains to show that the morphism $\chi^A_X\colon A \to \mathbf{R}\operatorname{Hom}_A(X,X)$ is an isomorphism in $\mathcal{D}(A)$ if and only if $\chi^B_{B\otimes^{\mathbf{L}}_A X}\colon B \to \mathbf{R}\operatorname{Hom}_B(B \otimes^{\mathbf{L}}_A X, B \otimes^{\mathbf{L}}_A X)$ is an isomorphism in $\mathcal{D}(B)$. It is routine to show that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\chi^A_X} & \mathbf{R}\operatorname{Hom}_A(X,X) \\ {\scriptstyle \varphi}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \omega} \\ B & \xrightarrow[\chi^B_{B\otimes^{\mathbf{L}}_A X}]{} & \mathbf{R}\operatorname{Hom}_B(B \otimes^{\mathbf{L}}_A X, B \otimes^{\mathbf{L}}_A X) \end{array}$$

where ω is the morphism from Remark A.12. As ω is an isomorphism by [**35**, Proposition 2.1], the desired equivalence follows. □

[†]Note here that the existence of a semidualizing DG A-module forces A to be homologically bounded. Conversely, if A is homologically bounded, then it has a semidualizing DG A-module, namely, A itself. See, for example, [**1**].

DEFINITION A.14. Let A be a DG R-algebra, and let M be a DG A-module. Given an integer n, the *nth soft left truncation of M* is the complex

$$\tau(M)_{(\leqslant n)} := \cdots \to 0 \to M_n/\operatorname{Im}(\partial^M_{n+1}) \to M_{n-1} \to M_{n-2} \to \cdots$$

with differential induced by ∂^M. In other words, $\tau(M)_{(\leqslant n)}$ is the quotient DG A-module M/M' where M' is the following DG submodule of M:

$$M' = \cdots \to M_{n+2} \to M_{n+1} \to \operatorname{Im}(\partial^M_{n+1}) \to 0.$$

Note that $M' \simeq 0$ if and only if $n \geqslant \sup(M)$, so the natural morphism $M \to \tau(M)_{(\leqslant n)}$ of DG A-modules is a quasi-isomorphism if and only if $n \geqslant \sup(M)$.

DEFINITION A.15. Let A be a local DG R-algebra with $k = A/\mathfrak{m}_A$, and let M be a homologically finite DG A-module. For each integer i, the ith *Betti number* and the ith *Bass number* are respectively

$$\beta^A_i(M) := \operatorname{rank}_k(\operatorname{Tor}^A_i(k, M)) \quad \mu^i_A(M) := \operatorname{rank}_k(\operatorname{Ext}^i_A(k, M)).$$

The *Poincaré series* and the *Bass series* of M are the formal Laurent series

$$P^M_A(t) := \sum_{i\in\mathbb{Z}} \beta^A_i(M)t^i \quad I^A_M(t) := \sum_{i\in\mathbb{Z}} \mu^i_A(M)t^i.$$

The inequality in the next lemma may be of independent interest.

LEMMA A.16. *Assume that R is local, and let C be a semidualizing R-complex such that* $\inf(C) = 0$. *Then one has $\beta^R_p(C) \leqslant \mu^{p+\operatorname{depth} R}_R(R)$ for all $p \geqslant 0$.*

Proof. The isomorphism $\mathbf{R}\mathrm{Hom}_R(C, C) \simeq R$ implies the following equality of power series $I^R_R(t) = P^C_R(t)I^R_C(t)$. See [**6**, (1.5.3)]. We conclude that for each m we have

$$\mu^m_R(R) = \sum_{i=0}^{m} \beta^R_i(C)\mu^{m-i}_R(C).$$

In particular, for $m < \operatorname{depth}(R)$, we have

$$0 = \mu^m_R(R) \geqslant \beta^R_0(C)\mu^m_R(C).$$

The equality $\inf(C) = 0$ implies that $\beta^R_0(C) \neq 0$ by [**13**, (1.7.1)], so it follows that $\mu^m_R(C) = 0$. For $m = \operatorname{depth}(R)$, we conclude from this that

$$0 \neq \mu^{\operatorname{depth}(R)}_R(R) = \beta^R_0(C)\mu^{\operatorname{depth}(R)}_R(C)$$

and hence $\mu^{\operatorname{depth}(R)}_R(C) \neq 0$. Similarly, for $m = p + \operatorname{depth}(R)$, we have

$$\mu^{p+\operatorname{depth}(R)}_R(R) \geqslant \beta^R_p(C)\mu^{\operatorname{depth}(R)}_R(C) \geqslant \beta^R_p(C)$$

as desired. □

We note that, over a non-local ring, the set $\mathfrak{S}(R)$ may not be finite. For instance, the *Picard group* $\operatorname{Pic}(R)$, consisting of finitely generated rank-1 projective R-modules, is contained in $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$, so $\mathfrak{S}(R)$ can be infinite even when R is a Dedekind domain. We use some notions from [**23**] to deal with this.

DEFINITION A.17. A *tilting R-complex* is a semidualizing R-complex of finite projective dimension. The *derived Picard group of R* is the set $\operatorname{DPic}(R)$ of isomoprhism classes in $\mathcal{D}(R)$ of tilting R-complexes. The isomorphism class of a tilting R-complex L is denoted $[L] \in \operatorname{DPic}(R)$.

REMARK A.18. A homologically finite R-complex L is tilting if and only if $L_{\mathfrak{m}} \sim R_{\mathfrak{m}}$ for all maximal (equivalently, for all prime) ideals $\mathfrak{m} \subset R$, by [**23**, Proposition 4.4 and Remark 4.7]. In [**9**] tilting complexes are called 'invertible'. The derived Picard group $\mathrm{DPic}(R)$ is an abelian group under the operation $[L][L'] := [L \otimes^{\mathbf{L}}_R L']$. The identity in $\mathrm{DPic}(R)$ is $[R]$, and $[L]^{-1} = [\mathbf{R}\mathrm{Hom}_R(L,R)]$. The classical Picard group $\mathrm{Pic}(R)$ is naturally a subgroup of $\mathrm{DPic}(R)$. The group $\mathrm{DPic}(R)$ acts on $\mathfrak{S}(R)$ in a natural way: $[L][C] := [L \otimes^{\mathbf{L}}_R C]$. See [**23**, Properties 4.3 and Remark 4.9]. This action restricts to an action of $\mathrm{Pic}(R)$ on $\mathfrak{S}_0(R)$ given by $[L][C] := [L \otimes_R C]$.

NOTATION A.19. The set of orbits in $\mathfrak{S}(R)$ under the action of $\mathrm{DPic}(R)$ is denoted $\overline{\mathfrak{S}}(R)$,[†] and the set of orbits in $\mathfrak{S}_0(R)$ under the action of $\mathrm{Pic}(R)$ is denoted $\overline{\mathfrak{S}_0}(R)$.

FACT A.20. Given semidualizing R-complexes A and B, the following conditions are equivalent by [**23**, Proposition 5.1]:

(i) the orbits $\mathrm{DPic}(R) \cdot A$ and $\mathrm{DPic}(R) \cdot B$ are equal, that is, there is an element $[P] \in \mathrm{DPic}(R)$ such that $B \simeq P \otimes^{\mathbf{L}}_R A$; and
(ii) $A_{\mathfrak{m}} \sim B_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subset R$ and $\mathrm{Ext}^i_R(A,B) = 0$ for $i \gg 0$.

It is straightforward to show that the natural inclusion $\mathfrak{S}_0(R) \subseteq \mathfrak{S}(R)$ gives an inclusion $\overline{\mathfrak{S}_0}(R) \subseteq \overline{\mathfrak{S}}(R)$.

LEMMA A.21. *Assume that R is Cohen–Macaulay (not necessarily local), and let C be a semidualizing R-complex. There is an element $[L] \in \mathrm{DPic}(R)$ such that $L \otimes^{\mathbf{L}}_R C$ is isomorphic in $\mathcal{D}(R)$ to a module. In other words, the orbit $\mathrm{DPic}(R) \cdot C$ contains a module.*

Proof. For each $\mathfrak{p} \in \mathrm{Spec}(R)$, the fact that R is Cohen–Macaulay implies that $\mathrm{amp}(C_{\mathfrak{p}}) = 0$ by [**13**, (3.4) Corollary], that is, $C_{\mathfrak{p}} \sim \mathrm{H}_i(C)_{\mathfrak{p}} \neq 0$ for some i. As $\mathrm{amp}(C) < \infty$, this implies that $\mathrm{Spec}(R)$ is the disjoint union

$$\mathrm{Spec}(R) = \bigcup_{i=\inf(C)}^{\sup(C)} \mathrm{Supp}_R(\mathrm{H}_i(C)).$$

It follows that each set $\mathrm{Supp}_R(\mathrm{H}_i(C))$ is both open and closed. So, if $\mathrm{Supp}_R(\mathrm{H}_i(C))$ is non-empty, then it is a union of connected components of $\mathrm{Spec}(R)$.

Let $e_1, \ldots, e_p$ be a 'complete set of orthogonal primitive idempotents of R' as in [**9**, 4.8]. Then $R \cong R_{e_1} \times \cdots \times R_{e_p}$ and each $\mathrm{Spec}(R_{e_i})$ is naturally homeomorphic to a connected component of $\mathrm{Spec}(R)$. From the previous paragraph, for $i = 1, \ldots, p$ we have $C_{e_i} \simeq \Sigma^{u_i} \mathrm{H}_{u_i}(C_{e_i})$, and $\mathrm{H}_{u_i}(C_{e_i})$ is a semidualizing R_{e_i}-module. Each R-module M has a natural decomposition $M \cong \oplus_{i=1}^p M_{e_i}$ that is compatible with the product decomposition of R, and it follows that $C \simeq \oplus_{i=1}^p \Sigma^{u_i} \mathrm{H}_{u_i}(C_{e_i})$.

Let $L = \oplus_{i=1}^p \Sigma^{-u_i} R_{e_i}$. Then L is a tilting R-complex by Remark A.18, and

$$\begin{aligned} L \otimes^{\mathbf{L}}_R C &\simeq (\oplus_{i=1}^p \Sigma^{-u_i} R_{e_i}) \otimes^{\mathbf{L}}_R (\oplus_{i=1}^p \Sigma^{u_i} \mathrm{H}_{u_i}(C_{e_i})) \\ &\simeq \oplus_{i=1}^p (\Sigma^{-u_i} R_{e_i}) \otimes^{\mathbf{L}}_{R_{e_i}} (\Sigma^{u_i} \mathrm{H}_{u_i}(C_{e_i})) \\ &\simeq \oplus_{i=1}^p \mathrm{H}_{u_i}(C_{e_i}). \end{aligned}$$

Since $\oplus_{i=1}^p \mathrm{H}_{u_i}(C_{e_i})$ is an R-module, this establishes the lemma. □

[†]Observe that the notations $\mathfrak{S}(R)$ and $\overline{\mathfrak{S}}(R)$ represent different sets in [**23**].

Acknowledgements. We are grateful to a-fortiori@mathoverflow.net, L. L. Avramov, A. Bertram, L. W. Christensen, D. Fulghesu, D. Happel, S. Iyengar, P. Jørgensen, B. Keller, P. McNamara, and U. Nagel for helpful discussions about this work. We are also grateful to the anonymous referees for many thoughtful suggestions, including the present proof of Theorem B.

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