

The Category of Factorization

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The Category of Factorization

Brandon Goodell¹ · Sean K. Sather-Wagstaff¹

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Abstract

We introduce and investigate the category of factorization of a multiplicative, commutative, cancellative, pre-ordered monoid A , which we denote $\mathcal{F}(A)$. The objects of $\mathcal{F}(A)$ are factorizations of elements of A , and the morphisms in $\mathcal{F}(A)$ encode combinatorial similarities and differences between the factorizations. We pay particular attention to the divisibility pre-order and to the monoid $A = D \setminus \{0\}$ where D is an integral domain. Among other results, we show that $\mathcal{F}(A)$ is a symmetric and strict monoidal category with weak equivalences and compute the associated category of fractions obtained by inverting the weak equivalences. Also, we use this construction to characterize various factorization properties of integral domains: atomicity, unique factorization, and so on.

Keywords Category · Factorization · Integral domain · Monoid · Pre-order

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1 Introduction

It is well known that factorization in an integral domain can be poorly behaved, e.g., via the failure of unique factorization. Various constructions and techniques have been used to understand the variety of behaviors that can occur. An example of this is the group of divisibility, which goes back at least to Krull [9], and serves as important motivation for the present work. See also, e.g. [7,8,10–13].

In this paper, we introduce a construction that tracks all factorizations of non-zero elements in an integral domain D simultaneously: the category of factorization. The objects of this category are the finite sequences/ordered tuples $(x_n)_{n=1}^N = (x_n)$ of non-zero elements of D , along with the empty tuple \mathbf{o} . Each sequence (x_n) can be thought of as a factorization of the product $x_1 \cdots x_N$, with the product of the empty sequence (i.e., the empty product) defined to be 1. In other words, the objects in this category are exactly the finite factorizations in $D \setminus \{0\}$.

We specify the morphisms in this category in Definition 2.1. Big picture, they track various divisibility properties between factorizations. As an example, one type of morphism comes from factoring the entries of a tuple. For instance, over $D = \mathbb{Z}$ we have the tuple $(6, 35)$. The factorizations $6 = 2 \cdot 3$ and $35 = 5 \cdot 7$ yield a morphism $(6, 35) \rightarrow (2, 3, 5, 7)$ as well as morphisms $(6, 35) \rightarrow (2, 5, 3, 7)$ and $(6, 35) \rightarrow (1, 7, -1, 2, 3, 5)$ and so on. We identify other important types of morphisms that monitor other information about the factorizations in Example 2.2 and Definition 3.10.

Note that we do not identify factorizations that one would usually think of as essentially the same, e.g., $(2, 3) \neq (3, 2) \neq (-3, -2)$. However, these tuples are isomorphic in this category. In fact, two tuples (x_n) and (y_m) are isomorphic if and only if they have the same length and there is a permutation σ such that for all i , the element y_i is a unit multiple of $x_{\sigma(i)}$. Thus, the morphisms in this category are defined so that the isomorphisms exactly monitor which factorizations are essentially the same.

As in many papers in the subject of factorization in integral domains, we work in the more general setting of a cancellative commutative monoid, periodically specializing to the particular case of the non-zero elements of an integral domain. See Sect. 2 for the formal definition of our category as well as some examples. Section 3 contains fundamental properties of the morphisms in this category, including characterizations of the monic morphisms, epic morphisms, and isomorphisms. Section 4 constructs several useful functors; in particular, we construct adjoint functors between the category of factorization and the original monoid, and we show in Theorem 4.13 that the category of factorization is a symmetric and strict monoidal category.

In Sect. 5 we introduce a class of weak equivalences that mirrors properties of invertible elements of A . Thus, the morphisms in $\mathcal{F}(A)$ not only track relations between factorizations in A , but they also see factorization properties of individual elements of A , a theme that we revisit in subsequent sections. In Theorem 5.6, we show that the category of fractions obtained by inverting the weak equivalences is exactly the original monoid A , considered as a category in a natural way.

Section 6 is a treatment of a notion of weak divisibility for morphisms in $\mathcal{F}(A)$. Our main purpose here is to be able to define and investigate what it means for a morphism to be weakly prime (the subject of Sect. 8) as a companion to the weakly irreducible morphisms of Sect. 7.

As the morphisms in our category encode relations between factorizations, weak divisibility of morphisms encodes divisibility conditions between these relations by Theorem 6.5. And the weakly prime and weakly irreducible morphisms encode prime and irreducible relations (loosely speaking), by Theorems 7.4 and 8.3, respectively.

The paper ends with Sect. 9 which includes several characterizations of factorization properties of integral domains: unique factorization, half-factorization and so on. For instance, part of Theorem 9.1 states that an integral domain is atomic if and only if every morphism $(x_n) \rightarrow (y_m)$ in $\mathcal{F}(D \setminus \{0\})$ between non-empty tuples decomposes as a finite composition $(x_n) \rightarrow \cdots \rightarrow (y_m)$ of weakly irreducible morphisms and weak equivalences; this is parallel to the definition in terms of decompositions of elements as products of irreducible elements and units.

One application of the ideas from this paper is in a new class of integral domains, the “irreducible divisor pair domains” or IDPDs. See [3] for more on these.

Assumptions Throughout this paper, let (A, \leq) be a multiplicative, commutative, cancellative, pre-ordered monoid. In particular, the relation \leq is reflexive and transitive (though it is neither symmetric nor antisymmetric in general), and it respects the monoid operation. Sometimes, we specify that A is a *divisibility monoid*; this means that A is a multiplicative, commutative, cancellative monoid where the partial order is the *divisibility order* $a \mid b$. Let **POM** denote the category of pre-ordered monoids where morphisms are monoid homomorphisms $\phi: A \rightarrow B$ that respect the pre-orders on A and B .

We set $\mathbb{N} = \{1, 2, \dots\}$. For all $N \in \mathbb{N}$, set $[N] = \{1, 2, \dots, N\}$. We denote the identity function on a set X as id_X . Also, **Cat** is the category of small categories and **Set** is the category of sets.

2 Definitions and Examples

We begin this section by defining the category of factorization.

Definition 2.1 The *category of factorization* of the monoid A , denoted $\mathcal{F}(A)$ has as its objects all ordered tuples of monoid elements

$$\text{Obj}(\mathcal{F}(A)) = \left\{ (x_n)_{n=1}^N \mid N \in \mathbb{N}, x_n \in A \right\} \cup \{o\}$$

including the *empty tuple* denoted o . (In other words, $\text{Obj}(\mathcal{F}(A))$ is the set underlying the free monoid on A . See Definition 4.11 for more about this.) Again, we follow the convention that the empty product is $1 = 1_A$. When we see no danger of confusion, we write a tuple $(x_n)_{n=1}^N$ as (x_n) , we write $(y_m)_{m=1}^M$ as (y_m) , and so on.

The morphisms $(x_n) \rightarrow (y_m)$ correspond to the functions $f: [M] \rightarrow [N]$ that are *order-constrained* by (x_n) and (y_m) , by which we mean that for every $n \in [N]$ we have $x_n \leq \prod_{m \in f^{-1}(n)} y_m$. Given such a function f , we usually denote the associated morphism in $\mathcal{F}(A)$ as $\hat{f}: (x_n) \rightarrow (y_m)$. Since the function $f: [M] \rightarrow [N]$ can represent many different morphisms in $\mathcal{F}(A)$ with different domains or codomains, we occasionally employ the notation $\hat{f}_{(y_m)}^{(x_n)}: (x_n) \rightarrow (y_m)$, but only when necessary.

Two morphisms $\hat{f}, \hat{h}: (x_n) \rightarrow (y_m)$ are equal if $f = h$. We define the composition of two morphisms $\hat{f}: (x_n) \rightarrow (y_m)$ and $\hat{g}: (y_m) \rightarrow (z_p)$ by setting $\hat{g} \circ \hat{f} = \widehat{f \circ g}$.

We could simply define $\text{Mor}_{\mathcal{F}(A)}((x_n), (y_m))$ to be the set of order-constrained functions $f: [M] \rightarrow [N]$, i.e., we could define $\text{Mor}_{\mathcal{F}(A)}((x_n), (y_m))$ as a subset of $\text{Mor}_{\text{Set}}([M], [N])$. However, we find it helpful to distinguish between f and \hat{f} .

In Theorem 2.3 below, we prove that $\mathcal{F}(A)$ is a small category under this definition of composition of morphisms. Before that, though, we discuss examples like those in the introduction to help familiarize the reader with our construction. In particular, we introduce some terminology informally in this example, which we formalize in Definition 3.10 below.

Example 2.2 Consider the divisibility monoid $A = \mathbb{Z} \setminus \{0\}$. The objects of $\mathcal{F}(A)$ are ordered tuples of non-zero integers.

Some morphisms in $\mathcal{F}(A)$ are “factorization morphisms” corresponding to factoring the elements x_n . For example, we have a morphism $\hat{f}: (6, 35) \rightarrow (2, 3, 5, 7)$ corresponding to the equalities $6 = 2 \cdot 3$ and $35 = 5 \cdot 7$. Formally, with $(x_n) = (6, 35)$ and $(y_m) = (2, 3, 5, 7)$ we have $N = 2$ and $M = 4$, and \hat{f} corresponds to the function $f: [4] \rightarrow [2]$ given by $f(1) = f(2) = 1$ and $f(3) = f(4) = 2$. Moreover, \hat{f} is the unique morphism $(6, 35) \rightarrow (2, 3, 5, 7)$; indeed, there are 2^4 functions $[4] \rightarrow [2]$, but one checks readily that the given function f is the only one that is order-constrained, so it is the only one that determines a morphism in $\mathcal{F}(A)$.

Some morphisms are “divisibility morphisms”, e.g., the morphism $\hat{g}: (2, 5) \rightarrow (6, 15)$ corresponding to the conditions $2 \mid 6$ and $5 \mid 15$. Formally, this comes from the identity function $g = \text{id}_{[2]}: [2] \rightarrow [2]$. Such morphisms are similar to factorization morphisms in that no tuple elements are “dropped” from the domain objects since f and g are surjective.

Some morphisms “drop invertible coordinates”. For instance, we have a morphism $\hat{h}: (6, 1, 1) \rightarrow (6)$, coming from the function $h: [1] \rightarrow [3]$ with $h(1) = 1$. The point here is that $h^{-1}(2) = \emptyset$, so for h to be order-constrained requires the second entry 1 to divide the empty product, and so on. On the other hand, there are no morphisms $(6, 2, 1) \rightarrow (6)$ nor $(6, 2, 1) \rightarrow (6, 1)$. Indeed, any such morphism would require 6 or 2 to divide the empty product 1. These are similar to divisibility morphisms in that the underlying functions g and h are injective.

Most morphisms in $\mathcal{F}(A)$ exhibit aspects of all of the above types. For example, consider the morphism $\hat{\alpha}: (2, 3, 1, 1) \rightarrow (2, 7, 3, 5)$ coming from the function $\alpha: [4] \rightarrow [4]$ defined by $\alpha(1) = \alpha(2) = 1$ and $\alpha(3) = \alpha(4) = 2$. One can decompose $\hat{\alpha}$ as the composition

$$(2, 3, 1, 1) \xrightarrow{\hat{e}} (2, 3) \xrightarrow{\widehat{\text{id}}_{[2]}} (14, 15) \xrightarrow{\hat{g}} (2, 7, 3, 5)$$

where \hat{e} drops the invertible coordinates, $\widehat{\text{id}}_{[2]}$ is a divisibility morphism that uses the relations $2 \mid 14$ and $3 \mid 15$, and \hat{g} is a factorization morphism that uses the factorizations $14 = 2 \cdot 7$ and $15 = 3 \cdot 5$. See Proposition 3.11 for a general factorization result for morphisms.

Theorem 2.3 *The construction $\mathcal{F}(A)$ is a small category.*

Proof We need to show that composition is well-defined in $\mathcal{F}(A)$. That is, given morphisms $\hat{f}: (x_n) \rightarrow (y_m)$ and $\hat{g}: (y_m) \rightarrow (z_p)$, we need to show that $\widehat{f \circ g}$ is a morphism in $\mathcal{F}(A)$. For each $n \in [N]$ we have $x_n \leq \prod_{m \in f^{-1}(n)} y_m$ since f is order-constrained, and similarly for each $m \in f^{-1}(n)$ we have $y_m \leq \prod_{p \in g^{-1}(m)} z_p$. Since the partial order on A respects the operation on A , it follows that

$$x_n \leq \prod_{m \in f^{-1}(n)} y_m \leq \prod_{m \in f^{-1}(n)} \left(\prod_{p \in g^{-1}(m)} z_p \right) = \prod_{p \in (f \circ g)^{-1}(n)} z_p$$

so $\widehat{f \circ g}$ is indeed a morphism.

It is straightforward to show that each identity function $\text{id}_{[N]}: [N] \rightarrow [N]$ is order-constrained with respect to (x_n) , determining a morphism $(x_n) \rightarrow (x_n)$. This yields an identity morphism since, for instance, $\hat{f} \circ \widehat{\text{id}_{[N]}} = \widehat{\text{id}_{[N]}} \circ \hat{f} = \hat{f}$. Also, since composition in $\mathcal{F}(A)$ is based on composition of functions in **Set**, our rule of composition is associative, so $\mathcal{F}(A)$ is a category.

By construction, the collection of objects of $\mathcal{F}(A)$ is a set, being the collection of all ordered tuples over the set A . Also, each collection $\text{Mor}_{\mathcal{F}(A)}((x_n), (y_m))$ is a finite set since it corresponds to a subcollection of the set $\text{Mor}_{\mathbf{Set}}([M], [N])$. Thus, $\mathcal{F}(A)$ is a small category, as desired. \square

For the last result of this section, recall the notations **Cat** and **POM** from our assumptions at the end of Sect. 1.

Proposition 2.4 *The category of factorization operation is a covariant functor $\mathcal{F}: \mathbf{POM} \rightarrow \mathbf{Cat}$.*

Proof Let $\phi: A \rightarrow B$ be a morphism in **POM**. Let $\mathcal{F}(\phi): \mathcal{F}(A) \rightarrow \mathcal{F}(B)$ be given on objects as $\mathcal{F}(\phi)(x_n) = (\phi(x_n))$. For each morphism $\hat{f}_{(y_m)}^{(x_n)}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(A)$, take the relation $x_n \leq \prod_{m \in f^{-1}(n)} y_m$ and apply ϕ to conclude that $\phi(x_n) \leq \phi\left(\prod_{m \in f^{-1}(n)} y_m\right) = \prod_{m \in f^{-1}(n)} \phi(y_m)$; from this, we conclude that $\hat{f}_{(\phi(y_m))}^{(\phi(x_n))}: (\phi(x_n)) \rightarrow (\phi(y_m))$ is a morphism in $\mathcal{F}(B)$, in other words, we have that $\hat{f}_{\mathcal{F}(\phi)(y_m)}^{\mathcal{F}(\phi)(x_n)}: \mathcal{F}(\phi)(x_n) \rightarrow \mathcal{F}(\phi)(y_m)$ is a morphism in $\mathcal{F}(B)$, and we define $\mathcal{F}(\phi)(\hat{f}_{(y_m)}^{(x_n)}) = \hat{f}_{\mathcal{F}(\phi)(y_m)}^{\mathcal{F}(\phi)(x_n)}$. It is straightforward to show that $\mathcal{F}(\text{id}_A) = \text{id}_{\mathcal{F}(A)}$ and $\mathcal{F}(\phi \circ \psi) = \mathcal{F}(\phi) \circ \mathcal{F}(\psi)$, so \mathcal{F} is a covariant functor, as desired. \square

Remark 2.5 Many examples of interest for us involve taking an integral domain D and considering the divisibility monoid $A = D \setminus \{0\}$ and its category of factorization $\mathcal{F}(D \setminus \{0\})$. If **Dom** is the category of integral domains where the morphisms are the ring monomorphisms, then the operation $D \mapsto D \setminus \{0\}$ describes a covariant functor **Dom** \rightarrow **POM**, where a morphism $\iota: D \hookrightarrow D'$ in **Dom** maps to the restriction $\iota': D \setminus \{0\} \rightarrow D' \setminus \{0\}$. Note that ι' is a morphism since ι is injective and respects products. Consequently, the composition operation $D \mapsto \mathcal{F}(D \setminus \{0\})$ describes a covariant functor **Dom** \rightarrow **Cat**.

3 Morphisms

In this section, we analyze morphisms in the category $\mathcal{F}(A)$, where A is still a multiplicative, commutative, cancellative, pre-ordered monoid. We begin with the morphisms in and out of the empty tuple \circ .

Proposition 3.1 *For each tuple (x_n) in $\mathcal{F}(A)$, we have*

$$|\text{Mor}_{\mathcal{F}(A)}(\circ, (x_n))| = \begin{cases} 1 & \text{if } (x_n) = \circ \\ 0 & \text{otherwise} \end{cases}$$

$$|\text{Mor}_{\mathcal{F}(A)}((x_n), \circ)| \leq 1.$$

If A is a divisibility monoid, then

$$|\text{Mor}_{\mathcal{F}(A)}((x_n), \circ)| = \begin{cases} 1 & \text{if } (x_n) = \circ \text{ or each } x_n \text{ is invertible} \\ 0 & \text{if } (x_n) \neq \circ \text{ and some } x_n \text{ is non-invertible.} \end{cases}$$

Proof If $(x_n) \neq \mathfrak{o}$, then there are no morphisms $\hat{f}: \mathfrak{o} \rightarrow (x_n)$, since there are no functions $f: [N] \rightarrow \emptyset$. On the other hand, for $(x_n) = \mathfrak{o}$, the unique function $f: \emptyset \rightarrow \emptyset$ vacuously determines the identity morphism $\text{id}_{\mathfrak{o}}: \mathfrak{o} \rightarrow \mathfrak{o}$. This justifies our formula for $|\text{Mor}_{\mathcal{F}(A)}(\mathfrak{o}, (x_n))|$ from the statement of the proposition.

The morphisms $\hat{f}: (x_n) \rightarrow \mathfrak{o}$ correspond exactly to the functions $f: \emptyset \rightarrow [N]$ such that $x_n \leq 1_A$ for all $n \in [N]$. In particular, there is at most one such morphism \hat{f} . This justifies our bound for $|\text{Mor}_{\mathcal{F}(A)}((x_n), \mathfrak{o})|$ from the statement.

In light of what we have shown, we assume for the rest of this proof that A is a divisibility monoid and $(x_n) \neq \mathfrak{o}$. Since A is a divisibility monoid, the unique function $f: \emptyset \rightarrow [N]$ satisfies $x_n \leq 1_A$ for all n if and only if $x_n \mid 1_A$ for all n , that is, if and only if each x_n is invertible. This gives the desired computation for $|\text{Mor}_{\mathcal{F}(A)}((x_n), \mathfrak{o})|$. \square

The next example shows that the divisibility monoid assumption is crucial for the last formula in Proposition 3.1. It applies similarly to subsequent results of this section, though we leave the details to the reader.

Example 3.2 Consider the real interval $(0, 1]$ under multiplication with the standard ordering \leq of real numbers. Then $a \leq 1$ for all $a \in (0, 1]$, but the only invertible element of $(0, 1]$ is 1. Thus, the proof of Proposition 3.1 shows that $|\text{Mor}_{\mathcal{F}(A)}((x_n), \mathfrak{o})| = 1$ for all (x_n) regardless of the invertibility of the x_n .

The proof of the next result is similar to the previous one. The point is that there is a unique function $[N] \rightarrow [1]$ and there are N functions $[1] \rightarrow [N]$.

Proposition 3.3 *If (y) is a 1-tuple in $\mathcal{F}(A)$ and (x_n) is an N -tuple in $\mathcal{F}(A)$, then*

$$|\text{Mor}_{\mathcal{F}(A)}((y), (x_n))| = \begin{cases} 1 & \text{if } y \leq x_1 \cdots x_N \\ 0 & \text{otherwise} \end{cases}$$

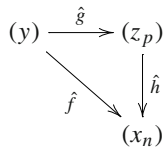
$$|\text{Mor}_{\mathcal{F}(A)}((x_n), (y))| \leq |\{n \in [N] \mid x_n \leq y\}| \leq N. \tag{3.1}$$

If in addition A is a divisibility monoid, then

$$|\text{Mor}_{\mathcal{F}(A)}((x_n), (y))| = |\{n_0 \in [N] \mid x_n \text{ is invertible for all } n \neq n_0 \text{ and } x_{n_0} \mid y\}|.$$

Note that the next result applies to diagrams of other shapes that can be described in terms of triangular diagrams.

Corollary 3.4 *Let morphisms $(y) \xrightarrow{\hat{f}} (x_n)$ and $(y) \xrightarrow{\hat{g}} (z_p) \xrightarrow{\hat{h}} (x_n)$ in $\mathcal{F}(A)$ be given where (y) is a 1-tuple. Then the following diagram commutes in $\mathcal{F}(A)$.*



Proof Formula (3.1) from Proposition 3.3 implies that there is at most one morphism $(y) \rightarrow (x_n)$. Our assumptions imply that there exists such a morphism, so all such morphisms must be equal. \square

Next, we show that $\mathcal{F}(A)$ can have several initial objects. Of course, they must be isomorphic, so this gives the first hint of our characterization of isomorphisms in Theorem 3.8 below.

Corollary 3.5 Assume that A is a divisibility monoid, and let $y \in A$. Then y is invertible in A if and only if the 1-tuple (y) is an initial object in $\mathcal{F}(A)$.

Proof The element $y \in A$ is invertible if and only if $y \mid 1$ if and only if $y \mid x$ for all $x \in A$, so the result follows directly from formula (3.1). \square

Proposition 3.6 Assume that A is a divisibility monoid. If A has a non-invertible element, then $\mathcal{F}(A)$ has no terminal object.

Proof We argue by contrapositive. Assume that (z_p) is terminal in $\mathcal{F}(A)$, and set $z = \prod_p z_p \in A$. Then every tuple (x_n) has a morphism of the form $(x_n) \rightarrow (z_p)$. In particular, for each $\alpha \in A$, there exists a morphism of the form $(\alpha) \rightarrow (z_p)$, hence $\alpha \mid z$. In particular, $z^2 \mid z$ so $z = az^2$ for some $a \in A$. Cancellativity implies that $1 = az$, so z is invertible. Thus, each element $\alpha \in A$ must divide the invertible element z , and it follows that each element $\alpha \in A$ is also invertible, as desired. \square

In general, the morphisms $(x_n) \rightarrow (y_m)$ in $\mathcal{F}(A)$ correspond to functions $[M] \rightarrow [N]$, so we have $|\text{Mor}_{\mathcal{F}(A)}((x_n), (y_m))| \leq N^M$. As we see next, this bound is not sharp in general, though it can be for specific examples.

Example 3.7 Consider the divisibility monoid $\mathbb{Z} \setminus \{0\}$, and let $M, N \in \mathbb{N}$. There are no morphisms from the N -tuple $(2, 2, \dots, 2)$ to the M -tuple $(3, 3, \dots, 3)$. On the other extreme, there are N^M morphisms from the N -tuple $(1, 1, \dots, 1)$ to the M -tuple $(3, 3, \dots, 3)$. Most tuples fit somewhere in the middle of this spectrum; for instance, there are exactly two morphisms $(1, 2) \rightarrow (1, 2)$.

Next, we characterize the isomorphisms in our category.

Theorem 3.8 Assume that A is a divisibility monoid. A morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(A)$ is an isomorphism if and only if (1) $N = M$, (2) f is a bijection, and (3) there is an N -tuple (u_n) of invertible monoid elements such that $u_n x_n = y_{f^{-1}(n)}$ for each $n \in [N]$.

Proof For the forward implication, let $\hat{f}: (x_n) \rightarrow (y_m)$ be an isomorphism in $\mathcal{F}(A)$ with inverse morphism $\hat{g}: (y_m) \rightarrow (x_n)$. It follows that $\text{id}_{[N]} = \text{id}_{(x_n)} = \hat{g} \circ \hat{f} = \widehat{f \circ g}$, and similarly $\text{id}_{[M]} = \widehat{g \circ f}$. We conclude that $g \circ f = \text{id}_{[M]}$ and $f \circ g = \text{id}_{[N]}$, so f and g are inverse bijections. In particular, we have $N = M$, and the functions f and g are bijections. Furthermore, it follows that the divisibility conditions making \hat{f} and \hat{g} morphisms in $\mathcal{F}(A)$ read as $x_n \mid y_{f^{-1}(n)}$ and $y_{f^{-1}(n)} \mid x_n$, respectively, for each n . Since A is cancellative, this implies that $u_n x_n = y_{f^{-1}(n)}$ for some invertible monoid element u_n , as desired.

For the converse, given N -tuples (x_n) and (u_n) such that each u_n is invertible, and given any permutation $f \in S_N$, it follows readily that the morphism $\hat{f}: (x_n) \rightarrow (u_{f(n)} x_{f(n)})$ is a well-defined morphism in $\mathcal{F}(A)$ with inverse $\hat{f}^{-1} = \widehat{f^{-1}}$. \square

Remark 3.9 Assume that A is a divisibility monoid, and consider a morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(A)$. Assume for this paragraph that $(y_m) \neq \circ$. Therefore, $(x_n) \neq \circ$ as well by Proposition 3.1. The divisibility condition making \hat{f} a morphism in $\mathcal{F}(A)$ says that $x_n \mid \prod_{m \in f^{-1}(n)} y_m$ for each $n \in [N]$, so there is an element $r_n \in A$ such that $r_n x_n = \prod_{m \in f^{-1}(n)} y_m$. In particular, the element $r := \prod_n r_n \in A$ satisfies

$$r \prod_{n=1}^N x_n = \prod_{n=1}^N r_n \cdot \prod_{n=1}^N x_n = \prod_{n=1}^N r_n x_n = \prod_{n=1}^N \prod_{m \in f^{-1}(n)} y_m = \prod_{m=1}^M y_m.$$

Moreover, the fact that A is cancellative implies that the elements r_n and r are unique in A with these properties.

In the case $(y_m) \neq \mathfrak{o}$, we again have a unique $r \in A$ such that $r \prod_{n=1}^N x_n = \prod_{m=1}^M y_m = 1$; moreover, r is invertible in A in this case. Indeed, Proposition 3.1 implies that $\prod_{n=1}^N x_n$ is invertible, so $r = (\prod_{n=1}^N x_n)^{-1}$ does the job.

Next, we formalize a decomposition result for morphisms suggested in Example 2.2. In light of Proposition 3.1, the non-empty assumption on (x_n) and (y_m) is not much of a restriction.

Definition 3.10 Assume that A is a divisibility monoid. A morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(A)$ between non-empty tuples is a *factorization morphism* provided that the underlying function $f: [M] \rightarrow [N]$ is surjective and for all $n \in [N]$ we have $x_n = \prod_{m \in f^{-1}(n)} y_m$. A *divisibility morphism* is any morphism $\widehat{\text{id}}_{[P]}: (z_p) \rightarrow (a_p z_p)$ with $P \geq 1$ and $(a_p) \in \mathcal{F}(A)$. (See 2.2 for specific examples.)

Proposition 3.11 Let A be a divisibility monoid. Let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\mathcal{F}(A)$ between non-empty tuples. Then \hat{f} decomposes as the composition

$$(x_n) \xrightarrow{\hat{\epsilon}} (z_p) \xrightarrow{\widehat{\text{id}}_{[P]}} (a_p z_p) \xrightarrow{\hat{\phi}} (y_m)$$

where we have the following:

- (1) ϵ is injective, and $\hat{\epsilon}: (x_n) \rightarrow (z_p)$ corresponds to dropping invertible elements and permuting the non-dropped elements;
- (2) $P = |\text{Im}(f)|$, and $\widehat{\text{id}}_{[P]}: (z_p) \rightarrow (a_p z_p)$ is a divisibility morphism; and
- (3) ϕ is surjective, and $\hat{\phi}: (a_p z_p) \rightarrow (y_m)$ is a factorization morphism.

Following this notation, we have $\prod_m y_m = \left(u \prod_p a_p\right) \left(\prod_n x_n\right)$ where u is the product of the invertible elements dropped by $\hat{\epsilon}$. That is, in the notation of Remark 3.9, we have $r = u \prod_p a_p$.

Before proving this result, we present an example illustrating the proof.

Example 3.12 We consider the divisibility monoid $\mathbb{Z} \setminus \{0\}$ and the morphism $(6, 1, 35) \xrightarrow{\hat{f}} (2, 7, 33, 65)$ corresponding to the divisibility conditions $6 \mid 2 \cdot 33$ and $35 \mid 7 \cdot 65$, along with the fact that 1 divides the empty product. Note that \hat{f} drops the invertible element 1 in $(6, 1, 35)$, so we start our factorization as

$$(6, 1, 35) \xrightarrow{\hat{\epsilon}} (6, 35) \xrightarrow{\hat{\phi}_{(2,7,33,65)}^{(6,35)}} (2, 7, 33, 65)$$

where $\hat{\epsilon}$ drops the invertible element, and $\hat{\phi}_{(2,7,33,65)}^{(6,35)}$ corresponds to the divisibility conditions $6 \mid 2 \cdot 33$ and $35 \mid 7 \cdot 65$. Recall the superscript/subscript notation for $\hat{\phi}$ from Definition 2.1.

Next, we write these divisibility conditions as $11 \cdot 6 = 2 \cdot 33$ and $13 \cdot 35 = 7 \cdot 65$, allowing us to rewrite \hat{f} in the form $\hat{\phi}_{(2,7,33,65)}^{(6-11,35-13)} \circ \widehat{\text{id}}_{[2]} \circ \hat{\epsilon}$ as follows

$$(6, 1, 35) \xrightarrow{\hat{\epsilon}} (6, 35) \xrightarrow{\widehat{\text{id}}_{[2]}} (6 \cdot 11, 35 \cdot 13) \xrightarrow{\hat{\phi}_{(2,7,33,65)}^{(6-11,35-13)}} (2, 7, 33, 65)$$

where $\widehat{\text{id}}_{[2]}$ corresponds to the divisibility conditions $6 \mid 6 \cdot 11$ and $35 \mid 35 \cdot 13$, and $\hat{\phi}_{(2,7,33,65)}^{(6-11,35-13)}$ corresponds to the factorization $6 \cdot 11 = 2 \cdot 33$ and $35 \cdot 13 = 7 \cdot 65$.

Formally, we have $P = 2$ and

$$\begin{array}{lclclcl}
 [4] \xrightarrow{f} [3] & 1 \mapsto 1 & 2 \mapsto 3 & 3 \mapsto 1 & 4 \mapsto 3 \\
 [2] \xrightarrow{\epsilon} [3] & 1 \mapsto 1 & 2 \mapsto 3 & & \\
 [4] \xrightarrow{\phi} [2] & 1 \mapsto 1 & 2 \mapsto 2 & 3 \mapsto 1 & 4 \mapsto 2
 \end{array}$$

Proof of Proposition 3.11 Let $\text{Im}(f) = \{n_1, \dots, n_P\}$ where $n_1 < \dots < n_P$. Define $\epsilon: [P] \rightarrow [N]$ by the formula $\epsilon(p) = n_p$. Note that for each $n \notin \text{Im}(f)$, the divisibility condition for x_n by definition says that x_n divides the empty product so x_n is invertible. It follows readily that $\hat{\epsilon}: (x_n) \rightarrow (x_{n_1}, \dots, x_{n_P})$ is a morphism in $\mathcal{F}(A)$ that drops the invertible elements x_n for all $n \notin \text{Im}(f)$.

Define $\phi: [M] \rightarrow [P]$ by the formula $\phi(m) = p$ where p is the unique element of $[P]$ such that $f(m) = n_p$. By construction, we have $\phi^{-1}(p) = f^{-1}(n_p)$. Thus, the divisibility condition $x_{n_p} \mid \prod_{m \in f^{-1}(n_p)} y_m$ reads as $x_{n_p} \mid \prod_{m \in \phi^{-1}(p)} y_m$. We conclude that $\hat{\phi}_{(y_m)}^{(x_{n_p})}: (x_{n_p}) = (x_{n_1}, \dots, x_{n_P}) \rightarrow (y_m)$ is a morphism in $\mathcal{F}(A)$. Furthermore, by construction we have $\epsilon \circ \phi = f$ since $\epsilon(\phi(m)) = \epsilon(p) = n_p = f(m)$, so we have $\hat{f} = \widehat{\epsilon \circ \phi} = \hat{\phi}_{(y_m)}^{(x_{n_p})} \circ \hat{\epsilon}$.

For $p = 1, \dots, P$ set $z_p = x_{n_p}$. Thus, we have $\hat{\epsilon}: (x_n) \rightarrow (z_p) = (x_{n_p})$ and $\hat{\phi}_{(y_m)}^{(z_p)}: (z_p) \rightarrow (y_m)$. The divisibility relation $z_p = x_{n_p} \mid \prod_{m \in f^{-1}(n_p)} y_m$ implies that there is an element $a_p \in A$ such that $a_p z_p = \prod_{m \in f^{-1}(n_p)} y_m$. Then $\widehat{\text{id}_{[P]}}: (z_p) \rightarrow (a_p z_p)$ is a morphism (a divisibility morphism) in $\mathcal{F}(A)$. Using the same ϕ as above, the equality $a_p z_p = \prod_{m \in f^{-1}(n_p)} y_m$ shows that $\hat{\phi}_{(y_m)}^{(a_p z_p)}: (a_p z_p) \rightarrow (y_m)$ is a morphism (a factorization morphism) in $\mathcal{F}(A)$. Furthermore, the equality $\text{id}_{[P]} \circ \phi = \phi$ implies that the morphism $\hat{\phi}_{(y_m)}^{(x_{n_p})}$ factors as $\hat{\phi}_{(y_m)}^{(x_{n_p})} = \widehat{\text{id}_{[P]}} \circ \phi = \hat{\phi}_{(y_m)}^{(a_p z_p)} \circ \widehat{\text{id}_{[P]}}$. Thus, in light of the equality $\hat{f} = \hat{\phi}_{(y_m)}^{(x_{n_p})} \circ \hat{\epsilon}$ from the previous paragraph, we have the factorization $\hat{f} = \hat{\phi}_{(y_m)}^{(x_{n_p})} \circ \hat{\epsilon} = \hat{\phi}_{(y_m)}^{(a_p z_p)} \circ \widehat{\text{id}_{[P]}} \circ \hat{\epsilon}$, as desired.

Lastly, we compute

$$\prod_m y_m = \prod_p a_p z_p = \left(\prod_p a_p \right) \left(\prod_p z_p \right) = \left(\prod_p a_p \right) u \left(\prod_n x_n \right)$$

where the steps in this sequence come from the construction of the morphisms $\hat{\phi}_{(y_m)}^{(a_p z_p)}$, $\widehat{\text{id}_{[P]}}$, and $\hat{\epsilon}$, respectively. □

The forward implications in the next result are fairly routine. However, the converses are a bit surprising, the fact that the order-constrained maps are rich enough to be able to see injectivity and surjectivity of the underlying functions.

Theorem 3.13 Assume that A is a divisibility monoid, and let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\mathcal{F}(A)$.

- (a) The function f is injective if and only if the morphism \hat{f} is epic.
- (b) The function f is surjective if and only if the morphism \hat{f} is monic.

Proof (a) For the forward implication, if f is injective then f cancels on the left as a function. Hence, if \hat{g} and \hat{h} are morphisms in $\mathcal{F}(A)$ such that $\hat{g} \circ \hat{f} = \hat{h} \circ \hat{f}$, then $\widehat{f \circ g} = \widehat{f \circ h}$, so $f \circ g = f \circ h$; the injectivity of f implies that $g = h$, so $\hat{g} = \hat{h}$.

For the converse, we assume that f is not injective (in particular, $M \geq 2$) and show that \hat{f} is not epic. Permuting the entries of the tuple (y_m) corresponds to composition of \hat{f} with an isomorphism, hence this does not affect the question of whether \hat{f} is epic, nor does it change the fact that f is not injective. Thus, we assume without loss of generality that $f(M - 1) = f(M)$.

To show that \hat{f} is not epic, we explicitly construct two morphisms $\hat{g}, \hat{h}: (y_m) \rightarrow (y_1, \dots, y_M, 1)$ such that $\hat{g} \neq \hat{h}$ but $\hat{g} \circ \hat{f} = \hat{h} \circ \hat{f}$. The morphism \hat{g} corresponds to the divisibility conditions $y_m \mid y_m$ for all $m < M$ and $y_M \mid 1 \cdot y_M$. In other words, we have $g: [M + 1] \rightarrow [M]$ defined by $g(m) = m$ for all $m \leq M$ and $g(M + 1) = M$. The morphism \hat{h} corresponds to the divisibility conditions $y_m \mid y_m$ for all $m < M - 1$ along with $y_{M-1} \mid 1 \cdot y_{M-1}$ and $y_M \mid y_M$. In other words, we have $h: [M + 1] \rightarrow [M]$ defined by $h(m) = m$ for all $m \leq M$ and $h(M + 1) = M - 1$.

Since $g(M + 1) = M \neq M - 1 = h(M + 1)$, we have $g \neq h$ hence $\hat{g} \neq \hat{h}$. On the other hand, we have $f \circ g = f \circ h$. Indeed, for $m \leq M$ we have $f(g(m)) = f(m) = f(h(m))$. Also, $f(g(M + 1)) = f(M) = f(M - 1) = f(h(M + 1))$, so we have $f \circ g = f \circ h$, and thus $\hat{g} \circ \hat{f} = \widehat{f \circ g} = \widehat{f \circ h} = \hat{h} \circ \hat{f}$, as desired.

- (b) The forward implication is established like the forward implication of part (a). For the converse, we assume that f is not surjective and show that \hat{f} is not monic. Permuting the entries of the tuple (x_n) , we assume without loss of generality that $N \notin \text{Im}(f)$. In particular, the divisibility condition for \hat{f} implies that x_N divides the empty product, so x_N is invertible.

In order to show that \hat{f} is not monic, we explicitly construct two morphisms $\hat{g}, \hat{h}: (x_1, \dots, x_N, x_N) \rightarrow (x_n)$ such that $\hat{g} \neq \hat{h}$ but $\hat{f} \circ \hat{g} = \hat{f} \circ \hat{h}$. The morphism \hat{g} corresponds to the divisibility conditions $x_n \mid x_n$ for all $n \leq N$ as well as the second copy of x_N in (x_1, \dots, x_N, x_N) dividing the empty product. In other words, we have $g: [N] \rightarrow [N + 1]$ defined by $g(n) = n$ for all $n \leq N$. The morphism \hat{h} corresponds to the divisibility conditions $x_n \mid x_n$ for $n < N$ as well as the first copy of x_N in (x_1, \dots, x_N, x_N) dividing the empty product and the second copy of x_N dividing x_N . In other words, we have $h: [N] \rightarrow [N + 1]$ defined by $h(n) = n$ for all $n < N$ and $h(N) = N + 1$.

As in part (a), it is straightforward to show that $g \neq h$ and $g \circ f = h \circ f$, so $\hat{g} \neq \hat{h}$ and $\hat{f} \circ \hat{g} = \hat{f} \circ \hat{h}$, as desired. □

Example 3.14 With Theorem 3.13 in hand, it is straightforward to say which of the morphisms from Example 2.2 are monic and which ones are epic, when A is a divisibility monoid. For instance, morphisms $(x_n) \rightarrow (y_m)$ that just drop invertible elements correspond to injective functions $[M] \rightarrow [N]$, so these morphisms are epic. For factorization morphisms, the underlying functions are surjective, so these morphisms are monic. And the functions underlying divisibility morphisms are bijections, so these morphisms are both epic and monic. In particular, in the decomposition $\hat{f} = \hat{\phi} \circ \widehat{\text{id}_{[P]}} \circ \hat{\epsilon}$ from Proposition 3.11, the morphisms $\hat{\epsilon}$ and $\widehat{\text{id}_{[P]}}$ are epic, while $\widehat{\text{id}_{[P]}}$ and $\hat{\phi}$ are monic.

Note that the divisibility morphisms therefore give plentiful examples of morphisms in $\mathcal{F}(A)$ that are both monic and epic but are not isomorphisms. For instance, in $\mathcal{F}(\mathbb{Z} \setminus \{0\})$ the divisibility morphism $(2) \rightarrow (6)$ is monic and epic, but not an isomorphism, by Theorem 3.8.

4 Functors and Symmetric Monoidal Structure

In this section, we investigate several natural functors involving $\mathcal{F}(A)$, and we show in Theorem 4.13 that $\mathcal{F}(A)$ is a symmetric and strict monoidal category, where A is still a multiplicative, commutative, cancellative, pre-ordered monoid.

Definition 4.1 Let $\mathfrak{F}: \mathcal{F}(A) \rightarrow \mathbf{Set}$ be the contravariant functor defined by mapping an N -tuple (x_n) to the set $[N]$, and by mapping each morphism $\hat{f}: (x_n) \rightarrow (y_m)$ to the underlying function $f: [M] \rightarrow [N]$.

Remark 4.2 It is straightforward to show that $\mathfrak{F}: \mathcal{F}(A) \rightarrow \mathbf{Set}$ is indeed a contravariant functor. Moreover, it is faithful because $\text{Mor}_{\mathcal{F}(A)}((x_n), (y_m))$ is defined as a subset of $\text{Mor}_{\mathbf{Set}}([M], [N])$.

On the other hand, \mathfrak{F} is not full in general. For instance, if A is a divisibility monoid with at least one non-invertible element y , then $\text{Mor}_{\mathcal{F}(A)}((y), (1)) = \emptyset \neq \text{Mor}_{\mathbf{Set}}([1], [1])$, so \mathfrak{F} is not full in this case.

Similarly, \mathfrak{F} is not representable in general. Indeed, continue with a divisibility monoid A with at least one non-invertible element y , and suppose that $(z_p) \in \mathcal{F}(A)$ is such that $[N] = \mathfrak{F}((x_n)) \cong \text{Mor}_{\mathcal{F}(A)}((x_n), (z_p))$ for all (x_n) in $\mathcal{F}(A)$. In particular, for each N -tuple (x_n) in $\mathcal{F}(A)$ we must have $|\text{Mor}_{\mathcal{F}(A)}((x_n), (z_p))| = N$. However, for the N -tuple $(1, \dots, 1)$ we have $|\text{Mor}_{\mathcal{F}(A)}((1, \dots, 1), (z_p))| = |\text{Mor}_{\mathbf{Set}}([P], [N])| = N^P$. The equation $N = N^P$ implies that $P = 1$, so $(z_p) = (z_1)$. For an arbitrary 1-tuple (x) , the condition $|\text{Mor}_{\mathcal{F}(A)}((x), (z_1))| = 1$ implies that $x \mid z_1$ for all $x \in A$; for example, we then have $y z_1 \mid z_1$, contradicting the assumption that y be non-invertible.

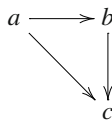
Next, we consider two functors between A and $\mathcal{F}(A)$.

Remark 4.3 Consider the monoid A as a small category with object set A and morphisms given by the inequalities in A :

$$\text{Mor}_A(x, y) = \begin{cases} \{x \leq y\} & \text{if } x \leq y \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, A is a small category where every morphism-set has at most one element. With this convention, when we feel it is helpful to consider things functorially, we consider the inclusion functor $\mathcal{I}: \mathbf{POM} \rightarrow \mathbf{Cat}$, so, for each pre-ordered monoid B , we denote by $\mathcal{I}(B)$ the corresponding small category where every morphism-set has at most one element.

As in Corollary 3.4, the fact that every morphism set in A has at most one element implies that any diagram



in A automatically commutes.

Definition 4.4 Let $\mathfrak{A}_A: \mathcal{F}(A) \rightarrow A$ be the covariant functor defined on objects by mapping an N -tuple (x_n) to the product $\prod_n x_n \in A$. For each morphism $\hat{f}: (x_n) \rightarrow (y_m)$, let $\mathfrak{A}_A(\hat{f})$ be the pre-order relation $\prod_n x_n \leq \prod_m y_m$ in A , which is satisfied because f is order-constrained.

Remark 4.5 It is straightforward to show that \mathfrak{A}_A is indeed a covariant functor. Unlike our previous functor, though, \mathfrak{A}_A is neither full nor faithful in general. Indeed, consider the divisibility monoid $A = \mathbb{Z} \setminus \{0\}$. It is straightforward to show that $\text{Mor}_{\mathcal{F}(\mathbb{Z} \setminus \{0\})}((4, 9), (6, 6)) = \emptyset \neq \text{Mor}_{\mathbb{Z} \setminus \{0\}}(36, 36)$, so \mathfrak{A}_A is not full. Similarly, we have $|\text{Mor}_{\mathcal{F}(\mathbb{Z} \setminus \{0\})}((2, 2), (2, 2))| = 2 > 1 = |\text{Mor}_{\mathbb{Z} \setminus \{0\}}(4, 4)|$, so \mathfrak{A}_A is not faithful.

Definition 4.6 Let $\mathfrak{B}_A : A \rightarrow \mathcal{F}(A)$ be the covariant functor defined by mapping an element $x \in A$ to the 1-tuple (x) , and by mapping each order relation $x \leq y$ to the morphism $\widehat{\text{id}}_{[1]} : (x) \rightarrow (y)$ which is a morphism in $\mathcal{F}(A)$ by definition.

Remark 4.7 It is straightforward to show that \mathfrak{B}_A is indeed a covariant functor. Unlike our previous functor, though, we have the following.

Proposition 4.8 *The functor $\mathfrak{B}_A : A \rightarrow \mathcal{F}(A)$ is fully faithful.*

Proof This follows directly from formula (3.1) from Proposition 3.3. □

Proposition 4.9 *The functors \mathfrak{A}_A and \mathfrak{B}_A are adjoints, that is, for all $y \in A$ and $(x_n) \in \mathcal{F}(A)$ we have natural bijections*

$$\text{Mor}_A(y, \mathfrak{A}_A((x_n))) \cong \text{Mor}_{\mathcal{F}(A)}(\mathfrak{B}_A(y), (x_n)).$$

In addition, the composition $\mathfrak{A}_A \mathfrak{B}_A : A \rightarrow A$ is the identity.

Proof To show that the composition $\mathfrak{A}_A \mathfrak{B}_A$ is the identity on objects, we compute directly: $\mathfrak{A}_A(\mathfrak{B}_A(y)) = \mathfrak{A}_A((y)) = y$. To show that $\mathfrak{A}_A \mathfrak{B}_A$ is the identity on morphisms, let $\phi : y \rightarrow z$ be a morphism in A . That is, we have $y \leq z$ in A and the arrow ϕ is the unique arrow in $\text{Mor}_A(y, z)$. The composition $\mathfrak{A}_A \mathfrak{B}_A$ maps arrows in $\text{Mor}_A(y, z)$ to arrows in $\text{Mor}_A(y, z)$, so $\mathfrak{A}_A(\mathfrak{B}_A(\phi))$ is the unique arrow in $\text{Mor}_A(y, z)$, i.e., we have $\mathfrak{A}_A(\mathfrak{B}_A(\phi)) = \phi$, as desired.

The adjoint bijection follows from the next display wherein the equalities are by definition and the bijection is from equation (3.1) in Proposition 3.3.

$$\begin{aligned} \text{Mor}_A(y, \mathfrak{A}_A((x_n))) &= \text{Mor}_A(y, \prod_n x_n) \\ &\cong \text{Mor}_{\mathcal{F}(A)}((y), (x_n)) \\ &= \text{Mor}_{\mathcal{F}(A)}(\mathfrak{B}_A(y), (x_n)) \end{aligned}$$

The naturality of the bijection here follows as in the preceding paragraph, using the fact that these morphism sets in A and $\mathcal{F}(A)$ have at most one element each. □

Our notations for \mathfrak{A}_A and \mathfrak{B}_A are intentionally suggestive of the next result.

Proposition 4.10 *Consider the inclusion functor $\mathcal{I} : \mathbf{POM} \rightarrow \mathbf{Cat}$ of Remark 4.3 and the category of factorization functor $\mathcal{F} : \mathbf{POM} \rightarrow \mathbf{Cat}$. Then the operations $\mathfrak{A} : \mathcal{F} \rightarrow \mathcal{I}$ and $\mathfrak{B} : \mathcal{I} \rightarrow \mathcal{F}$ are natural transformations.*

Proof First, observe that for each pre-ordered monoid A , we have $\mathcal{I}(A) = A$, and the operations $\mathfrak{A}_A : \mathcal{F}(A) \rightarrow \mathcal{I}(A)$ and $\mathfrak{B}_A : \mathcal{I}(A) \rightarrow \mathcal{F}(A)$ are covariant functors, that is, they are morphisms in \mathbf{Cat} . Furthermore, for each morphism $\phi : A \rightarrow B$ in \mathbf{POM} , we see next that the following diagram commutes

$$\begin{array}{ccccc} A & \xrightarrow{\mathfrak{B}_A} & \mathcal{F}(A) & \xrightarrow{\mathfrak{A}_A} & A \\ \phi \downarrow & & \mathcal{F}(\phi) \downarrow & & \phi \downarrow \\ B & \xrightarrow{\mathfrak{B}_B} & \mathcal{F}(B) & \xrightarrow{\mathfrak{A}_B} & B \end{array}$$

$$\begin{array}{ccc}
 y & \xrightarrow{\quad} & (y) \\
 \downarrow & & \downarrow \\
 \phi(y) & \xrightarrow{\quad} & (\phi(y))
 \end{array}
 \qquad
 \begin{array}{ccc}
 (x_n) & \xrightarrow{\quad} & \prod_n x_n \\
 \downarrow & & \downarrow \\
 (\phi(x_n)) & \xrightarrow{\quad} & \prod_n \phi(x_n)
 \end{array}$$

so we indeed have natural transformations. □

We use the following concatenation tensor product in our treatment of weak divisibility and weak primeness in Sects. 6, 8, and 9 below. Proposition 4.12 and Theorem 4.13 document some of its properties.

Definition 4.11 Fix non-empty tuples (x_n) and (y_m) in $\mathcal{F}(A)$, and define their concatenation tensor product as

$$(x_n) \otimes (y_m) = (x_1, \dots, x_N, y_1, \dots, y_M).$$

We also define $(x_n) \otimes \mathfrak{o} = (x_n) = \mathfrak{o} \otimes (x_n)$ and $\mathfrak{o} \otimes \mathfrak{o} = \mathfrak{o}$. (In other words, the tensor product here is the binary operation on the free monoid on A .) Note that this makes the length of the tensor product of two tuples equal to the sum of the lengths of the factors. Note further that the tuples $(x_n) \otimes (y_m)$ and $(y_m) \otimes (x_n)$ are isomorphic in $\mathcal{F}(A)$, by permuting the entries, and similarly for tensor products involving \mathfrak{o} .

We now make this into a bifunctor $\mathcal{F}(A) \times \mathcal{F}(A) \rightarrow \mathcal{F}(A)$. Let $\hat{f}: (x_n) \rightarrow (w_q)$ and $\hat{g}: (y_m) \rightarrow (z_p)$ be morphisms in $\mathcal{F}(A)$ between non-empty tuples. Intuitively, the induced morphism

$$(x_n) \otimes (y_m) = (x_1, \dots, x_N, y_1, \dots, y_M) \rightarrow (w_1, \dots, w_Q, z_1, \dots, z_P) = (w_q) \otimes (z_p)$$

is given by the conditions $x_n \leq \prod_{q \in f^{-1}(n)} w_q$ and $y_m \leq \prod_{p \in g^{-1}(m)} z_p$ coming from the fact that \hat{f} and \hat{g} are morphisms in $\mathcal{F}(A)$. Formally, the induced morphism $\hat{f} \otimes \hat{g}: (x_n) \otimes (y_m) \rightarrow (w_q) \otimes (z_p)$ is defined to be $\widehat{f * g}$ where $f * g: [Q + P] \rightarrow [N + M]$ is defined as $(f * g)(q) = f(q)$ for $q \in [Q]$, and $(f * g)(Q + p) = N + g(p)$ for $p \in [P]$. Following the intuitive description of $\hat{f} \otimes \hat{g}$ above, one checks readily that the fact that \hat{f} and \hat{g} are morphisms in $\mathcal{F}(A)$ implies that $\widehat{f * g}$ is also a morphism. Note that this formal definition also applies to morphisms with domain or codomain \mathfrak{o} , using $M = 0, P = 0, N = 0, \text{ or } Q = 0$, as necessary. It is straightforward to show that this gives $\mathfrak{o} \otimes \hat{f} = \hat{f} = \hat{f} \otimes \mathfrak{o}$.

Proposition 4.12 *The above tensor product is a bifunctor $\mathcal{F}(A) \times \mathcal{F}(A) \rightarrow \mathcal{F}(A)$ such that $\mathfrak{o} \otimes - = - \otimes \mathfrak{o}$ is the identity functor.*

Proof The operation $\mathfrak{o} \otimes - = - \otimes \mathfrak{o}$ is the identity functor, by definition. To see that \otimes respects identity morphisms $\widehat{\text{id}}_{[N]}: (x_n) \rightarrow (x_n)$ and $\widehat{\text{id}}_{[M]}: (y_m) \rightarrow (y_m)$, notice that $\text{id}_{[N]} * \text{id}_{[M]} = \text{id}_{[N+M]}: [N + M] \rightarrow [N + M]$, and so

$$\text{id}_{(x_n)} \otimes \text{id}_{(y_m)} = \widehat{\text{id}}_{[N]} \otimes \widehat{\text{id}}_{[M]} = \widehat{\text{id}}_{[N+M]} = \text{id}_{(x_n) \otimes (y_m)}.$$

To check that this respects compositions, consider morphisms $(x_n) \xrightarrow{\hat{f}} (w_q) \xrightarrow{\hat{h}} (u_j)$ and $(y_m) \xrightarrow{\hat{g}} (z_p) \xrightarrow{\hat{k}} (v_i)$. Then $\widehat{f \circ h} = \widehat{f} \otimes \widehat{h}$ and $\widehat{g \circ k} = \widehat{g} \otimes \widehat{k}$. We show that $(f * g) \circ (h * k) = (f \circ h) * (g \circ k)$. To this end, for $j \in [J]$ the first equality in the next display is by definition, as is the second one using the condition $h(j) \in [Q]$, and similarly for the other equalities.

$$(f * g)((h * k)(j)) = (f * g)(h(j)) = f(h(j))$$

$$\begin{aligned}
 &= (f \circ h)(j) = ((f \circ h) * (g \circ k))(j) \\
 (f * g)((h * k)(J + i)) &= (f * g)(Q + k(i)) = N + g(k(i)) \\
 &= N + (g \circ k)(i) = ((f \circ h) * (g \circ k))(J + i)
 \end{aligned}$$

This explains the third equality in the next display

$$\begin{aligned}
 (\widehat{h} \otimes \widehat{k}) \circ (\widehat{f} \otimes \widehat{g}) &= \widehat{h * k} \circ \widehat{f * g} = [(f * g) \circ (h * k)]^\wedge = [(f \circ h) * (g \circ k)]^\wedge \\
 &= \widehat{f \circ h} \otimes \widehat{g \circ k} = (\widehat{h} \circ \widehat{f}) \otimes (\widehat{k} \circ \widehat{g})
 \end{aligned}$$

and the other equalities are by definition. □

Theorem 4.13 *With the above tensor product, $\mathcal{F}(A)$ is a symmetric and strict monoidal category.*

Proof We have already seen that $\circ \otimes - = - \otimes \circ$ is the identity functor. Also, it follows by definition that $(x_n) \otimes [(y_m) \otimes (z_p)] = [(x_n) \otimes (y_m)] \otimes (z_p)$. From this, it follows readily that the following diagrams commute

$$\begin{array}{ccc}
 [(x_n) \otimes (y_m)] \otimes (z_p) \otimes (w_q) & \xrightarrow[\text{id}]{\text{id} \otimes \text{id}} & (x_n) \otimes [(y_m) \otimes (z_p)] \otimes (w_q) \\
 \downarrow \text{id} & & \downarrow \text{id} \\
 [(x_n) \otimes (y_m)] \otimes [(z_p) \otimes (w_q)] & \xrightarrow{\text{id}} & (x_n) \otimes [(y_m) \otimes [(z_p) \otimes (w_q)]] \\
 & & \downarrow \text{id} \otimes \text{id} = \text{id} \\
 & & (x_n) \otimes [(y_m) \otimes (z_p)] \otimes (w_q) \\
 & & \downarrow \text{id} \\
 & & (x_n) \otimes [(y_m) \otimes (z_p)] \otimes (w_q)
 \end{array}$$

$$\begin{array}{ccc}
 [(x_n) \otimes \circ] \otimes (z_p) & \xrightarrow{\text{id}} & (x_n) \otimes [\circ \otimes (z_p)] \\
 \searrow \text{id} \otimes \text{id} = \text{id} & & \downarrow \text{id} \otimes \text{id} = \text{id} \\
 & & (x_n) \otimes (z_p)
 \end{array}$$

so $\mathcal{F}(A)$ is a strict monoidal category.

For the symmetric structure, we need to define commutativity isomorphisms $B_{(x_n), (y_m)} : (x_n) \otimes (y_m) \rightarrow (y_m) \otimes (x_n)$ as in the next display

$$(x_n) \otimes (y_m) = (x_1, \dots, x_n, y_1, \dots, y_m) \rightarrow (y_1, \dots, y_m, x_1, \dots, x_n) = (y_m) \otimes (x_n).$$

Formally, this is the morphism $B_{(x_n), (y_m)} = \widehat{\zeta_{M,N}} : (x_n) \otimes (y_m) \rightarrow (y_m) \otimes (x_n)$ where $\zeta_{M,N} : [M + N] \rightarrow [N + M]$ is defined to be $\zeta_{M,N}(m) = N + m$ and $\zeta_{M,N}(M + n) = n$ for $m \in [M]$ and $n \in [N]$. Note that this makes sense even when $N = 0$ or $M = 0$. It is straightforward to show that $\widehat{\zeta_{M,N}}$ is a morphism in $\mathcal{F}(A)$, and furthermore that $\zeta_{N,M} \circ \zeta_{M,N} = \text{id}_{[M+N]}$ so

$$\widehat{\zeta_{M,N}} \circ \widehat{\zeta_{N,M}} = [\zeta_{N,M} \circ \zeta_{M,N}]^\wedge = \widehat{\text{id}_{[M+N]}} = \text{id}_{(y_m) \otimes (x_n)}.$$

As in the proof of Proposition 4.12, the next diagram commutes

$$\begin{array}{ccccc}
 [(N + M) + P] & \xleftarrow{\text{id}} & [N + (M + P)] & \xleftarrow{\zeta_{M+P,N}} & [(M + P) + N] \\
 \zeta_{M,N} * \text{id} \uparrow & & & & \uparrow \text{id} \\
 [(M + N) + P] & \xleftarrow{\text{id}} & [M + (N + P)] & \xleftarrow{\text{id} * \zeta_{P,N}} & [M + (P + N)]
 \end{array}$$

which yields the following commutative hexagonal diagram

$$\begin{array}{ccc}
 [(x_n) \otimes (y_m)] \otimes (z_p) & \xrightarrow{\text{id}} & (x_n) \otimes [(y_m) \otimes (z_p)] \xrightarrow[\widehat{=B}]{\zeta_{M+P,N}} [(y_m) \otimes (z_p)] \otimes (x_n) \\
 \widehat{\zeta_{M,N}} \otimes \text{id} \downarrow \widehat{=B} \otimes \text{id} & & \downarrow \text{id} \\
 [(y_m) \otimes (x_n)] \otimes (z_p) & \xrightarrow{\text{id}} & (y_m) \otimes [(x_n) \otimes (z_p)] \xrightarrow[\widehat{=id} \otimes B]{\text{id} \otimes \zeta_{P,N}} (y_m) \otimes [(z_p) \otimes (x_n)]
 \end{array}$$

thus completing the proof. □

5 Weak Equivalences

Assumption In this section, A is a divisibility monoid.

Here we identify an important set of morphisms in $\mathcal{F}(A)$, denoted $\mathcal{W}(A)$. We then prove in Theorem 5.6 that the associated category of fractions $\mathcal{F}(A)[\mathcal{W}(A)^{-1}]$ is (naturally isomorphic to) A . See [2,6] for background on this construction. Then we exhibit some general properties of the set $\mathcal{W}(A)$, e.g., Propositions 5.8 and 5.10 show that $\mathcal{W}(A)$ is a collection of weak equivalences and admits a calculus of fractions.

Our definition for the set $\mathcal{W}(A)$ is motivated by the fact that, for elements $a, b \in A$, the natural morphisms $(1, ab) \rightarrow (ab) \rightarrow (a, b)$ are not isomorphisms, even though they do not carry any new factorization information about A .

Definition 5.1 Let $\mathcal{W}(A)$ be the set of morphisms $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(A)$ such that either (1) we have $(y_m) = \mathfrak{o}$, or (2) we have $(y_m) \neq \mathfrak{o}$, hence $(x_n) \neq \mathfrak{o}$, and for each $n \in [N]$, the element $r_n \in A$ from Remark 3.9 satisfying $r_n x_n = \prod_{m \in f^{-1}(n)} y_m$ is invertible in A .

Remark 5.2 If $\hat{\epsilon}: (x_n) \rightarrow (z_p)$ corresponds to dropping invertible elements from (x_n) and permuting the non-dropped elements, then $\hat{\epsilon} \in \mathcal{W}(A)$ because $r_n = 1$ for each $n \in \text{Im}(\hat{\epsilon})$ and $r_n = x_n^{-1}$ for each $n \in [N] \setminus \text{Im}(\hat{\epsilon})$. Similarly, every factorization morphism is in $\mathcal{W}(A)$. Each isomorphism is in $\mathcal{W}(A)$, and a divisibility morphism is in $\mathcal{W}(A)$ if and only if it is an isomorphism, by Theorem 3.8.

Next we show some of how the morphisms in $\mathcal{W}(A)$ can see factorization properties in A . We revisit this theme in subsequent sections.

Lemma 5.3 For an element $r \in A$, the following conditions are equivalent.

- (i) r is invertible in A ,
- (ii) each morphism $\hat{f}: (x_n) \rightarrow (y_m)$ such that $r \cdot \prod_n x_n = \prod_m y_m$ is in $\mathcal{W}(A)$, and
- (iii) some morphism $\hat{f}: (x_n) \rightarrow (y_m)$ such that $r \cdot \prod_n x_n = \prod_m y_m$ is in $\mathcal{W}(A)$.

Proof The implication (ii) \implies (iii) is straightforward.

(i) \implies (ii) Assume that r is invertible in A , and consider a morphism $\hat{f}: (x_n) \rightarrow (y_m)$ such that $r \cdot \prod_n x_n = \prod_m y_m$. If $(y_m) = \mathfrak{o}$, then $\hat{f} \in \mathcal{W}(A)$ by definition, so assume that $(y_m) \neq \mathfrak{o}$, and therefore $(x_n) \neq \mathfrak{o}$. For all $n \in [N]$, let $r_n \in A$ be such that $r_n x_n = \prod_{m \in f^{-1}(n)} y_m$. Then $r = \prod_n r_n$ by Remark 3.9. Since r is invertible, each r_n must be invertible as well, so $\hat{f} \in \mathcal{W}(A)$ by definition.

(iii) \implies (i) Assume that some morphism $\hat{f}: (x_n) \rightarrow (y_m)$ such that $r \cdot \prod_n x_n = \prod_m y_m$ is in $\mathcal{W}(A)$. If $(y_m) = \mathfrak{o}$, then r is invertible by Remark 3.9. So, assume that $(y_m) \neq \mathfrak{o}$, hence $(x_n) \neq \mathfrak{o}$. The elements r_n from Definition 5.1 therefore satisfy $r = \prod_n r_n$. The condition $\widehat{\text{id}}_{[M]} \in \mathcal{W}(A)$ implies that each r_n is invertible in A , therefore so is r , as desired. \square

The set $\mathcal{W}(A)$ has a convenient description in terms of the factorization result in Proposition 3.11, as we show next.

Proposition 5.4 *Let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\mathcal{F}(A)$ between non-empty tuples. Consider the decomposition $\hat{f} = \hat{\phi} \circ \widehat{\text{id}}_{[P]} \circ \hat{\epsilon}$ from Proposition 3.11*

$$(x_n) \xrightarrow{\hat{\epsilon}} (z_p) \xrightarrow{\widehat{\text{id}}_{[P]}} (a_p z_p) \xrightarrow{\hat{\phi}} (y_m).$$

Then the morphism \hat{f} is in $\mathcal{W}(A)$, if and only if the divisibility morphism $\widehat{\text{id}}_{[P]}$ is an isomorphism (see Theorem 3.8).

Proof By Remark 3.9, let $r \in A$ be the unique element of A such that $\prod_m y_m = r \prod_n x_n$. Proposition 3.11 implies that $r = u \cdot \prod_p a_p$ where u is an invertible element of A . It follows that a_p is invertible if and only if r is invertible. Thus $\widehat{\text{id}}_{[P]}$ is an isomorphism if and only if $\hat{f} \in \mathcal{W}(A)$ by Theorem 3.8 and Lemma 5.3. \square

For the next result of this section, we recall the following notion from [6].

Definition 5.5 Let \mathcal{C} be a category with a collection of morphisms \mathcal{W} . The *category of fractions* of $(\mathcal{C}, \mathcal{W})$, if it exists, is a category $\mathcal{C}[\mathcal{W}^{-1}]$ with a universal functor $\mathfrak{R}: \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$ such that

- (1) for each $f \in \mathcal{W}$ the morphism $\mathfrak{R}(f)$ is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$, and
- (2) for each category \mathcal{D} and each functor $\mathfrak{L}: \mathcal{C} \rightarrow \mathcal{D}$ such that for each $f \in \mathcal{W}$ the morphism $\mathfrak{L}(f)$ is an isomorphism in \mathcal{D} , there is a unique functor $\mathfrak{L}': \mathcal{C}[\mathcal{W}^{-1}] \rightarrow \mathcal{D}$ making the following diagram commute

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathfrak{R}} & \mathcal{C}[\mathcal{W}^{-1}] \\ & \searrow \mathfrak{L} & \downarrow \exists! \mathfrak{L}' \\ & & \mathcal{D}. \end{array}$$

Theorem 5.6 *The category A is (naturally isomorphic to) the category of fractions $\mathcal{F}(A)[\mathcal{W}(A)^{-1}]$, where the functor $\mathfrak{A}_A: \mathcal{F}(A) \rightarrow A$ from Definition 4.4 plays the role of the universal functor \mathfrak{R} from Definition 5.5.*

Proof First, let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\mathcal{W}(A)$; we show that $\mathfrak{A}_A(\hat{f})$ is an isomorphism in A . Lemma 5.3 says that there is an invertible $u \in A$ such that $\prod_m y_m = u \prod_n x_n$.

The fact that u is invertible implies that $\prod_m y_m \mid \prod_n x_n$ and $\prod_n x_n \mid \prod_m y_m$ so the morphism $\prod_n x_n \mid \prod_m y_m$ in A is an isomorphism. That is, the morphism $\mathfrak{A}_A(\hat{f})$ is invertible, as desired.

Next, let \mathcal{D} be a category, and let $\mathfrak{L}: \mathcal{F}(A) \rightarrow \mathcal{D}$ be a functor such that for each $\hat{f} \in \mathcal{W}(A)$ the morphism $\mathfrak{L}(\hat{f})$ is an isomorphism in \mathcal{D} . For each monoid element $x \in A$, define $\mathfrak{L}'(x) = \mathfrak{L}((x))$ where (x) is the associated 1-tuple in $\mathcal{F}(A)$. In other words, we define $\mathfrak{L}' = \mathfrak{L}\mathfrak{B}_A$ where $\mathfrak{B}_A: A \rightarrow \mathcal{F}(A)$ is the functor from Definition 4.6. This yields the next commutative diagram

$$\begin{array}{ccc}
 \mathcal{F}(A) & \xrightarrow{\mathfrak{A}_A} & A \\
 & \searrow \mathfrak{L} & \downarrow \mathfrak{L}\mathfrak{B}_A = \mathfrak{L}' \\
 & & \mathcal{D}
 \end{array} \tag{5.1}$$

as follows. First, we need to show that for each object (x_n) in $\mathcal{F}(A)$, we have $\mathfrak{L}(\mathfrak{B}_A(\mathfrak{A}_A((x_n)))) \cong \mathfrak{L}((x_n))$. By the definitions of \mathfrak{B}_A and \mathfrak{A}_A , this is tantamount to showing that $\mathfrak{L}((\prod_n x_n)) \cong \mathfrak{L}((x_n))$. Note that by Remark 5.2, the factorization morphism $\hat{f}: (\prod_n x_n) \rightarrow (x_n)$ in $\mathcal{F}(A)$ is in $\mathcal{W}(A)$. Thus, by assumption the induced morphism $\mathfrak{L}(\hat{f}): \mathfrak{L}((\prod_n x_n)) \rightarrow \mathfrak{L}((x_n))$ is an isomorphism in \mathcal{D} , as desired.

Next, we need to show that for each morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(A)$, we have $\mathfrak{L}(\mathfrak{B}_A(\mathfrak{A}_A(\hat{f}))) \cong \mathfrak{L}(\hat{f})$. As in the previous paragraph, the natural morphisms $(\prod_n x_n) \rightarrow (x_n)$ and $(\prod_m y_m) \rightarrow (y_m)$ in $\mathcal{F}(A)$ are in $\mathcal{W}(A)$. Also, the divisibility relation $x_n \mid \prod_{m \in f^{-1}(n)} y_m$ implies that $\prod_n x_n \mid \prod_m y_m$. By Corollary 3.4, this yields a commutative diagram in $\mathcal{F}(A)$ with 1-tuples on the left

$$\begin{array}{ccc}
 (\prod_n x_n) & \longrightarrow & (x_n) \\
 \mathfrak{B}_A(\mathfrak{A}_A(\hat{f})) \downarrow & & \downarrow \hat{f} \\
 (\prod_m y_m) & \longrightarrow & (y_m)
 \end{array}$$

where the horizontal morphisms are in $\mathcal{W}(A)$. An application of \mathfrak{L} turns the horizontal morphisms into isomorphisms in \mathcal{D} , by assumption, so we have the next commutative diagram in \mathcal{D} .

$$\begin{array}{ccc}
 \mathfrak{L}((\prod_n x_n)) & \xrightarrow{\cong} & \mathfrak{L}((x_n)) \\
 \mathfrak{L}(\mathfrak{B}_A(\mathfrak{A}_A(\hat{f}))) = \mathfrak{L}'(\mathfrak{A}_A(\hat{f})) \downarrow & & \downarrow \mathfrak{L}(\hat{f}) \\
 \mathfrak{L}((\prod_m y_m)) & \xrightarrow{\cong} & \mathfrak{L}((y_m))
 \end{array}$$

This is exactly what it means to have $\mathfrak{L}(\mathfrak{B}_A(\mathfrak{A}_A(\hat{f}))) \cong \mathfrak{L}(\hat{f})$. This shows that \mathfrak{L}' is a functor making the diagram in (5.1) commute.

Lastly, we need to show that \mathfrak{L}' is the unique functor (up to natural isomorphism, of course) making diagram (5.1) commute. Suppose that \mathfrak{L}'' is another functor making the next diagram commute.

$$\begin{array}{ccc}
 \mathcal{F}(A) & \xrightarrow{\mathfrak{A}_A} & A \\
 & \searrow \mathfrak{L} & \downarrow \mathfrak{L}'' \\
 & & \mathcal{D}
 \end{array} \tag{5.2}$$

In the next display, the first equality is from Proposition 4.9.

$$\mathcal{L}'' = \mathcal{L}'' \mathfrak{A}_A \mathfrak{B}_A = \mathcal{L} \mathfrak{B}_A = \mathcal{L}'$$

The second equality is from diagram (5.2), and the third equality is by the definition of \mathcal{L}' . This establishes the desired uniqueness. \square

Next, we exhibit further properties of the set $\mathcal{W}(A)$, beginning with some terminology from [5, Chap. 2].

Definition 5.7 Let \mathbf{C} be a category. A collection of morphisms \mathcal{W} from \mathbf{C} is a collection of *weak equivalences* if (1) every isomorphism is in \mathcal{W} , and (2) \mathcal{W} satisfies the following 2-of-3 property: if any two of the morphisms $f, g, g \circ f$ are in \mathcal{W} , then the third is also.

Proposition 5.8 *The set $\mathcal{W}(A)$ is a collection of weak equivalences.*

Proof The isomorphisms of $\mathcal{F}(A)$ are all in $\mathcal{W}(A)$ by Remark 5.2. For the 2-of-3 condition, let $\hat{f}: (x_n) \rightarrow (y_m)$ and $\hat{g}: (y_m) \rightarrow (z_\ell)$ be morphisms in $\mathcal{F}(A)$. Let $r, s \in A$ be such that $\prod_m y_m = r \prod_n x_n$ and $\prod_\ell z_\ell = s \prod_m y_m = rs \prod_n x_n$. The product rs is invertible in A if and only if r and s are both invertible in A . So, using the cancellativity of A , one applies Lemma 5.3 to conclude that $\hat{g} \circ \hat{f} \in \mathcal{W}(A)$ if and only if both \hat{f} and \hat{g} are in $\mathcal{W}(A)$. This implies the 2-of-3 property \square

Next, we turn our attention to the calculus of fractions of [6].

Definition 5.9 Let \mathbf{C} be a category. A collection of morphisms \mathcal{W} from \mathbf{C} admits a *calculus of right fractions* if it satisfies the following conditions:

- (1) \mathcal{W} contains all identities and is closed under composition;
- (2) (right Ore condition) for each morphism $f: x \rightarrow y$ in \mathcal{W} and every morphism $g: z \rightarrow y$ in \mathbf{C} , there are morphisms $f': w \rightarrow z$ in \mathcal{W} and $g': w \rightarrow x$ in \mathbf{C}

$$\begin{array}{ccc}
 w & \xrightarrow{g'} & x \\
 f' \downarrow & & \downarrow f \\
 z & \xrightarrow{g} & y
 \end{array}$$

making the above diagram commute; and

- (3) (right cancellability) for every pair of morphisms $f, f': x \rightarrow y$ in \mathbf{C} , if there is a morphism $g: y \rightarrow z$ in \mathcal{W} such that $g \circ f = g \circ f'$, then there is a morphism $h: w \rightarrow x$ in \mathcal{W} such that $f \circ h = f' \circ h$.

Proposition 5.10 *The set $\mathcal{W}(A)$ admits a calculus of right fractions.*

Proof The set $\mathcal{W}(A)$ contains all identities and is closed under composition by Proposition 5.8. For the right Ore condition, let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\mathcal{W}(A)$, and let $\hat{g}: (z_p) \rightarrow (y_m)$ be a morphism in $\mathcal{F}(A)$. The morphism condition for \hat{g} implies that $\prod_p z_p \mid \prod_m y_m$. And according to Lemma 5.3 the fact that \hat{f} is in $\mathcal{W}(A)$ implies that $\prod_m y_m = r \prod_n x_n$ for an invertible element $r \in A$. It follows that $\prod_p z_p \mid \prod_n x_n$, so the unique map $g': [N] \rightarrow [1]$ defines a morphism $\hat{g}': (\prod_p z_p) \rightarrow (x_n)$. Also, the unique map $f': [P] \rightarrow [1]$ defines a factorization morphism $\hat{f}': (\prod_p z_p) \rightarrow (z_p)$ in $\mathcal{W}(A)$ by

Remark 5.2. Corollary 3.4 shows that these morphisms make the following diagram commute

$$\begin{array}{ccc}
 (\prod_p z_p) \xrightarrow{\hat{g}'} & (x_n) \\
 \hat{f}' \downarrow & \downarrow \hat{f} \\
 (z_p) \xrightarrow{\hat{g}} & (y_m)
 \end{array}$$

so the right Ore condition is satisfied.

For right cancellability, let $\hat{f}, \hat{f}': (x_n) \rightarrow (y_m)$ be morphisms in $\mathcal{F}(A)$, and let $\hat{g}: (y_m) \rightarrow (z_p)$ be a morphism in $\mathcal{W}(A)$ such that $\hat{g} \circ \hat{f} = \hat{g} \circ \hat{f}'$. As in the preceding paragraph, one shows that the unique morphism $\hat{h}: (\prod_n x_n) \rightarrow (x_n)$, which is in $\mathcal{W}(A)$, satisfies $\hat{f} \circ \hat{h} = \hat{f}' \circ \hat{h}$, as desired. \square

Remark 5.11 Because of [6, Sections 1.2–3] one of the benefits of $\mathcal{W}(A)$ admitting a calculus of right fractions is that it allows for a simple representation of morphisms in $\mathcal{F}(A)[\mathcal{W}(A)^{-1}]$ as simple fractions instead of more complicated zig-zag diagrams. Furthermore, it provides a straightforward characterization of equality of parallel morphisms; and the natural functor $\mathcal{F}(A) \rightarrow \mathcal{F}(A)[\mathcal{W}(A)^{-1}]$ is flat, moreover, it is uniformly flat [2, Section 6]. In particular, the properties described in [2, Section 1] are satisfied, including that $\mathcal{W}(A)$ is focal. Since we already have a simple description of $\mathcal{F}(A)[\mathcal{W}(A)^{-1}]$ in Theorem 5.6, we resist the urge to say more about these conditions here.

6 Weak Divisibility

Assumptions In this section, again A is a divisibility monoid. In addition, we say that a morphism \hat{f} is a *weak equivalence* provided that $\hat{f} \in \mathcal{W}(A)$.

Here we introduce a notion of weak divisibility of morphisms for use with our treatment of weak primeness in Sect. 8; see Definition 6.2. It may not be clear initially why this definition is useful; see however the application of weak primeness in Theorem 9.2. For some motivation/perspective on this definition, we begin with a motivating example.

Example 6.1 Consider the divisibility monoid $A = \mathbb{Z} \setminus \{0\}$ and the morphisms $\widehat{\text{id}}_{[1](6)}^{(2)}: (2) \rightarrow (6)$ and $\widehat{\text{id}}_{[1](105)}^{(5)}: (5) \rightarrow (105)$. Intuitively, we think of the morphism \hat{f} as multiplication by 3, while \hat{g} is multiplication by 21. With this guiding thought, it seems reasonable that \hat{f} should divide \hat{g} and that \hat{g} should not divide \hat{f} .

A general idea says that \hat{f} should divide \hat{g} when \hat{g} is in a nice subset $\langle \hat{f} \rangle \subseteq \text{Mor}(\mathcal{F}(A))$ that behaves like an ideal generated by \hat{f} ; the difficulty is to decide what $\langle \hat{f} \rangle$ should be. In our opinion, it should contain \hat{f} , and it should be closed under meaningful compositions. So, for instance, one could define $\langle \hat{f} \rangle$ to be the set of all meaningful compositions $\hat{\alpha} \circ \hat{f} \circ \hat{\beta}$. With this first approximation, in order for \hat{f} to divide \hat{g} in our example, we would require the existence of morphisms in $\mathcal{F}(A)$ making the following diagram commute.

$$\begin{array}{ccc}
 (5) & \dashrightarrow & (2) \\
 \hat{g} \downarrow & & \downarrow \hat{f} \\
 (5 \cdot 7 \cdot 3) & \ll - - & (2 \cdot 3)
 \end{array} \tag{6.1}$$

It is straightforward to show that no such morphisms exist, e.g., because $5 \nmid 2$. In other words, the failure of existence of such a diagram lies in the entries of the domains and codomains of the morphisms \hat{f} and \hat{g} , not necessarily in the factorization properties encoded in the morphisms.

One can come close to remedying this by tensoring (i.e., concatenating, as in Definition 4.11) appropriately to make the domains line up as in the following diagram wherein the top horizontal arrow is the natural isomorphism.

$$\begin{array}{ccc}
 (2, 5) \cong (2) \otimes (5) & \xrightarrow{\quad} & (5) \otimes (2) \cong (5, 2) \\
 \downarrow (2) \otimes \hat{g} & & \downarrow (5) \otimes \hat{f} \\
 (2, 105) = (2) \otimes (5 \cdot 7 \cdot 3) & \dashleftarrow \quad \dashrightarrow & (5) \otimes (2 \cdot 3) = (5, 6).
 \end{array}$$

However, one again checks readily that there is no morphism for the dashed arrow, let alone one making the diagrams commute.

Taking a cue from the category of fractions, one could return to diagram (6.1) and replace the horizontal arrows with zig-zags as in the following

$$\begin{array}{ccccccc}
 (5) & \xleftarrow[\in \mathcal{W}(A)]{\hat{f}_1} & (c_{n_1}) & \xrightarrow{\hat{f}_2} & (c_{n_2}) & \xleftarrow[\in \mathcal{W}(A)]{\hat{f}_3} \cdots & \xrightarrow{\hat{f}_{2r}} & (2) \\
 \downarrow \hat{g} & & & & & & & \downarrow \hat{f} \\
 (5 \cdot 7 \cdot 3) & \xleftarrow[\in \mathcal{W}(A)]{\hat{g}_{2s}} \cdots & \xrightarrow{\hat{g}_3} & (d_{m_2}) & \xleftarrow[\in \mathcal{W}(A)]{\hat{g}_2} & (d_{m_1}) & \xrightarrow[\in \mathcal{W}(A)]{\hat{g}_1} & (2 \cdot 3)
 \end{array}$$

where each row consists of a finite number of morphisms, and the odd-indexed morphisms $\widehat{f_{2k+1}}$ and $\widehat{g_{2k+1}}$ are weak equivalences. (Note that we make no assumption about commutativity in this diagram, because the shape makes commutativity meaningless.) However, there is no such diagram. Indeed, were such a diagram to exist, Remark 6.4 would provide the following commutative diagram in A

$$\begin{array}{ccc}
 5 & \dashrightarrow & 2 \\
 \downarrow & & \downarrow \\
 5 \cdot 7 \cdot 3 & \dashleftarrow \quad \dashrightarrow & 2 \cdot 3
 \end{array}$$

but there is no such diagram (let alone a commutative one), e.g., because $5 \nmid 2$.

In light of Example 6.1, we introduce the following which is a fusion of the two inadequate ways we suggested that one might address the deficiencies of diagram (6.1).

Definition 6.2 Recall the concatenation tensor product from Definition 4.11. Let $\hat{f}: (v_i) \rightarrow (w_j)$ and $\hat{g}: (x_n) \rightarrow (y_m)$ be morphisms in $\mathcal{F}(A)$. We say that \hat{f} weakly divides \hat{g} , denoted $\hat{f} \mid_w \hat{g}$, provided that there are tuples (a_p) and (b_q) in $\mathcal{F}(A)$ and a diagram in $\mathcal{F}(A)$

$$\begin{array}{ccccccc}
 (a_p) \otimes (x_n) & \xleftarrow[\in \mathcal{W}(A)]{\hat{f}_1} & (c_{n_1}) & \xrightarrow{\hat{f}_2} & (c_{n_2}) & \xleftarrow[\in \mathcal{W}(A)]{\hat{f}_3} \cdots & \xrightarrow{\hat{f}_{2r}} & (b_q) \otimes (v_i) \\
 \downarrow (a_p) \otimes \hat{g} & & & & & & & \downarrow (b_q) \otimes \hat{f} \\
 (a_p) \otimes (y_m) & \xleftarrow[\in \mathcal{W}(A)]{\hat{g}_{2s}} \cdots & \xrightarrow{\hat{g}_3} & (d_{m_2}) & \xleftarrow[\in \mathcal{W}(A)]{\hat{g}_2} & (d_{m_1}) & \xrightarrow[\in \mathcal{W}(A)]{\hat{g}_1} & (b_q) \otimes (w_j)
 \end{array} \tag{6.2}$$

such that each row consists of a finite number of morphisms, and the odd-indexed morphisms $\widehat{f_{2k+1}}$ and $\widehat{g_{2k+1}}$ are weak equivalences.

As in Example 6.1, note that we do not say that the diagram (6.2) is commutative in $\mathcal{F}(A)$, since the shape of the diagram makes commutativity meaningless. Some readers will notice that Proposition 5.10 can be used to simplify the diagram (6.2). We do not make this explicit here because of the simplification in Theorem 6.5 below.

Example 6.3 Consider the divisibility monoid $A = \mathbb{Z} \setminus \{0\}$ and the morphisms $\widehat{\text{id}}_{[1](6)}^{(2)} : (2) \rightarrow (6)$ and $\widehat{\text{id}}_{[1](105)}^{(5)} : (5) \rightarrow (105)$ from Example 6.1. We claim that the following diagram shows that $\widehat{\text{id}}_{[1](6)}^{(2)} \mid_w \widehat{\text{id}}_{[1](105)}^{(5)}$.

$$\begin{array}{ccccc}
 (2) \otimes (5) & \xleftarrow[\in \mathcal{W}(A)]{\widehat{f}_1} & (10) & \xrightarrow{\widehat{f}_2} & (5) \otimes (2) \\
 \downarrow \widehat{\text{id}}_{[1](105)}^{(5)} & & & & \downarrow \widehat{\text{id}}_{[1](6)}^{(2)} \\
 (2) \otimes (105) & \xleftarrow[\in \mathcal{W}(A)]{\widehat{g}_2} & (30) & \xrightarrow[\in \mathcal{W}(A)]{\widehat{g}_1} & (5) \otimes (6)
 \end{array}$$

Here the horizontal morphisms are all factorization morphisms; see Example 2.2 and Definition 3.10. Indeed, by definition, this diagram has the following form

$$\begin{array}{ccccc}
 (2, 5) & \xleftarrow[\in \mathcal{W}(A)]{\widehat{f}_1} & (10) & \xrightarrow{\widehat{f}_2} & (5, 2) \\
 \widehat{g}' \downarrow & & & & \downarrow \widehat{f}' \\
 (2, 105) & \xleftarrow[\in \mathcal{W}(A)]{\widehat{g}_2} & (30) & \xrightarrow[\in \mathcal{W}(A)]{\widehat{g}_1} & (5, 6)
 \end{array} \tag{6.3}$$

where the factorization morphisms \widehat{f}_1 and \widehat{g}_1 are weak equivalences by Remark 5.2.

Remark 6.4 Let $\widehat{f} : (v_i) \rightarrow (w_j)$ and $\widehat{g} : (x_n) \rightarrow (y_m)$ be morphisms in $\mathcal{F}(A)$ such that $\widehat{f} \mid_w \widehat{g}$. The diagram (6.2) provided by Definition 6.2 induces the following diagram in A

$$\begin{array}{ccccccc}
 \prod_p a_p \cdot \prod_n x_n & \xrightarrow{\mathfrak{A}_A(\widehat{f}_1)^{-1}} & \prod_{n_1} c_{n_1} & \xrightarrow{\mathfrak{A}_A(\widehat{f}_2)} & \prod_{n_2} c_{n_2} & \xrightarrow{\mathfrak{A}_A(\widehat{f}_3)^{-1}} & \dots & \xrightarrow{\mathfrak{A}_A(\widehat{f}_{2r})} & \prod_q b_q \cdot \prod_i v_i \\
 \downarrow \mathfrak{A}_A((a_p) \otimes \widehat{g}) & & & & & & & & \downarrow \mathfrak{A}_A((b_q) \otimes \widehat{f}) \\
 \prod_p a_p \cdot \prod_m y_m & \xleftarrow{\mathfrak{A}_A(\widehat{g}_2)} & \dots & \xleftarrow{\mathfrak{A}_A(\widehat{g}_3)^{-1}} & \prod_{m_2} d_{m_2} & \xleftarrow{\mathfrak{A}_A(\widehat{g}_2)} & \prod_{m_1} d_{m_1} & \xleftarrow{\mathfrak{A}_A(\widehat{g}_1)^{-1}} & \prod_q b_q \cdot \prod_j w_j
 \end{array}$$

where $\mathfrak{A}_A : \mathcal{F}(A) \rightarrow A$ is the the functor from Definition 4.4. Note that the morphisms $\mathfrak{A}_A(\widehat{f_{2k+1}})$ and $\mathfrak{A}_A(\widehat{g_{2k+1}})$ in this diagram are isomorphisms in $\mathcal{F}(A)$ since the morphisms $\widehat{f_{2k+1}}$ and $\widehat{g_{2k+1}}$ in the original diagram (6.2) are weak equivalences; see Theorem 5.6. In particular, in A we have $\prod_p a_p \cdot \prod_n x_n \mid \prod_q b_q \cdot \prod_i v_i$ and $\prod_q b_q \cdot \prod_j w_j \mid \prod_p a_p \cdot \prod_m y_m$.

Note furthermore that this new diagram in A commutes by Remark 4.3.

Lastly, we observe that one can detect weak divisibility using diagrams of shapes different from (6.2) by inserting identity morphisms in the horizontal sequences. For instance, if one has a diagram of the following shape in $\mathcal{F}(A)$

$$\begin{array}{ccccc}
 (a_p) \otimes (x_n) & \xleftarrow[\in \mathcal{W}(A)]{\hat{f}'} & (c_{n_1}) & \xleftarrow[\in \mathcal{W}(A)]{\hat{f}''} & (b_q) \otimes (v_i) \\
 (a_p) \otimes \hat{g} \downarrow & & & & \downarrow (b_q) \otimes \hat{f} \\
 (a_p) \otimes (y_m) & \xleftarrow[\in \mathcal{W}(A)]{\hat{g}_2} & (d_{m_1}) & \xrightarrow[\in \mathcal{W}(A)]{\hat{g}_1} & (b_q) \otimes (w_j)
 \end{array}$$

then one obtains the next diagrams to conclude that $\hat{f} \mid_w \hat{g}$.

$$\begin{array}{ccccccc}
 & & \hat{f}' \circ \hat{f}'' & & & & \\
 & & \in \mathcal{W}(A) & & & & \\
 & \swarrow & & \searrow & \xrightarrow{\text{id}_{(b_q) \otimes (v_i)}} & & \\
 (a_p) \otimes (x_n) & \xleftarrow[\in \mathcal{W}(A)]{\hat{f}'} & (c_{n_1}) & \xleftarrow[\in \mathcal{W}(A)]{\hat{f}''} & (b_q) \otimes (v_i) & \xrightarrow{\text{id}_{(b_q) \otimes (v_i)}} & (b_q) \otimes (v_i) \\
 (a_p) \otimes \hat{g} \downarrow & & & & & & \downarrow (b_q) \otimes \hat{f} \\
 (a_p) \otimes (y_m) & \xleftarrow[\in \mathcal{W}(A)]{\hat{g}_2} & (d_{m_1}) & \xrightarrow[\in \mathcal{W}(A)]{\hat{g}_1} & (b_q) \otimes (w_j) & & \\
 & & & & & & \\
 (a_p) \otimes (x_n) & \xleftarrow[\in \mathcal{W}(A)]{\hat{f}' \circ \hat{f}''} & (b_q) \otimes (v_i) & \xrightarrow{\text{id}_{(b_q) \otimes (v_i)}} & (b_q) \otimes (v_i) & & \\
 (a_p) \otimes \hat{g} \downarrow & & & & & & \downarrow (b_q) \otimes \hat{f} \\
 (a_p) \otimes (y_m) & \xleftarrow[\in \mathcal{W}(A)]{\hat{g}_2} & (d_{m_1}) & \xrightarrow[\in \mathcal{W}(A)]{\hat{g}_1} & (b_q) \otimes (w_j) & &
 \end{array}$$

In the next result, condition (v) makes detection of weak divisibility quite easy. And it gives another indication of how the morphisms in $\mathcal{W}(A)$ can see factorization properties in A . Compare condition (ii) to diagram (6.3) in Example 6.3.

Theorem 6.5 *Let $\hat{f}: (v_i) \rightarrow (w_j)$ and $\hat{g}: (x_n) \rightarrow (y_m)$ be morphisms in $\mathcal{F}(A)$. Then the following conditions are equivalent.*

- (i) We have $\hat{f} \mid_w \hat{g}$.
- (ii) There are elements $a, b \in A$ and a diagram in $\mathcal{F}(A)$

$$\begin{array}{ccc}
 (a) \otimes (x_n) & \xleftarrow[\in \mathcal{W}(A)]{\hat{\mu}} & (a \prod_n x_n) \xrightarrow{\hat{\alpha}} (b) \otimes (v_i) \\
 (a) \otimes \hat{g} \downarrow & & \downarrow (b) \otimes \hat{f} \\
 (a) \otimes (y_m) & \xleftarrow[\in \mathcal{W}(A)]{\hat{\beta}} & (b \prod_j w_j) \xrightarrow{\hat{\eta}} (b) \otimes (w_j)
 \end{array}$$

where the middle tuple in each row has length 1, and the morphisms $\hat{\mu}$ and $\hat{\eta}$ are factorization morphisms.

- (iii) We have $\prod_n x_n \cdot \prod_j w_j \mid \prod_i v_i \cdot \prod_m y_m$.
- (iv) There are elements $a, b \in A$ such that $a \prod_n x_n \mid b \prod_i v_i$ and $b \prod_j w_j \mid a \prod_m y_m$.
- (v) The monoid elements $r, s \in A$ such that $\prod_m y_m = r \prod_n x_n$ and $\prod_j w_j = s \prod_i v_i$ satisfy $s \mid r$.

Proof The implication (ii) \implies (i) follows by definition of $\hat{f} \mid_w \hat{g}$, and the implication (i) \implies (iv) is from the first paragraph of Remark 6.4 with $a = \prod_p a_p$ and $b = \prod_q b_q$.

(iv) \implies (iii) Assume that there are elements $a, b \in A$ such that $a \prod_n x_n \mid b \prod_i v_i$ and $b \prod_j w_j \mid a \prod_m y_m$. Multiply these divisibility relations by $\prod_j w_j$ and $\prod_i v_i$, respectively, to conclude that

$$a \prod_n x_n \cdot \prod_j w_j \mid b \prod_i v_i \cdot \prod_j w_j \mid a \prod_i v_i \cdot \prod_m y_m.$$

It follows that $a \prod_n x_n \cdot \prod_j w_j \mid a \prod_i v_i \cdot \prod_m y_m$, hence $\prod_n x_n \cdot \prod_j w_j \mid \prod_i v_i \cdot \prod_m y_m$ since A is cancellative.

(iii) \implies (ii) Assume that $\prod_n x_n \cdot \prod_j w_j \mid \prod_i v_i \cdot \prod_m y_m$. This yields the morphism $\widehat{\text{id}}_{[1]}$ in the following diagram in $\mathcal{F}(A)$.

$$\begin{array}{ccccc} (\prod_i v_i) \otimes (x_n) & \xleftarrow[\in \mathcal{W}(A)]{\hat{\mu}} & (\prod_i v_i \cdot \prod_n x_n) & \xrightarrow{\hat{\alpha}} & (\prod_n x_n) \otimes (v_i) \\ \downarrow (\prod_i v_i) \otimes \hat{g} & & & & \downarrow (b) \otimes \hat{f} \\ (\prod_i v_i) \otimes (y_m) & \xleftarrow{\hat{\gamma}} & (\prod_i v_i \cdot \prod_m y_m) & \xleftarrow{\widehat{\text{id}}_{[1]}} & (\prod_n x_n \cdot \prod_j w_j) \xrightarrow[\in \mathcal{W}(A)]{\hat{\eta}} (\prod_n x_n) \otimes (w_j) \end{array}$$

The morphisms $\hat{\mu}$, $\hat{\alpha}$, $\hat{\eta}$, and $\hat{\gamma}$ are factorization morphisms. Thus, condition (ii) is satisfied with $a = \prod_i v_i$ and $b = \prod_n x_n$.

(iii) \iff (v) From the conditions $\prod_m y_m = r \prod_n x_n$ and $\prod_j w_j = s \prod_i v_i$ it is straightforward to show that $\prod_n x_n \cdot \prod_j w_j \mid \prod_i v_i \cdot \prod_m y_m$ if and only if $s \mid r$. \square

Next, we present some applications of Theorem 6.5. In the first one, parts (a)–(b) say that weak divisibility is a pre-order on the set $\text{Mor}(\mathcal{F}(A))$. Part (c) says that weak equivalences are strongly minimal in this ordering; see Theorem 6.7 for more about this. It is not hard to prove Proposition 6.6 by definition, without invoking Theorem 6.5, but that proof is longer, and this one showcases Theorem 6.5 nicely.

Proposition 6.6 *Let $\hat{f}: (v_i) \rightarrow (w_j)$ and $\hat{g}: (x_n) \rightarrow (y_m)$ and $\hat{h}: (z_\ell) \rightarrow (t_k)$ be morphisms in $\mathcal{F}(A)$.*

- (a) *We have $\hat{f} \mid_w \hat{f}$.*
- (b) *If $\hat{f} \mid_w \hat{g}$ and $\hat{g} \mid_w \hat{h}$, then $\hat{f} \mid_w \hat{h}$.*
- (c) *If \hat{f} is a weak equivalence, e.g., if \hat{f} is an isomorphism, then $\hat{f} \mid_w \hat{g}$.*

Proof Let $r, s, q \in A$ be such that $\prod_m y_m = r \prod_n x_n$ and $\prod_j w_j = s \prod_i v_i$ and $\prod_k t_k = q \prod_\ell z_\ell$. Part (a) follows from Theorem 6.5 because $s \mid s$. Part (b) follows similarly: if $\hat{f} \mid_w \hat{g}$ and $\hat{g} \mid_w \hat{h}$, then $s \mid r$ and $r \mid q$, hence $s \mid q$, so $\hat{f} \mid_w \hat{h}$.

(c) Assume that \hat{f} is a weak equivalence. Lemma 5.3 implies that s is invertible, so $s \mid r$, hence $\hat{f} \mid_w \hat{g}$ by Theorem 6.5. \square

Our next application of Theorem 6.5 complements Proposition 6.6(c).

Theorem 6.7 *Let $\hat{f}: (v_i) \rightarrow (w_j)$ be a morphism in $\mathcal{F}(A)$. The following conditions are equivalent.*

- (i) *We have $\hat{f} \mid_w \hat{g}$ for all morphisms \hat{g} in $\mathcal{F}(A)$.*
- (ii) *We have $\hat{f} \mid_w \hat{g}$ for some weak equivalence \hat{g} in $\mathcal{F}(A)$.*
- (iii) *The morphism \hat{f} is a weak equivalence.*

Proof The implication (i) \implies (ii) is trivial, and (iii) \implies (i) is from Proposition 6.6(c).

(ii) \implies (iii) Assume that there is a weak equivalence $\hat{g}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(A)$ such that $\hat{f} \mid_w \hat{g}$. Let $r, s \in A$ be such that $\prod_m y_m = r \prod_n x_n$ and $\prod_j w_j = s \prod_i v_i$. Lemma 5.3 implies that r is invertible. Theorem 6.5 says that s divides the invertible element r , so s is invertible as well, so another application of Lemma 5.3 implies that \hat{f} is a weak equivalence. \square

Proposition 6.6 shows that the weak divisibility relation on the set of morphisms in $\mathcal{F}(A)$ is a pre-order. Our next definition is the induced equivalence relation.

Definition 6.8 Let $\hat{f}: (v_i) \rightarrow (w_j)$ and $\hat{g}: (x_n) \rightarrow (y_m)$ be morphisms in $\mathcal{F}(A)$. We say that \hat{f} and \hat{g} are *weakly associate* if $\hat{f} \mid_w \hat{g}$ and $\hat{g} \mid_w \hat{f}$.

The notions from this section provide several characterizations of the weakly associate relation. We single out the next ones, which follows directly from Theorem 6.5, for perspective in Theorem 9.6 below.

Proposition 6.9 Let $\hat{f}: (v_i) \rightarrow (w_j)$ and $\hat{g}: (x_n) \rightarrow (y_m)$ be morphisms in $\mathcal{F}(A)$. Let $r, s \in A$ be such that $\prod_m y_m = r \prod_n x_n$ and $\prod_j w_j = s \prod_i v_i$. Then the following conditions are equivalent.

- (i) The morphisms \hat{f} and \hat{g} are weakly associate.
- (ii) The elements r and s divide each other in A , i.e., we have $r \mid s$ and $s \mid r$.
- (iii) We have $r = vs$ for some invertible element $v \in A$.
- (iv) The elements $\prod_n x_n \cdot \prod_j w_j$ and $\prod_i v_i \cdot \prod_m y_m$ divide each other in A .
- (v) We have $\prod_n x_n \cdot \prod_j w_j = u \cdot \prod_i v_i \cdot \prod_m y_m$ for some invertible element $u \in A$.

7 Weak Irreducibility

Assumptions In this section, again A is a divisibility monoid, and we say that a morphism \hat{f} is a weak equivalence provided that $\hat{f} \in \mathcal{W}(A)$.

Here we treat a notion of irreducibility for morphisms and objects in $\mathcal{F}(A)$. Note that the definition of an irreducible tuple fits with a fundamental idea behind our construction, that factorization in A is tracked by morphisms in $\mathcal{F}(A)$.

Definition 7.1 A monoid element $a \in A$ is *irreducible* if it is not invertible and only has trivial factorizations in A : if $a = bc$ in A , then either b or c is invertible in A . We shall say that a morphism \hat{f} in $\mathcal{F}(A)$ is *weakly irreducible* when \hat{f} is not a weak equivalence and for every factorization $\hat{f} = \hat{g} \circ \hat{h}$ either \hat{g} or \hat{h} is a weak equivalence. We say an object $(x_n) \in \mathcal{F}(A)$ is *weakly irreducible* if the morphism $(1) \rightarrow (x_n)$ is weakly irreducible.

Remark 7.2 If $\hat{f}: (x_n) \rightarrow (y_m)$ is a morphism in $\mathcal{F}(A)$ with $(x_n) = \mathfrak{o}$ or $(y_m) = \mathfrak{o}$, then \hat{f} is a weak equivalence, so in particular \hat{f} is not weakly irreducible. For example, this shows that \mathfrak{o} is not weakly irreducible in $\mathcal{F}(A)$.

The term “weakly irreducible morphism” is meant to suggest the usual notion of irreducibility, modified by weak equivalences. Of course, one could introduce variations on this, for instance, a stronger notion declaring a morphism \hat{f} in $\mathcal{F}(A)$ to be “irreducible” when \hat{f} is not an isomorphism and for every factorization $\hat{f} = \hat{g} \circ \hat{h}$ either \hat{g} or \hat{h} is an isomorphism. However, this version is too strong for our purposes.

Lemma 7.3 Consider morphisms in $\mathcal{F}(A)$

$$(v_i) \xrightarrow[\in \mathcal{W}(A)]{\hat{e}} (r_a) \xrightarrow{\hat{\eta}} (s_b) \xrightarrow[\in \mathcal{W}(A)]{\hat{\phi}} (w_j)$$

such that \hat{e} and $\hat{\phi}$ are weak equivalences. If the composition $\hat{f} = \hat{\phi} \circ \hat{\eta} \circ \hat{e}$ is weakly irreducible, then so is $\hat{\eta}$.

Proof Assume that \hat{f} is weakly irreducible. Then \hat{f} is not a weak equivalence, so the 2-of-3 condition for $\mathcal{W}(A)$ implies that $\hat{\eta}$ is not a weak equivalence. Suppose by way of contradiction that $\hat{\eta}$ is not weakly irreducible, so that it factors as $\hat{\eta} = \hat{h} \circ \hat{g}$ where neither \hat{h} nor \hat{g} is a weak equivalence. Plug this factorization into the definition of \hat{f} to conclude that

$$\hat{f} = \hat{\phi} \circ \hat{\eta} \circ \hat{\epsilon} = (\hat{\phi} \circ \hat{h}) \circ (\hat{g} \circ \hat{\epsilon})$$

Since $\hat{\epsilon}$ and $\hat{\phi}$ are both weak equivalences, the 2-of-3 condition for $\mathcal{W}(A)$ implies that neither $\hat{\phi} \circ \hat{h}$ nor $\hat{g} \circ \hat{\epsilon}$ is a weak equivalence, contradicting the fact that \hat{f} is weakly irreducible. This establishes the result. \square

The next result shows, e.g., that the morphism $\widehat{\text{id}}_{[1]}: (2) \rightarrow (6)$ from Example 6.1 is weakly irreducible. And it gives another indication of how the morphisms in $\mathcal{W}(A)$ can see factorization properties in A .

Theorem 7.4 *For an element $r \in A$, the following conditions are equivalent.*

- (i) r is irreducible in A ,
- (ii) each morphism $\hat{f}: (x_n) \rightarrow (y_m)$ where $r \cdot \prod_n x_n = \prod_m y_m$ is weakly irreducible,
- (iii) the 1-tuple (r) is weakly irreducible in $\mathcal{F}(A)$, and
- (iv) some morphism $\hat{f}: (x_n) \rightarrow (y_m)$ with $r \cdot \prod_n x_n = \prod_m y_m$ is weakly irreducible.

Proof The implications (ii) \implies (iii) \implies (iv) are straightforward.

(i) \implies (ii) Assume that r is irreducible in A . Consider a morphism $\hat{f}: (x_n) \rightarrow (y_m)$ such that $r \cdot \prod_n x_n = \prod_m y_m$. The fact that r is not invertible implies that the given morphism is not a weak equivalence by Lemma 5.3. To show that \hat{f} is weakly irreducible, decompose \hat{f} as a composition

$$(x_n) \xrightarrow{\hat{g}} (z_\ell) \xrightarrow{\hat{h}} (y_m).$$

Let $a, b \in A$ be such that $\prod_\ell z_\ell = a \prod_n x_n$ and

$$r \cdot \prod_n x_n = \prod_m y_m = b \prod_\ell z_\ell = ab \prod_n x_n.$$

Cancellation implies that $r = ab$. Since r is irreducible, we conclude that a or b is invertible in A , so \hat{g} or \hat{h} is a weak equivalence by Lemma 5.3, as desired.

(iv) \implies (i) Assume that $\hat{f}: (x_n) \rightarrow (y_m)$ is a weakly irreducible morphism in $\mathcal{F}(A)$ with $r \cdot \prod_n x_n = \prod_m y_m$. In particular, \hat{f} is not a weak equivalence, so $(y_m) \neq \circ$, and hence $(x_n) \neq \circ$. Consider the decomposition $\hat{f} = \hat{\phi} \circ \widehat{\text{id}}_{[P]}^{(z_p)} \circ \hat{\epsilon}$ from Proposition 3.11

$$(x_n) \xrightarrow[\in \mathcal{W}(A)]{\hat{\epsilon}} (z_p) \xrightarrow[\widehat{\text{id}}_{[P]}^{(z_p)}]{(a_p z_p)} (a_p z_p) \xrightarrow[\in \mathcal{W}(A)]{\hat{\phi}} (y_m). \tag{7.1}$$

Recall the superscript/subscript notation for $\widehat{\text{id}}_{[P]}$ from Definition 2.1.

Claim 1: There is an element $p_0 \in [P]$ such that a_p is invertible in A for all $p \neq p_0$. Indeed, Proposition 5.4 implies that some a_p is non-invertible in A . Let p_0 be the smallest such $p \in [P]$, so that a_{p_0} is non-invertible in A but a_p is invertible in A for all $p < p_0$. The morphism $\widehat{\text{id}}_{[P]}^{(z_p)}$ factors as the composition of the next two morphisms.

$$(z_p) \xrightarrow[\notin \mathcal{W}(A)]{\widehat{\text{id}}_{[P]}^{(z_p)}} \underbrace{(z_1, \dots, z_{p_0-1}, a_{p_0} z_{p_0}, z_{p_0+1}, \dots, z_P)}_{=(z'_p)} \xrightarrow[\widehat{\text{id}}_{[P]}^{(z'_p)}]{(a_p z_p)} (a_p z_p) \tag{7.2}$$

The fact that a_{p_0} is non-invertible implies that the first factor $\widehat{\text{id}}_{[P]}^{(z_p)}(z'_p)$ is not a weak equivalence. Since $\hat{\epsilon}$ is a weak equivalence, the 2-of-3 condition implies that $\widehat{\text{id}}_{[P]}^{(z_p)}(z'_p) \circ \hat{\epsilon}$ is not a weak equivalence. Combining the decompositions (7.1) and (7.2), we obtain the next decomposition of \hat{f} .

$$(x_n) \xrightarrow[\notin \mathcal{W}(A)]{\widehat{\text{id}}_{[P]}^{(z_p)}(z'_p) \circ \hat{\epsilon}} (z'_p) \xrightarrow{\hat{\phi} \circ \widehat{\text{id}}_{[P]}^{(z'_p)}(a_p z_p)} (y_m).$$

Since \hat{f} is weakly irreducible and the first factor here is not a weak equivalence, it follows that the second factor must be a weak equivalence. Proposition 5.4 implies that a_p is invertible in A for all $p \neq p_0$. This establishes Claim 1.

Because of (7.1), the morphism $\widehat{\text{id}}_{[P]}^{(z_p)}(a_p z_p)$ is weakly irreducible by Lemma 7.3. However, it decomposes in (7.2), so another application of Lemma 7.3 shows that the morphism $\widehat{\text{id}}_{[P]}^{(z_p)}(z'_p)$ is weakly irreducible.

Claim 2: the element $a_{p_0} \in A$ is irreducible. By Claim 1, this element is non-invertible. Suppose that $a_{p_0} = bc$ for some $b, c \in A$. Then the weakly irreducible morphism $\widehat{\text{id}}_{[P]}^{(z_p)}(z'_p)$ from (7.2) factors as

$$(z_p) \xrightarrow{\widehat{\text{id}}_{[P]}} (z_1, \dots, z_{p_0-1}, bz_{p_0}, \dots, z_p) \xrightarrow{\widehat{\text{id}}_{[P]}} (z_1, \dots, z_{p_0-1}, bc z_{p_0}, \dots, z_p)$$

so the fact that this composition is weakly irreducible implies that one of the morphisms in the composition is a weak equivalence. By definition, it follows that b is invertible in A or c is invertible. This establishes Claim 2.

Recall that Proposition 3.11 implies that $r = u \prod_p a_p = a_{p_0} (u \prod_{p \neq p_0} a_p)$ for some invertible $u \in A$. Claim 1 implies that the element $u \prod_{p \neq p_0} a_p$ is invertible in A . Since a_{p_0} is irreducible in A by Claim 2 and $u \prod_{p \neq p_0} a_p$ is invertible, their product r is irreducible in A , as desired. \square

Our next result is for use in Theorem 7.6.

Proposition 7.5 *Let $P \in \mathbb{N}$, and consider a divisibility morphism $\widehat{\text{id}}_{[P]}: (z_p) \rightarrow (a_p z_p)$ in $\mathcal{F}(A)$. Such a morphism is weakly irreducible if and only if there is an integer $p_0 \in [P]$ such that a_{p_0} is irreducible in A and for all $p \neq p_0$ the element a_p is invertible in A , that is, if and only if the element $r = \prod_p a_p$ is irreducible in A .*

Proof The element r satisfies $\prod_p (a_p z_p) = r \prod_p z_p$, so Theorem 7.4 says that $\widehat{\text{id}}_{[P]}$ is weakly irreducible if and only if r is irreducible in A , that is, if and only if there is an integer $p_0 \in [P]$ such that a_{p_0} is irreducible in A and for all $p \neq p_0$ the element a_p is invertible in A . \square

Next, we give a characterization of irreducibility akin to Proposition 5.4.

Theorem 7.6 *Let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\mathcal{F}(A)$ between non-empty tuples. Consider the decomposition $\hat{f} = \hat{\phi} \circ \widehat{\text{id}}_{[P]} \circ \hat{\epsilon}$ from Proposition 3.11*

$$(x_n) \xrightarrow{\hat{\epsilon}} (z_p) \xrightarrow{\widehat{\text{id}}_{[P]}} (a_p z_p) \xrightarrow{\hat{\phi}} (y_m).$$

Then the morphism \hat{f} is weakly irreducible if and only if the divisibility morphism $\widehat{\text{id}}_{[P]}$ is weakly irreducible (see Proposition 7.5).

Proof Assume that the divisibility morphism $\widehat{\text{id}}_{[P]}$ is weakly irreducible. Proposition 7.5 implies that $\prod_p a_p$ is irreducible in A . By Remark 3.9, let $r \in A$ be the unique element of A such that $\prod_m y_m = r \prod_n x_n$. Since $r = u \prod_n a_n$ for some invertible $u \in A$ by Proposition 3.11, we conclude that r is irreducible in A , so Theorem 7.4 says that \hat{f} is weakly irreducible.

The converse is proved similarly, or by Lemma 7.3. □

Corollary 7.7 *A tuple $(x_n) \neq o$ is weakly irreducible in $\mathcal{F}(A)$ if and only if there is an integer $n_0 \in [N]$ such that x_{n_0} is irreducible in A and for all $n \neq n_0$ the element x_n is invertible in A , that is, if and only if $\prod_n x_n$ is irreducible in A .*

Proof By definition, a tuple $(x_n) \neq o$ is weakly irreducible in $\mathcal{F}(A)$ if and only if the morphism $\hat{f}: (1) \rightarrow (x_n)$ is weakly irreducible. Theorem 7.4 says that this holds if and only if $\prod_n x_n$ is irreducible in A , that is, if and only if there is an integer $n_0 \in [N]$ such that x_{n_0} is irreducible in A and for all $n \neq n_0$ the element x_n is invertible in A . □

The next lemma is for use in the proof of Theorem 9.1.

Lemma 7.8 *Consider two tuples $(x_n), (y_m)$ in $\mathcal{F}(A)$, and assume that in $\mathcal{F}(A)$ there is a finite sequence $(x_n) \rightarrow \dots \rightarrow (y_m)$ of weakly irreducible morphisms and weak equivalences. If $\prod_n x_n$ is invertible or has an irreducible factorization in A , then $\prod_m y_m$ is invertible or has an irreducible factorization in A .*

Proof Let q denote the number of morphisms in the given sequence $(x_n) \rightarrow \dots \rightarrow (y_m)$. By Lemma 5.3 and Theorem 7.4 there is a sequence r_1, \dots, r_q of invertible elements and irreducible elements such that $\prod_m y_m = r_1 \dots r_q \prod_n x_n$. As $\prod_n x_n$ is assumed to be invertible or to have an irreducible factorization in A , it follows from this formula that $\prod_m y_m$ is invertible or has an irreducible factorization in A . □

8 Weak Primeness

Assumptions In this section, again A is a divisibility monoid, and we say that a morphism \hat{f} is a weak equivalence provided that $\hat{f} \in \mathcal{W}(A)$.

Here we introduce and study a notion of weak primeness for objects and morphisms in $\mathcal{F}(A)$, based on the idea of weak divisibility from Sect. 6. Our treatment here is roughly parallel to our discussion of weak irreducibility.

Definition 8.1 A monoid element $a \in A$ is *prime* if it is not invertible and for all $b, c \in A$ if $a \mid bc$ in A , then either $a \mid b$ or $a \mid c$ in A . We shall say that a morphism \hat{f} in $\mathcal{F}(A)$ is *weakly prime* when \hat{f} is not a weak equivalence and for every weak divisibility relation $\hat{f} \mid_w \hat{g} \circ \hat{h}$ either $\hat{f} \mid_w \hat{g}$ or $\hat{f} \mid_w \hat{h}$. We say an object $(x_n) \in \mathcal{F}(A)$ is *weakly prime* if the morphism $(1) \rightarrow (x_n)$ is weakly prime.

Remark 8.2 If $\hat{f}: (x_n) \rightarrow (y_m)$ is a morphism in $\mathcal{F}(A)$ with $(x_n) = o$ or $(y_m) = o$, then \hat{f} is not weakly prime; so o is not weakly prime in $\mathcal{F}(A)$.

Similar to the previous section, the term “weakly prime morphism” is meant to suggest the usual notion of primeness, modified by weak equivalences and weak divisibility. One could introduce variations on this, for instance, using isomorphisms or one of the divisibility notions discussed in Example 6.1. However, these versions do not suit our purposes.

The next result shows, e.g., that the morphism $\widehat{\text{id}}_{[1]}: (2) \rightarrow (6)$ from Example 6.1. is weakly prime. And it gives yet another indication of how the morphisms in $\mathcal{W}(A)$ can see factorization properties in A .

Theorem 8.3 *For an element $r \in A$, the following conditions are equivalent.*

- (i) r is prime in A ,
- (ii) each morphism $\hat{f}: (x_n) \rightarrow (y_m)$ where $r \cdot \prod_n x_n = \prod_m y_m$ is weakly prime,
- (iii) the 1-tuple (r) is weakly prime in $\mathcal{F}(A)$, and
- (iv) some morphism $\hat{f}: (x_n) \rightarrow (y_m)$ with $r \cdot \prod_n x_n = \prod_m y_m$ is weakly prime.

Proof The implications (ii) \implies (iii) \implies (iv) are trivial.

(iv) \implies (i) Assume that $\hat{f}: (x_n) \rightarrow (y_m)$ is a weakly prime morphism with $r \cdot \prod_n x_n = \prod_m y_m$. In particular, r is not invertible in A , otherwise \hat{f} would be a weak equivalence by Lemma 5.3. Let $a, b \in A$ be such that $r \mid ab$. Consider the following morphisms between 1-tuples.

$$(1) \xrightarrow[\hat{h}]{\widehat{\text{id}}_{[1]}} (a) \xrightarrow[\hat{g}]{\widehat{\text{id}}_{[1]}} (ab)$$

Because of Theorem 6.5, the assumption $r \mid ab$ implies that $\hat{f} \mid_w \hat{h} \circ \hat{g}$. Since \hat{f} is weakly prime, we have $\hat{f} \mid_w \hat{h}$ or $\hat{f} \mid_w \hat{g}$. In the case $\hat{f} \mid_w \hat{h}$, another application of Theorem 6.5 implies that $r \mid a$. In the case $\hat{f} \mid_w \hat{g}$, we conclude similarly that $r \mid b$. Thus, the element r is prime in A , as desired.

(i) \implies (ii) Assume that r is prime in A . In particular, the element r is not invertible in A . Let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism such that $r \cdot \prod_n x_n = \prod_m y_m$. Lemma 5.3 implies that \hat{f} is not a weak equivalence. If $\hat{f} \mid_w \hat{\alpha} \circ \hat{\beta}$ for some morphisms $\hat{\alpha}$ and $\hat{\beta}$, then reverse the steps of the previous paragraph to conclude that $\hat{f} \mid_w \hat{\alpha}$ or $\hat{f} \mid_w \hat{\beta}$, so that \hat{f} is weakly prime, as desired. \square

The next result is included for use in Theorem 8.5.

Proposition 8.4 *With $I \in \mathbb{N}$, consider a divisibility morphism $\widehat{\text{id}}_{[I]}: (v_i) \rightarrow (a_i v_i)$ in $\mathcal{F}(A)$. Such a morphism is weakly prime if and only if there is an integer $i_0 \in [I]$ such that a_{i_0} is prime in A and for all $i \neq i_0$ the element a_i is invertible in A , that is, if and only if the element $\tilde{a} = \prod_i a_i$ is prime in A .*

Proof Argue as for Proposition 7.5, using Theorem 8.3. \square

Next, we give a characterization of weak primeness akin to Proposition 5.4 and Theorem 7.6.

Theorem 8.5 *Let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\mathcal{F}(A)$ between non-empty tuples. Consider the decomposition $\hat{f} = \hat{\phi} \circ \widehat{\text{id}}_{[P]} \circ \hat{\epsilon}$ from Proposition 3.11*

$$(x_n) \xrightarrow{\hat{\epsilon}} (z_p) \xrightarrow{\widehat{\text{id}}_{[P]}} (a_p z_p) \xrightarrow{\hat{\phi}} (y_m).$$

Then the morphism \hat{f} is weakly prime if and only if the divisibility morphism $\widehat{\text{id}}_{[P]}$ is weakly prime (see Proposition 8.4).

Proof Argue as for Theorem 7.6, using Theorem 8.3 and Proposition 8.4. \square

Corollary 8.6 *A tuple $(x_n) \neq \mathbf{o}$ is weakly prime in $\mathcal{F}(A)$ if and only if there is an integer $n_0 \in [N]$ such that x_{n_0} is prime in A and for all $n \neq n_0$ the element x_n is invertible in A , that is, if and only if $\prod_n x_n$ is prime in A .*

Proof Argue as for Corollary 7.7, using Theorem 8.3. □

Corollary 8.7 *If a morphism \hat{f} is weakly prime in $\mathcal{F}(A)$, then it is weakly irreducible in $\mathcal{F}(A)$. If a tuple (x_n) is weakly prime in $\mathcal{F}(A)$, then it is weakly irreducible in $\mathcal{F}(A)$.*

Proof Since prime elements of A are irreducible, the first implication follows from Theorems 7.4 and 8.3, and the second implication is by Corollaries 7.7 and 8.6. □

9 Factorization in Integral Domains

Assumptions In this section, Let D be an integral domain. We say that a morphism \hat{f} in $\mathcal{F}(D \setminus \{0\})$ is a weak equivalence provided that $\hat{f} \in \mathcal{W}(D \setminus \{0\})$.

Next, we interpret some properties of integral domains in terms of the category of factorization. This entire section fits with the theme that factorization properties of our monoid are mirrored in the morphisms of its category of factorization. We begin with the atomic property. Recall that our integral domain D is *atomic* if every non-zero non-unit in D factors as a finite product of irreducible elements of D , also known as “atoms”.

Theorem 9.1 *The following conditions are equivalent:*

- (i) *the integral domain D is atomic,*
- (ii) *every morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(D \setminus \{0\})$ between non-empty tuples decomposes as a finite composition $(x_n) \rightarrow \dots \rightarrow (y_m)$ of weakly irreducible morphisms and weak equivalences,*
- (iii) *every tuple $(y_m) \neq \mathbf{o}$ in $\mathcal{F}(D \setminus \{0\})$ admits a finite chain $(1) \rightarrow \dots \rightarrow (y_m)$ of weakly irreducible morphisms and weak equivalences, and*
- (iv) *every 1-tuple (y) admits a finite chain $(1) \rightarrow \dots \rightarrow (y)$ of weakly irreducible morphisms and weak equivalences, and*
- (v) *every 1-tuple (y) admits a finite chain $(1) \rightarrow \dots \rightarrow (y)$ of weakly irreducible morphisms and weak equivalences consisting entirely of 1-tuples.*

Proof (i) \implies (ii) Assume that D is atomic.

Claim: each divisibility morphism $\widehat{\text{id}}_{[P]}: (z_p) \rightarrow (a_p z_p)$ between P -tuples decomposes as a finite composition $(z_p) \rightarrow \dots \rightarrow (a_p z_p)$ of weakly irreducible divisibility morphisms and weak equivalences, consisting entirely of P -tuples. Indeed, since D is atomic, each a_p factors as $a_p = u_p q_{p,1} \dots q_{p,K_p}$ with $K_p \geq 0$, where u_p is a unit and each $q_{p,k}$ is irreducible. We induct on $n = \sum_{p=1}^P K_p$. In the base case $n = 0$, each a_p is a unit, so $\widehat{\text{id}}_{[P]}$ is a weak equivalence by Proposition 5.4. For the induction step, assume that $n \geq 1$, so we have $K_{p_1} \geq 1$ for some $p_1 \in [P]$. Then the morphism $\widehat{\text{id}}_{[P]}$ factors as follows

$$(z_p) \xrightarrow[\text{w.i.}]{\widehat{\text{id}}_{[P]}} (q_{1,1} z_1, z_2, \dots, z_P) \xrightarrow[(\dagger)]{\widehat{\text{id}}_{[P]}} (a_p z_p).$$

The morphism labeled “w.i.” is weakly irreducible by Theorem 7.4. The morphism labeled (\dagger) satisfies our induction hypothesis, so it decomposes as a finite composition of weakly

irreducible morphisms and weak equivalences, consisting entirely of P -tuples. Thus, the same is true of the original morphism $\widehat{\text{id}}_{[P]}$.

Now, consider a morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(D \setminus \{0\})$ between non-empty tuples. Consider the decomposition $\hat{f} = \hat{\phi} \circ \widehat{\text{id}}_{[P]} \circ \hat{e}$ from Proposition 3.11

$$(x_n) \xrightarrow{\hat{e}} (z_p) \xrightarrow{\widehat{\text{id}}_{[P]}} (a_p z_p) \xrightarrow{\hat{\phi}} (y_m).$$

The Claim implies that the divisibility morphism $\widehat{\text{id}}_{[P]}$ decomposes as a finite composition $(z_p) \rightarrow \dots \rightarrow (a_p z_p)$ of weakly irreducible divisibility morphisms and weak equivalences. Since \hat{e} and $\hat{\phi}$ are weak equivalences by Remark 5.2, the desired conclusion for \hat{f} follows from the displayed factorization.

(ii) \implies (iii) Condition (iii) is the special case $(x_n) = (1)$ of condition (ii), so this implication is trivial.

(iii) \implies (iv) Condition (iv) is the special case $N = 1$ of condition (iii), so this implication is trivial.

(iv) \implies (i) Assume that every 1-tuple (y) admits a finite chain $(1) \rightarrow \dots \rightarrow (y)$ of weakly irreducible morphisms and weak equivalences. Let $x \in D$ be a non-zero non-unit. By assumption, there is a finite chain $(1) \rightarrow \dots \rightarrow (x)$ of weakly irreducible morphisms and weak equivalences. Since $1 \in A$ is invertible, Lemma 7.8 implies that x is invertible or has an irreducible factorization as well. Thus, D is atomic, as desired.

(i) \implies (v) The natural morphism $(1) \rightarrow (y)$ is a divisibility morphism between 1-tuples, so the Claim in the proof of the preceding implication gives the desired conclusion.

(v) \implies (iv) This implication is trivial. □

Next, we have a similar characterization of UFD's (unique factorization domains). Compare conditions (iv)–(v) to [4, Theorem 5.1]. It is worth noting that conditions (iv) and (v) do not include commutativity requirements on the diagrams they contain.

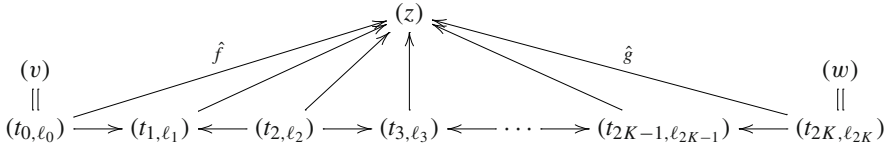
Theorem 9.2 *The following conditions are equivalent.*

- (i) *the integral domain D is a UFD.*
- (ii) (a) *Every tuple $(x_n) \neq 0$ in $\mathcal{F}(D \setminus \{0\})$ admits a finite chain $(1) \rightarrow \dots \rightarrow (x_n)$ of weakly irreducible morphisms and weak equivalences, and (b) every weakly irreducible tuple in $\mathcal{F}(D \setminus \{0\})$ is weakly prime.*
- (iii) (a) *Every morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(D \setminus \{0\})$ between non-empty tuples decomposes as a finite composition $(x_n) \rightarrow \dots \rightarrow (y_m)$ of weakly irreducible morphisms and weak equivalences, and (b) every weakly irreducible morphism in $\mathcal{F}(D \setminus \{0\})$ is weakly prime.*
- (iv) (a) *Every morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(D \setminus \{0\})$ between non-empty tuples decomposes as a finite composition $(x_n) \rightarrow \dots \rightarrow (y_m)$ of weakly irreducible morphisms and weak equivalences; and (b) for every pair of morphisms $(v_i) \xrightarrow{\hat{f}} (z_p) \xleftarrow{\hat{g}} (w_j)$ in $\mathcal{F}(D \setminus \{0\})$ with (v_i) and (w_j) weakly irreducible, either there is a weak equivalence $(v_i) \rightarrow (w_j)$ or there are morphisms*

$$\begin{array}{ccc}
 & (z_p) & \\
 \hat{f} \nearrow & \uparrow & \nwarrow \hat{g} \\
 (v_i) & \longrightarrow (t) & \longleftarrow (w_j)
 \end{array} \tag{9.1}$$

where each morphism in the bottom row of the diagram is weakly irreducible and (t) is a 1-tuple such that t is associate to $\prod_i v_i \cdot \prod_j w_j$ in D .

- (v) (a) Every tuple $(x_n) \neq \mathbf{o}$ in $\mathcal{F}(D \setminus \{0\})$ admits a finite chain $(1) \rightarrow \dots \rightarrow (x_n)$ of weakly irreducible morphisms and weak equivalences; and (b) for every pair of morphisms $(v) \xrightarrow{\hat{f}} (z) \xleftarrow{\hat{g}} (w)$ between 1-tuples in $\mathcal{F}(D \setminus \{0\})$ with (v) and (w) weakly irreducible, either there is a weak equivalence $(v) \rightarrow (w)$ or there is a positive integer K such that there are morphisms



where each morphism in the bottom row is weakly irreducible, and for each $k \in [K]$ the elements $\prod_{\ell_{2k-2}} t_{2k-2, \ell_{2k-2}} \cdot \prod_{\ell_{2k}} t_{2k, \ell_{2k}}$ and $\prod_{\ell_{2k-1}} t_{2k-1, \ell_{2k-1}}$ are associates.

Proof (i) \iff (iii) The integral domain D is a UFD if and only if it is atomic and every irreducible element of D is prime. Theorem 9.1 shows that D being atomic is equivalent to condition (iii)(a). If every weakly irreducible morphism in $\mathcal{F}(D \setminus \{0\})$ is weakly prime, then every irreducible element of D is prime by Theorems 7.4 and 8.3; and conversely.

(i) \iff (ii) Argue as in the previous paragraph using Corollaries 7.7 and 8.6.

(i) \implies (iv) Assume that D is a UFD. In particular, D is atomic, so Theorem 9.1 shows

that condition (a) holds. For condition (b), consider a pair of morphisms $(v_i) \xrightarrow{\hat{f}} (z_p) \xleftarrow{\hat{g}} (w_j)$ in $\mathcal{F}(D \setminus \{0\})$ with (v_i) and (w_j) weakly irreducible, and assume that there is not a weak equivalence $(v_i) \rightarrow (w_j)$. Since (v_i) and (w_j) are weakly irreducible, Corollary 7.7 implies that there are integers $i_0 \in [I]$ and $j_0 \in [J]$ such that v_{i_0} and w_{j_0} are irreducible in D , for all $i \in [I] \setminus \{i_0\}$ the element v_i is a unit in D , and for all $j \in [J] \setminus \{j_0\}$ the element w_j is a unit in D . The existence of the morphism \hat{f} implies that $v_{i_0} \mid z_{p_1}$ for some $p_1 \in [P]$, and similarly $w_{j_0} \mid z_{p_2}$ for some $p_2 \in [P]$, as the assumption that D is a UFD implies that v_{i_0} and w_{j_0} are prime in D .

Claim: The elements v_{i_0} and w_{j_0} are not associates in D . By way of contradiction, suppose that v_{i_0} and w_{j_0} were associates in D , say $u \in D$ is a unit such that $w_{j_0} = uv_{i_0}$. Consider the next morphisms in $\mathcal{F}(D \setminus \{0\})$ where \hat{e} drops all the units v_i with $i \neq i_0$ and $\hat{\sigma}$ is the morphism induced by the condition $w_{j_0} \mid \prod_j w_j$.

$$(v_i) \xrightarrow{\hat{e}} (v_{i_0}) \xrightarrow{\widehat{\text{id}}_{[1]}} (uv_{i_0}) = (w_{j_0}) \xrightarrow{\hat{\sigma}} (w_j)$$

Remark 5.2 implies that the three morphisms in this display are weak equivalences. Thus, the composition $\hat{\sigma} \circ \widehat{\text{id}}_{[1]} \circ \hat{e}: (v_i) \rightarrow (w_j)$ is also a weak equivalence, by the 2-of-3 condition in Proposition 5.8. This contradicts our assumption that there is not a weak equivalence $(v_i) \rightarrow (w_j)$, establishing the Claim.

Set $(t) = (v_{i_0} w_{j_0})$. Let $\hat{\alpha}$ be the composition of the following morphisms in $\mathcal{F}(D \setminus \{0\})$ where \hat{e} drops all the units v_i with $i \neq i_0$

$$(v_i) \xrightarrow{\hat{e}} (v_{i_0}) \xrightarrow{\widehat{\text{id}}_{[1]}} (v_{i_0} w_{j_0}) = (t).$$

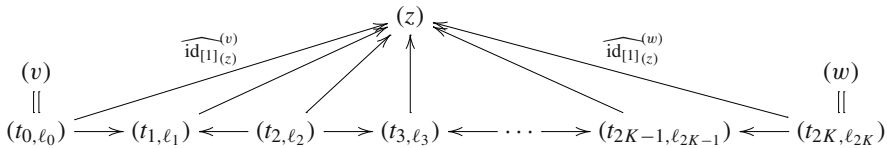
Since w_{j_0} is irreducible, Theorem 7.6 implies that $\hat{\alpha}$ is weakly irreducible. Similarly, there is a weakly irreducible morphism $(w_j) \rightarrow (t)$. To complete the diagram (9.1), it suffices to

exhibit a morphism $(t) = (v_{i_0}w_{j_0}) \rightarrow (z_p)$. In the case $p_1 \neq p_2$, such a morphism comes from the conditions $t = v_{i_0}w_{j_0} \mid z_{p_1}z_{p_2} \mid \prod_p z_p$. In the case $p_1 = p_2$, we have $v_{i_0}, w_{j_0} \mid z_{p_1}$. Since v_{i_0} and w_{j_0} are non-associate irreducibles in D , we conclude that $v_{i_0}w_{j_0} \mid z_{p_1} \mid \prod_p z_p$ because D is a UFD. This yields a morphism $(t) = (v_{i_0}w_{j_0}) \rightarrow (z_p)$, thus completing the diagram (9.1) in this case, as well as the proof of this implication.

(iv) \implies (v) Theorem 9.1 shows that condition (iv)(a) implies condition (v)(a). Since condition (v)(b) is a special case of (iv)(b), this implication is established.

(v) \implies (i) Assume that (v) holds. Then condition (v)(a) implies that D is atomic by Theorem 9.1. We use [4, Theorem 5.1] to show that D is a UFD. To this end, let $z, v, w \in D$ be such that v and w are non-associate irreducible divisors of z . It suffices to exhibit irreducible divisors a_1, \dots, a_m of z such that the following products all divide z : $va_1, a_1a_2, \dots, a_{m-1}a_m$, and a_mw . (In the notation of [4, Theorem 5.1], this will show that $G(z)$ is connected.)

The assumptions $v \mid z$ and $w \mid z$ provide morphisms $(v) \xrightarrow{\widehat{\text{id}}_{[1](z)}} (z) \xleftarrow{\widehat{\text{id}}_{[1](z)}} (w)$ in $\mathcal{F}(D \setminus \{0\})$. As v and w are non-associate irreducibles, there is not a weak equivalence $(v) \rightarrow (w)$. So, condition (v)(b) provides a positive integer K and morphisms



where each morphism in the bottom row is weakly irreducible and for each $k \in [K]$ the element $\prod_{\ell_{2k-2}} t_{2k-2, \ell_{2k-2}} \cdot \prod_{\ell_{2k}} t_{2k, \ell_{2k}}$ is associate to $\prod_{\ell_{2k-1}} t_{2k-1, \ell_{2k-1}}$. Thus, Theorem 7.4 provides irreducible elements $b_1, b_2, \dots, b_{2K} \in D$ such that for all $k \in [K]$ we have

$$b_{2k-1} \prod_{\ell_{2k-2}} t_{2k-2, \ell_{2k-2}} = \prod_{\ell_{2k-1}} t_{2k-1, \ell_{2k-1}} = b_{2k} \prod_{\ell_{2k}} t_{2k, \ell_{2k}}.$$

For $i = 0, \dots, 2K$, set $t_i = \prod_{\ell_i} t_i, \ell_i$. Then the above equalities involving b_i read

$$b_{2k-1}t_{2k-2} = t_{2k-1} = b_{2k}t_{2k} \tag{9.2}$$

for $k \in [K]$. Furthermore, the vertical/diagonal arrows in the above diagram in $\mathcal{F}(D \setminus \{0\})$ show that $t_i \mid z$ for all $i \in [2K - 1]$; in particular, this shows that $t_{2k-1} \mid z$ for all $k \in [K]$.

The Eqs. (9.2) in the case $k = 1$ show that $b_1t_0 = t_1 = b_2t_2$. By assumption, we have $t_1 \sim t_0t_2$, where \sim is the associate relation. Combining these relations, we find that $b_1t_0 \sim t_0t_2 \sim b_2t_2$ and hence $b_1 \sim t_2$ and $b_2 \sim t_0$. In particular, since b_1 is irreducible, so is t_2 , that is, t_2 is an irreducible divisor of z such that $t_2v = t_2t_0 \sim t_1 \mid z$, by the previous paragraph. Arguing similarly, we see that for all $k \in [K]$ the element t_{2k} is an irreducible divisor of z such that $t_{2k-2}t_{2k} \sim t_{2k-1} \mid z$. For $k = K$, this reads as $t_{2K-2}w \sim t_{2K-1} \mid z$.

In summary, this shows that we have irreducible divisors t_2, \dots, t_{2K-2} of z such that the following products all divide z : $vt_2, t_2t_4, \dots, t_{2K-4}t_{2K-2}$, and $t_{2K-2}w$. In other words, the elements $a_k = t_{2k}$ for $k \in [K - 1]$ satisfy the conditions described in the first paragraph of this part of the proof. So D is a UFD by [4, Theorem 5.1]. \square

We continue with similar characterizations of other properties from [1], beginning with the ACCP (ascending chain condition on principal ideals) property.

Theorem 9.3 *The following conditions are equivalent:*

- (i) *the integral domain D satisfies ACCP,*

- (ii) every chain of morphisms $(x_{1,n})_{n=1}^{N_1} \leftarrow (x_{2,n})_{n=1}^{N_2} \leftarrow \dots$ in $\mathcal{F}(D \setminus \{0\})$ stabilizes to a chain of weak equivalences, that is, for every such chain there is an index i such that each morphism $(x_{i,n})_{n=1}^{N_i} \leftarrow (x_{i+1,n})_{n=1}^{N_{i+1}} \leftarrow \dots$ is a weak equivalence, and
- (iii) every chain of morphisms $(x_1) \leftarrow (x_2) \leftarrow \dots$ of 1-tuples in $\mathcal{F}(D \setminus \{0\})$ stabilizes to a chain of isomorphisms (equivalently, weak equivalences).

Proof First, compare Theorem 3.8 and Definition 5.1 to see that a morphism $(x) \rightarrow (y)$ between 1-tuples in $\mathcal{F}(D \setminus \{0\})$ is a weak equivalence if and only if it is an isomorphism.

(i) \implies (ii) Assume that D satisfies ACCP, and consider a chain of morphisms

$$(x_{1,n})_{n=1}^{N_1} \xleftarrow{\widehat{f}_1} (x_{2,n})_{n=1}^{N_2} \xleftarrow{\widehat{f}_2} \dots$$

in $\mathcal{F}(D \setminus \{0\})$. This provides a chain of divisibility relations

$$\dots \mid \prod_{n=1}^{N_2} x_{2,n} \mid \prod_{n=1}^{N_1} x_{1,n}$$

hence a chain of principal ideals $\langle \prod_{n=1}^{N_1} x_{1,n} \rangle \subseteq \langle \prod_{n=1}^{N_2} x_{2,n} \rangle \subseteq \dots$. ACCP implies that this chain stabilizes, so there is an integer i such that for each $j \geq i$ there is a unit $u_j \in D$ such that $\prod_{n=1}^{N_j} x_{j,n} = u_j \prod_{n=1}^{N_{j+1}} x_{j+1,n}$. Lemma 5.3 implies that \widehat{f}_j is a weak equivalence for each $j \geq i$, as desired.

(ii) \implies (iii) Condition (iii) is the special case of condition (ii) where $N_i = 1$ for all i . Thus, this implication is clear.

(iii) \implies (i) Assume that condition (iii) is satisfied. A chain of principal ideals $(x_1) \subseteq (x_2) \subseteq \dots$ yields a chain of divisibility morphisms $(x_1) \xleftarrow{\widehat{\text{id}}_{[1]}} (x_2) \xleftarrow{\widehat{\text{id}}_{[1]}} \dots$ which stabilizes by assumption. As in the proof of (i) \implies (ii), one concludes that the original chain of principal ideals also stabilizes. \square

Recall that D is a half-factorial domain (HFD) provided that it is atomic and for every pair of irreducible factorizations $p_1 \cdots p_m = q_1 \cdots q_n$ in D , we have $m = n$. (Note that originally HFD's were not assumed to be atomic.) Our next result characterizes HFDs as in Theorem 9.1; see also Theorem 9.7.

Theorem 9.4 *The following conditions are equivalent:*

- (i) the integral domain D is an HFD;
- (ii) every morphism $\widehat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(D \setminus \{0\})$ between non-empty tuples decomposes as a finite composition $(x_n) \rightarrow \dots \rightarrow (y_m)$ of weakly irreducible morphisms and weak equivalences, and for every such morphism \widehat{f} every such decomposition for \widehat{f} has the same number of weakly irreducible morphisms;
- (iii) every tuple $(x_n) \neq 0$ in $\mathcal{F}(D \setminus \{0\})$ admits a finite chain $(1) \rightarrow \dots \rightarrow (x_n)$ of weakly irreducible morphisms and weak equivalences, and every such chain for (x_n) has the same number of weakly irreducible morphisms;
- (iv) every 1-tuple (x) admits a finite chain $(1) \rightarrow \dots \rightarrow (x)$ of weakly irreducible morphisms and weak equivalences, and every such chain for (x) has the same number of weakly irreducible morphisms; and
- (v) every 1-tuple (x) admits a finite chain $(1) \rightarrow \dots \rightarrow (x)$ of weakly irreducible morphisms and weak equivalences consisting entirely of 1-tuples, and every such chain for (x) has the same number of weakly irreducible morphisms.

Proof We prove the implication (i) \implies (ii); the remainder of the proof follows as for Theorem 9.1. Assume that D is an HFD, and let $\widehat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\mathcal{F}(D \setminus \{0\})$

between non-empty tuples. Since D is atomic, Theorem 9.1 implies that \hat{f} decomposes as a finite chain $(x_n) \rightarrow \cdots \rightarrow (y_m)$ of weakly irreducible morphisms and weak equivalences. Let $a \in A$ be such that $a \prod_n x_n = \prod_m y_m$ and factor a as a product of irreducibles $a = \prod_{i=1}^I r_i$. The decomposition of \hat{f} in the proof of Theorem 9.1 has exactly I weakly irreducible morphisms.

Let $(x_n) = (x_n)_{n=1}^{N_1} \xrightarrow{\hat{f}_1} (x_{1,n_1})_{n_1=1}^{N_1} \xrightarrow{\hat{f}_2} \cdots \xrightarrow{\hat{f}_K} (x_{K,n_K})_{n_K=1}^{N_K} = (y_m)$ be a decomposition of \hat{f} as a finite chain of weakly irreducible morphisms and weak equivalences. We need to show that the number of weakly irreducible morphisms in this chain is I . Since each \hat{f}_k is a morphism, there is an element $s_k \in A$ such that $\prod_{n_k} x_{k,n_k} = s_k \prod_{n_{k-1}} x_{k-1,n_{k-1}}$. Composing the \hat{f}_k 's we obtain

$$\prod_k s_k \prod_n x_n = \prod_m y_m = a \prod_n x_n$$

so cancellation implies that $a = \prod_k s_k$. Since the \hat{f}_k 's are weak equivalences and weakly irreducible morphisms, Lemma 5.3 and Theorem 7.4 imply that the corresponding s_k 's are units and irreducibles, respectively. Since D is an HFD, the number of irreducible s_k 's must be exactly I . Since the number of weakly irreducible \hat{f}_k 's is the same as the number of irreducible s_k 's, our proof is complete. \square

The following characterizations of bounded factorization domains (BFDs) and finite factorization domains (FFDs) are proved like Theorem 9.4. Recall that D is a BFD if it is atomic, and for every non-zero non-unit $x \in D$ there is a bound (depending only on x) on the numbers of factors in each irreducible factorization of x . Also, D is an FFD if it is atomic, and each non-zero non-unit in D has only finitely many irreducible factors (up to associates).

Theorem 9.5 *The following conditions are equivalent:*

- (i) *the integral domain D is a BFD;*
- (ii) *every morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(D \setminus \{0\})$ between non-empty tuples decomposes as a finite composition $(x_n) \rightarrow \cdots \rightarrow (y_m)$ of weakly irreducible morphisms and weak equivalences, and for every such morphism \hat{f} there is a bound (depending only on \hat{f}) on the number of weakly irreducible morphisms in each such decomposition of \hat{f} ;*
- (iii) *every tuple $(x_n) \neq \mathfrak{o}$ in $\mathcal{F}(D \setminus \{0\})$ admits a finite chain $(1) \rightarrow \cdots \rightarrow (x_n)$ of weakly irreducible morphisms and weak equivalences, and for every such tuple (x_n) there is a bound (depending only on (x_n)) on the number of weakly irreducible morphisms in each such chain for (x_n) ;*
- (iv) *every 1-tuple (x) admits a finite chain $(1) \rightarrow \cdots \rightarrow (x)$ of weakly irreducible morphisms and weak equivalences, and for every such 1-tuple (x) there is a bound (depending only on (x)) on the number of weakly irreducible morphisms in each such chain for (x) ; and*
- (v) *every 1-tuple (x) admits a finite chain $(1) \rightarrow \cdots \rightarrow (x)$ of weakly irreducible morphisms and weak equivalences consisting entirely of 1-tuples, and for every such 1-tuple (x) there is a bound (depending only on (x)) on the number of weakly irreducible morphisms in each such chain for (x) .*

Theorem 9.6 *The following conditions are equivalent:*

- (i) *the integral domain D is an FFD;*
- (ii) *every morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(D \setminus \{0\})$ between non-empty tuples decomposes as a finite chain $(x_n) \rightarrow \cdots \rightarrow (y_m)$ of weakly irreducible morphisms and weak*

equivalences, and every such morphism \hat{f} has a finite number of non-weakly-associate weak divisors;

- (iii) every tuple $(x_n) \neq \mathfrak{o}$ in $\mathcal{F}(D \setminus \{0\})$ admits a finite chain $(1) \rightarrow \dots \rightarrow (x_n)$ of weakly irreducible morphisms and weak equivalences, and the morphism $(1) \rightarrow (x_n)$ has a finite number of non-weakly-associate weak divisors;
- (iv) every 1-tuple (x) admits a finite chain $(1) \rightarrow \dots \rightarrow (x)$ of weakly irreducible morphisms and weak equivalences, and the morphism $(1) \rightarrow (x)$ has a finite number of non-weakly-associate weak divisors; and
- (v) every 1-tuple (x) admits a finite chain $(1) \rightarrow \dots \rightarrow (x)$ of weakly irreducible morphisms and weak equivalences consisting entirely of 1-tuples, and the morphism $(1) \rightarrow (x)$ has a finite number of non-weakly-associate weak divisors.

We conclude by souping up a characterization of HFDs due to Zaks [14]. For this result, set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem 9.7 *The following conditions are equivalent.*

- (i) *The integral domain D is an HFD and not a field.*
- (ii) *There is a function $\ell_{\text{mor}}: \text{Mor}(\mathcal{F}(D \setminus \{0\})) \rightarrow \mathbb{N}_0$ such that*
 - (ii-1) $\text{Im}(\ell_{\text{mor}}) = \mathbb{N}_0$,
 - (ii-2) $\ell_{\text{mor}}(\hat{f}) = 0$ *if and only if* \hat{f} *is a weak equivalence in* $\mathcal{F}(D \setminus \{0\})$,
 - (ii-3) $\ell_{\text{mor}}(\hat{f}) = 1$ *if and only if* \hat{f} *is weakly irreducible in* $\mathcal{F}(D \setminus \{0\})$, *and*
 - (ii-4) $\ell_{\text{mor}}(\hat{g} \circ \hat{f}) = \ell_{\text{mor}}(\hat{g}) + \ell_{\text{mor}}(\hat{f})$ *for all* \hat{f} *and* \hat{g} *such that* $\hat{g} \circ \hat{f}$ *is defined.*

(iii) *There is a function $\ell_{\text{obj}}: \text{Obj}(\mathcal{F}(D \setminus \{0\})) \rightarrow \mathbb{N}_0$ such that*

- (iii-1) $\text{Im}(\ell_{\text{obj}}) = \mathbb{N}_0$,
- (iii-2) $\ell_{\text{obj}}((x_n)) = 1$ *if and only if* (x_n) *is weakly irreducible in* $\mathcal{F}(D \setminus \{0\})$,
- (iii-3) $\ell_{\text{obj}}((x_n) \otimes (y_m)) = \ell_{\text{obj}}((x_n)) + \ell_{\text{obj}}((y_m))$ *for all* (x_n) , (y_m) , *and*
- (iii-4) *for each morphism* $\hat{f}: (x_n) \rightarrow (y_m)$ *we have* $\ell_{\text{obj}}((x_n)) \leq \ell_{\text{obj}}((y_m))$, *with equality holding if and only if* \hat{f} *is a weak equivalence in* $\mathcal{F}(D \setminus \{0\})$.

Proof (i) \implies (ii) Assume that D is an HFD and not a field. In particular, D is atomic. For each $x \in D \setminus \{0\}$, let $\ell_{\text{elt}}(x) = n$ where $x = up_1 \cdots p_n$ with u a unit of D and each p_i an irreducible. The fact that D is an HFD implies that ℓ_{elt} is well defined (i.e., independent of the choice of factorization). By the proof of [14, Lemma 1.3], this function has the following properties:

- (i-1) $\text{Im}(\ell_{\text{elt}}) = \mathbb{N}_0$,
- (i-2) $\ell_{\text{elt}}(x) = 0$ if and only if x is a unit of D ,
- (i-3) $\ell_{\text{elt}}(x) = 1$ if and only if x is irreducible, and
- (i-4) $\ell_{\text{elt}}(xy) = \ell_{\text{elt}}(x) + \ell_{\text{elt}}(y)$ for all non-zero $x, y \in D$.

Condition (i-1) is where we use the assumption that D is not a field.

For each morphism $\hat{f}: (x_n) \rightarrow (y_m)$ in $\mathcal{F}(D \setminus \{0\})$, recall that there is a unique element $a \in D \setminus \{0\}$ such that $\prod_m y_m = a \prod_n x_n$; define $\ell_{\text{mor}}(\hat{f}) = \ell_{\text{elt}}(a) \in \mathbb{N}_0$. If $\hat{g}: (y_m) \rightarrow (z_p)$ is another morphism, the element $b \in D \setminus \{0\}$ satisfying $\prod_p z_p = b \prod_m y_m$ therefore also satisfies $\prod_p z_p = ab \prod_n x_n$, so we have

$$\ell_{\text{mor}}(\hat{g} \circ \hat{f}) = \ell_{\text{elt}}(ab) = \ell_{\text{elt}}(a) + \ell_{\text{elt}}(b) = \ell_{\text{mor}}(\hat{f}) + \ell_{\text{mor}}(\hat{g}).$$

Thus, condition (ii-4) is satisfied.

For condition (ii-1), since we have $\text{Im}(\ell_{\text{elt}}) = \mathbb{N}_0$, let $n \in \mathbb{N}_0$ be given and let $x \in D \setminus \{0\}$ be such that $\ell_{\text{elt}}(x) = n$. By definition, it follows that the morphism $\widehat{\text{id}}_{[1]}: (1) \rightarrow (x)$ satisfies $\ell_{\text{mor}}(\widehat{\text{id}}_{[1]}) = \ell_{\text{elt}}(x) = n$. Thus, we have $\text{Im}(\ell_{\text{mor}}) = \mathbb{N}_0$.

For condition (ii-3), consider a morphism $\hat{f}: (x_n) \rightarrow (y_m)$, with $a \in D \setminus \{0\}$ such that $\prod_m y_m = a \prod_n x_n$. By Theorem 7.4, the morphism \hat{f} is weakly irreducible if and only if a is irreducible in D , that is, if and only if $\ell_{\text{mor}}(\hat{f}) = \ell_{\text{elt}}(a) = 1$.

Condition (ii-2) is verified similarly, with Lemma 5.3. This completes the proof of the implication (i) \implies (ii).

(i) \implies (iii) Assume that D is an HFD, and not a field. We use the function ℓ_{elt} and the properties (i-1)–(i-4) from the proof of the implication (i) \implies (iii).

For each tuple (x_n) of $\mathcal{F}(D \setminus \{0\})$, set $\ell_{\text{obj}}((x_n)) = \ell_{\text{elt}}(\prod_n x_n)$. Thus, the definition $(x_n) \otimes (y_m) = (x_1, \dots, x_N, y_1, \dots, y_M)$ explains the first equality in the next display, and the second equality is from the properties of ℓ_{elt} .

$$\begin{aligned} \ell_{\text{obj}}((x_n) \otimes (y_m)) &= \ell_{\text{elt}}(\prod_n x_n \cdot \prod_m y_m) \\ &= \ell_{\text{elt}}(\prod_n x_n) + \ell_{\text{elt}}(\prod_m y_m) \\ &= \ell_{\text{obj}}((x_n)) + \ell_{\text{obj}}((y_m)) \end{aligned}$$

The third equality is by definition. This explains condition (iii-3).

For condition (iii-1), let $n \in \mathbb{N}_0$ be given and let $x \in D \setminus \{0\}$ be such that $\ell_{\text{elt}}(x) = n$. By definition, it follows that the 1-tuple (x) satisfies $\ell_{\text{obj}}((x)) = \ell_{\text{elt}}(x) = n$. Thus, we have $\text{Im}(\ell_{\text{mor}}) = \mathbb{N}_0$.

For condition (iii-2), consider a tuple (x_n) . Corollary 7.7 implies that (x_n) is weakly irreducible in $\mathcal{F}(D \setminus \{0\})$ if and only if $\prod_n x_n$ is irreducible in D , that is, if and only if $\ell_{\text{obj}}((x_n)) = \ell_{\text{elt}}(\prod_n x_n) = 1$.

For condition (iii-4), consider a morphism $\hat{f}: (x_n) \rightarrow (y_m)$, and let $a \in D \setminus \{0\}$ be such that $\prod_m y_m = a \prod_n x_n$. Thus, we have

$$\begin{aligned} \ell_{\text{obj}}((y_m)) &= \ell_{\text{elt}}(\prod_m y_m) = \ell_{\text{elt}}(a \prod_n x_n) \\ &= \ell_{\text{elt}}(a) + \ell_{\text{elt}}(\prod_n x_n) \geq \ell_{\text{elt}}(\prod_n x_n) = \ell_{\text{obj}}((x_n)). \end{aligned}$$

Moreover, equality holds in the penultimate step here if and only if $\ell_{\text{elt}}(a) = 0$, that is, if and only if a is a unit, that is, if and only if \hat{f} is a weak equivalence by Lemma 5.3. This completes the proof of the implication (i) \implies (iii).

(ii) \implies (i) Let ℓ_{mor} be given as in condition (ii). We argue as in the proof of [14, Lemma 1.3] to show that D is an HFD, and not a field.

Claim: D is atomic. Let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\text{Mor}(\mathcal{F}(D \setminus \{0\}))$. According to Theorem 9.1, we need to show that \hat{f} factors as a composition of finitely many weakly irreducible morphisms and weak equivalences. We verify this condition by strong induction on $L = \ell_{\text{mor}}(\hat{f})$. If $L = 0$ or 1 , then we are done by condition (ii-2) or (ii-3), respectively. This addresses the base case. For the induction step, assume that $L = \ell_{\text{mor}}(\hat{f}) \geq 2$ and that each morphism \hat{g} in $\text{Mor}(\mathcal{F}(D \setminus \{0\}))$ such that $\ell_{\text{mor}}(\hat{g}) < L$ factors as a composition of finitely many weakly irreducible morphisms and weak equivalences. Conditions (ii-2) and (ii-3) imply that \hat{f} is not a weak equivalence and is not weakly irreducible. Thus, there are morphisms \hat{g} and \hat{h} in $\text{Mor}(\mathcal{F}(D \setminus \{0\}))$ that are not weak equivalences and such that $\hat{f} = \hat{h} \circ \hat{g}$. Our assumptions about ℓ_{mor} imply that $\ell_{\text{mor}}(\hat{g}), \ell_{\text{mor}}(\hat{h}) \geq 1$ and $\ell_{\text{mor}}(\hat{h}) + \ell_{\text{mor}}(\hat{g}) = \ell_{\text{mor}}(\hat{h} \circ \hat{g}) = \ell_{\text{mor}}(\hat{f}) \geq 2$. In particular, we have $\ell_{\text{mor}}(\hat{g}), \ell_{\text{mor}}(\hat{h}) < \ell_{\text{mor}}(\hat{f}) = L$, so by our induction assumption, the morphisms \hat{g} and \hat{h} both factor as compositions of finitely many

weakly irreducible morphisms and weak equivalences, hence, so does their composition \hat{f} . This confirms the claim.

Now, let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\text{Mor}(\mathcal{F}(D \setminus \{0\}))$, and use what we have just established to find a finite list $\widehat{p}_1, \dots, \widehat{p}_L$ of weakly irreducible morphisms and weak equivalences such that $\hat{f} = \widehat{p}_1 \circ \dots \circ \widehat{p}_L$. Using our assumptions about ℓ_{mor} , we have $\ell_{\text{mor}}(\hat{f}) = \ell_{\text{mor}}(\widehat{p}_1) + \dots + \ell_{\text{mor}}(\widehat{p}_L)$ and this equals the number of weakly irreducible morphisms in the list $\widehat{p}_1, \dots, \widehat{p}_L$. In particular, the number of weakly irreducible morphisms in this list is independent of the choice of factorization of \hat{f} . Thus, Theorem 9.4 implies that D is an HFD.

To show that D is not a field, let $\hat{f}: (x_n) \rightarrow (y_m)$ be a morphism in $\mathcal{F}(D \setminus \{0\})$ such that $\ell_{\text{mor}}(\hat{f}) = 1$, since $\text{Im}(\ell_{\text{mor}}) = \mathbb{N}_0$. Condition (ii-3) implies that \hat{f} is weakly irreducible; in particular, it is not a weak equivalence. Lemma 5.3 implies that the non-zero element $a \in D$ such that $\prod_m y_m = a \prod_n x_n$ is not a unit in D , so D is not a field. This concludes the proof of this implication.

(iii) \implies (i) Let ℓ_{obj} be given as in item (iii). For all $x \in D \setminus \{0\}$, set $\ell_{\text{elt}}(x) = \ell_{\text{obj}}((x))$. Since for all $x, y \in D \setminus \{0\}$, the factorization morphism $(xy) \rightarrow (x, y) = (x) \otimes (y)$ is a weak equivalence, we have

$$\ell_{\text{elt}}(xy) = \ell_{\text{obj}}((xy)) = \ell_{\text{obj}}((x) \otimes (y)) = \ell_{\text{obj}}((x)) + \ell_{\text{obj}}((y)) = \ell_{\text{elt}}(x) + \ell_{\text{elt}}(y).$$

Thus, condition (i-4) from the first part of this proof is satisfied. Furthermore, this implies that $\ell_{\text{elt}}(1) = \ell_{\text{elt}}(1 \cdot 1) = \ell_{\text{elt}}(1) + \ell_{\text{elt}}(1)$ and thus $\ell_{\text{elt}}(1) = 0$.

Next, we show that an element $x \in D \setminus \{0\}$ is a unit if and only if $\ell_{\text{obj}}((x)) = 0$. Indeed, consider the morphism $\widehat{\text{id}}_{[1]}: (1) \rightarrow (x)$. By assumption, $\ell_{\text{obj}}((1)) = 0 \leq \ell_{\text{obj}}((x))$. Moreover, x is a unit in D if and only if $\widehat{\text{id}}_{[1]}$ is a weak equivalence by Lemma 5.3, if and only if $\ell_{\text{obj}}((x)) = \ell_{\text{obj}}((1)) = 0$ by condition (iii-4). Thus, condition (i-2) from the first part of this proof is satisfied.

We verify condition (i-3) similarly: Because of Corollary 7.7, an element $x \in D \setminus \{0\}$ is irreducible in D if and only if the 1-tuple (x) is weakly irreducible in $\mathcal{F}(D \setminus \{0\})$, that is, if and only if $\ell_{\text{elt}}(x) = \ell_{\text{obj}}((x)) = 1$.

Next, we verify condition (i-1). By assumption, there exists a tuple (x_n) in $\mathcal{F}(D \setminus \{0\})$ such that $\ell_{\text{obj}}((x_n)) = 1$ by what we have already shown. Condition (iii-2) implies that (x_n) is weakly irreducible in $\mathcal{F}(D \setminus \{0\})$, so the element $x = \prod_n x_n$ is irreducible in D by Corollary 7.7. Hence, $\ell_{\text{elt}}(x) = 1$. Condition (i-4) implies that $\ell_{\text{elt}}(x^k) = k$ for all $k \in \mathbb{N}$. As $\ell_{\text{elt}}(1) = 0$, it follows that $\text{Im}(\ell_{\text{elt}}) = \mathbb{N}_0$, as desired.

Define a function ℓ from the set of non-zero non-units of D to \mathbb{N} by the formula $\ell(x) = \ell_{\text{elt}}(x)$, i.e., ℓ is obtained from ℓ_{elt} by restricting the domain and codomain. Conditions (iii-1)–(iii-4) show that ℓ is well-defined and satisfy the following:

- (1) $\text{Im}(\ell) = \mathbb{N}$,
- (2) $\ell(x) = 1$ if and only if x is irreducible, and
- (3) $\ell(xy) = \ell(x) + \ell(y)$ for all non-zero $x, y \in D$.

These are exactly the conditions that allow us to apply [14, Lemma 1.3] to conclude that D is an HFD. (Note that conditions (1) and (2) imply that D has an irreducible element, in particular, a non-unit, so D is not a field.) □

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