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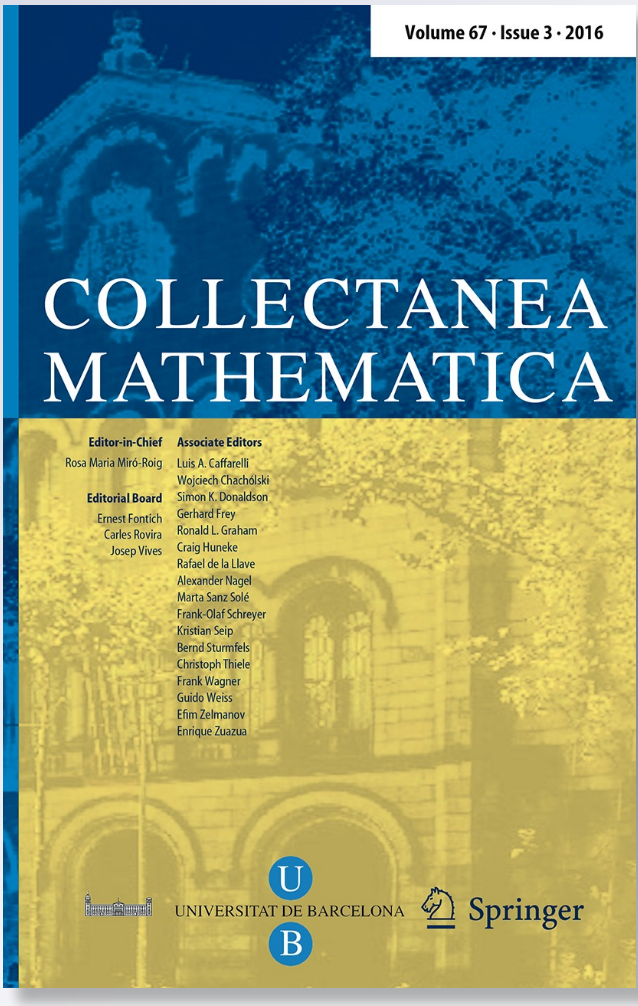


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# Testing for the Gorenstein property

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**Abstract** We answer a question of Celikbas, Dao, and Takahashi by establishing the following characterization of Gorenstein rings: a commutative noetherian local ring $(R, \mathfrak{m})$ is Gorenstein if and only if it admits an integrally closed $\mathfrak{m}$-primary ideal of finite Gorenstein dimension. This is accomplished through a detailed study of certain test complexes. Along the way we construct such a test complex that detect finiteness of Gorenstein dimension, but not that of projective dimension.

**Keywords** Integrally closed ideals · G-dimension · Projective dimension · Semidualizing complexes · Test complexes

**Mathematics Subject Classification** 13B22 · 13D02 · 13D07 · 13D09 · 13H10

## 1 Introduction

Throughout this paper $R$ denotes a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field $k$.

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A celebrated theorem of Auslander et al. [3,33] tells us that $R$ is regular if and only if $k$ has finite projective dimension. Burch [8, p. 947, Corollary 3] extended this by proving that $R$ is regular if and only if $\mathrm{pd}(R/I) < \infty$ for some integrally closed $\mathfrak{m}$-primary ideal $I$ of $R$.

Auslander and Bridger [1,2] introduced the G-dimension as a generalization of projective dimension. (See Sect. 2.3 for the definition.) Analogous to the regular setting, the finiteness of $\text{G-dim}_R(k)$ characterizes the Gorensteinness of $R$. In our local setting, Goto and Hayasaka [21] studied Gorenstein dimension of integrally closed $\mathfrak{m}$-primary ideals and, analogous to Burch's result, established the following; see the question of Yoshida stated in the discussion following [21, (1.1)].

**1.1** Goto and Hayasaka [21, (1.1)] Let $I$ be an integrally closed $\mathfrak{m}$-primary ideal of $R$. Assume $I$ contains a non-zerodivisor of $R$, or $R$ satisfies Serre's condition $(S_1)$. Then $R$ is Gorenstein if and only if $\text{G-dim}_R(R/I) < \infty$.

Our aim in this paper is to remove the hypothesis "$I$ contains a non-zerodivisor of $R$, or $R$ satisfies Serre's condition $(S_1)$" from 1.1. We accomplish this in the following result and hence obtain a complete generalization of Burch's aforementioned result; see also Corollary 4.7 for a further generalization.

**Theorem 1.1** *Let $I \subseteq R$ be an integrally closed ideal with* $\mathrm{depth}(R/I) = 0$*, e.g., such that $I$ is $\mathfrak{m}$-primary. Then $R$ is Gorenstein if and only if* $\text{G-dim}_R(R/I) < \infty$.

Our argument is quite different from that of Goto and Hayasaka [21] since it uses G-dim-test complexes. For this part of the introduction, we focus on the case of H-dim-test modules, defined next. Note that the pd-test modules (i.e., the case where H-dim = pd) are from [9].

**Definition 1.2** Let H-dim denote either projective dimension pd or G-dimension G-dim. Let $M$ be a finitely generated $R$-module. Then $M$ is an H-dim-*test module* over $R$ if the following condition holds for all finitely generated $R$-modules $N$: If $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, then $\text{H-dim}_R(N) < \infty$.

It is straightforward to show that if $M$ is a pd-test $R$-module, then it is also G-dim-test. Example 3.11 shows that the converse of this statement fails in general.

As part of our proof of Theorem 1.1, we also answer the following questions; see Corollaries 3.8(c) and 3.9.

**Question 1.3** Let $M$ be a pd-test module over $R$.

(a) Must $\widehat{M}$ be a pd-test module over $\widehat{R}$?
(b) Celikbas et al. [9, (3.5)] If $\text{G-dim}_R(M) < \infty$, must $R$ be Gorenstein?

Affirmative answers to Question 1.3(b) under additional hypotheses are in [9, (1.3)] and [10, (2.15)]. Also, Majadas [28] gives an affirmative answer to a version of Question 1.3(a) that uses a more restrictive version of test modules.

Theorem 4.8 follows from the next, significantly stronger result.

**Theorem** 4.4. *Let $M$ be a* G-dim-*test $R$-module such that* $\mathrm{Ext}^i_R(M, R) = 0$ *for $i \gg 0$. Then $R$ is Gorenstein.*

In turn, this follows from the much more general Theorem 4.1 and Corollary 4.2, which are results for detecting dualizing complexes.

We conclude this introduction by summarizing the contents of this paper. Section 2 consists of background material for use throughout the paper, and contains some technical lemmas for later use. In Sect. 3, we develop foundational properties of various H-dim-test objects, and answer Question 1.3. And Sect. 4 contains the theorems highlighted above.

## 2 Derived categories and semidualizing complexes

**2.1.** Throughout this paper we work in the derived category $\mathcal{D}(R)$ whose objects are the chain complexes of $R$-modules, i.e., the $R$-complexes $X$ with homological differential $\partial_i^X : X_i \to X_{i-1}$. References for this include [13,23,36,37]. Our notation is consistent with [12]. In particular, $\mathbf{R}\mathrm{Hom}_R(X, Y)$ and $X \otimes_R^{\mathbf{L}} Y$ are the derived Hom-complex and derived tensor product of two $R$-complexes $X$ and $Y$. Isomorphisms in $\mathcal{D}(R)$ are identified by the symbol $\simeq$. The projective dimension and flat dimension of an $R$-complex $X \in \mathcal{D}_b(R)$ are denoted $\mathrm{pd}_R(X)$ and $\mathrm{fd}_R(X)$.

The subcategory of $\mathcal{D}(R)$ consisting of homologically bounded $R$-complexes (i.e., complexes $X$ such that $\mathrm{H}_i(X) = 0$ for $|i| \gg 0$) is $\mathcal{D}_b(R)$. The subcategory of $\mathcal{D}(R)$ consisting of homologically finite $R$-complexes (i.e., complexes $X$ such that $\mathrm{H}(X) := \bigoplus_{i \in \mathbb{Z}} \mathrm{H}_i(X)$ is finitely generated) is denoted $\mathcal{D}_b^f(R)$.

**2.2.** A homologically finite $R$-complex $C$ is *semidualzing* if the natural morphism $R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ in $\mathcal{D}(R)$ is an isomorphism. For example, an $R$-module is semidualizing if and only if $\mathrm{Hom}_R(C, C) \cong R$ and $\mathrm{Ext}_R^i(C, C) = 0$ for $i \geqslant 1$. In particular, $R$ is a semidualizing $R$-module. A *dualizing $R$-complex* is a semidualizing $R$-complex of finite injective dimension.

**2.2.1.** If $R$ is a homomorphic image of a local Gorenstein ring $Q$, then $R$ has a dualizing complex, by [23, V.10]. (The converse holds by work of Kawasaki [27].) In particular, the Cohen Structure Theorem shows that the completion $\widehat{R}$ has a dualizing complex. When $R$ has a dualizing complex $D$, and $C$ is a semidualizing $R$-complex, the dual $\mathbf{R}\mathrm{Hom}_R(C, D)$ is also semidualizing over $R$, by [12, (2.12)].

**2.2.2.** Let $\varphi \colon R \to S$ be a flat local ring homomorphism, and let $C$ be a semidualizing $R$-complex. Then the $S$-complex $S \otimes_R^{\mathbf{L}} C$ is semidualizing, by [12, (5.6)]. If the closed fibre $S/\mathfrak{m}S$ is Gorenstein and $R$ has a dualizing complex $D^R$, then $D^S := S \otimes_R^{\mathbf{L}} D^R$ is dualizing for $S$ by [5, (5.1)].

Dualizing complexes were introduced by Grothendieck and Harshorne [23]. The more general semidualizing complexes originated in special cases, e.g., in [6,16,19,35], with the general version premiering in [12]. The notion of G-dimension, summarized next, started with the work of Auslander and Bridger [1,2] for modules. Foxby and Yassemi [38] recognized the connection with derived reflexivity, with the general situation given in [12].

**2.3.** Let $C$ be a semidualizing $R$-complex and $X \in \mathcal{D}_b^f(R)$. Write $\mathrm{G}_C\text{-}\dim_R(X) < \infty$ when $X$ is "derived $C$-reflexive", i.e., when $\mathbf{R}\mathrm{Hom}_R(X, C) \in \mathcal{D}_b(R)$ and the natural morphism $X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$ in $\mathcal{D}(R)$ is an isomorphism. In the case $C = R$, we write $\mathrm{G}\text{-}\dim_R(X) < \infty$ instead of $\mathrm{G}_R\text{-}\dim_R(X) < \infty$.

**2.3.1.** The complex $C$ is dualizing if and only if every $R$-complex in $\mathcal{D}_b^f(R)$ is derived $C$-reflexive, by [12, (8.4)]. In particular, $R$ is Gorenstein if and only if every $R$-complex $X \in \mathcal{D}_b^f(R)$ has $\mathrm{G}\text{-}\dim_R(X) < \infty$.

**2.3.2.** Let $R \to S$ be a flat local ring homomorphism. Given an $R$-complex $X \in \mathcal{D}_b^f(R)$, one has $\mathrm{G}_C\text{-}\dim_R(X) < \infty$ if and only if $\mathrm{G}_{S \otimes_R^{\mathbf{L}} C}\text{-}\dim_S(S \otimes_R^{\mathbf{L}} X) < \infty$, by [12, (5.10)]; see 2.2.2.

Auslander and Bass classes, defined next, arrived in special cases in [6,16], again with the general case described in [12].

**2.4.** Let $C$ be a semidualizing $R$-complex. The *Auslander class* $\mathcal{A}_C(R)$ consists of the $R$-complexes $X \in \mathcal{D}_b(R)$ such that $C \otimes^{\mathbf{L}}_R X \in \mathcal{D}_b(R)$ and the natural morphism $\gamma^C_X : X \to \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)$ in $\mathcal{D}(R)$ is an isomorphism. The *Bass class* $\mathcal{B}_C(R)$ consists of all the $R$-complexes $X \in \mathcal{D}_b(R)$ such that $\mathbf{R}\mathrm{Hom}_R(C, X) \in \mathcal{D}_b(R)$ and such that the natural morphism $\xi^C_X : C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, X) \to X$ in $\mathcal{D}(R)$ is an isomorphism.

**2.4.1.** When $R$ has a dualizing complex $D$, given an $R$-complex $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, one has $\mathrm{G}_C\text{-}\dim_R(X) < \infty$ if and only if $X \in \mathcal{A}_{\mathbf{R}\mathrm{Hom}_R(C,D)}(R)$, by [12, (4.7)]; this uses 2.2.1 and 2.3.1, which imply that $\mathbf{R}\mathrm{Hom}_R(C, D)$ is semidualizing and $C \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C, D), D)$.

**2.4.2.** Let $R \to S$ be a flat local ring homormorphism. Given an $S$-complex $X$, one has $X \in \mathcal{A}_C(R)$ if and only if $X \in \mathcal{A}_{S\otimes^{\mathbf{L}}_R C}(S)$, by [12, (5.3.a)].

The following two lemmas are proved like [17, (4.4)] and [25, (7.3)], respectively.

**Lemma 2.1** *Let $X, P \in \mathcal{D}^f_b(R)$ such that $P \not\simeq 0$ and* $\mathrm{pd}_R(P) < \infty$. *Let $C$ be a semidualizing $R$-complex.*

(a) *The following conditions are equivalent*:

  (i) $X \in \mathcal{A}_C(R)$,
  (ii) $P \otimes^{\mathbf{L}}_R X \in \mathcal{A}_C(R)$, *and*
  (iii) $\mathbf{R}\mathrm{Hom}_R(P, X) \in \mathcal{A}_C(R)$.

(b) *The following conditions are equivalent*:

  (i) $X \in \mathcal{B}_C(R)$,
  (ii) $P \otimes^{\mathbf{L}}_R X \in \mathcal{B}_C(R)$, *and*
  (iii) $\mathbf{R}\mathrm{Hom}_R(P, X) \in \mathcal{B}_C(R)$.

**Lemma 2.2** *Let $R \to S$ be a flat local ring homomorphism such that $S/\mathfrak{m}S$ is Gorenstein. Let $X \in \mathcal{D}^f_b(S)$ such that each homology module* $\mathrm{H}_i(X)$ *is finitely generated over $R$.*

(a) *One has* $\mathrm{G}_C\text{-}\dim_R(X) < \infty$ *if and only if* $\mathrm{G}_{S\otimes^{\mathbf{L}}_R C}\text{-}\dim_S(X) < \infty$.
(b) *One has* $\mathrm{G}\text{-}\dim_R(X) < \infty$ *if and only if* $\mathrm{G}\text{-}\dim_S(X) < \infty$.

The next result, essentially from [8, Theorem 5(ii)], is key for Theorem 4.8.

**Lemma 2.3** *Let $I$ be an integrally closed ideal such that* $\mathrm{depth}(R/I) = 0$, *and let $M$ be a finitely generated $R$-module. If* $\mathrm{Tor}^R_i(R/I, M) = 0 = \mathrm{Tor}^R_{i+1}(R/I, M)$ *for some $i \geqslant 1$, then* $\mathrm{pd}_R(M) \leqslant i$. *In particular, $R/I$ is a* pd*-test module over $R$.*

*Proof* Assume that $\mathrm{Tor}^R_i(R/I, M) = 0 = \mathrm{Tor}^R_{i+1}(R/I, M)$, and suppose that $\mathrm{pd}_R(M) > i$. The Auslander–Buchsbaum–Serre Theorem states that $k$ is a pd-test $R$-module, so we assume without loss of generality that $I \subsetneq \mathfrak{m}$. Hence, we have $I : \mathfrak{m} \neq R$ so $I : \mathfrak{m} \subseteq \mathfrak{m}$. From [8, Theorem 5(ii)], we conclude that $\mathfrak{m}I = \mathfrak{m}(I : \mathfrak{m})$.

**Claim** $I : \mathfrak{m} = I$. One containment ($\supseteq$) is standard. For the reverse containment, let $r \in I : \mathfrak{m} \subseteq \mathfrak{m}$. To show that $r$ is in $I$, it suffices to show that $r$ is integral over $I$, since $I$ is integrally closed. To this end, we use the "determinantal trick" from [24, (1.1.8)]: it suffices to show that (1) we have $r\mathfrak{m} \subseteq I\mathfrak{m}$, and (2) whenever $a\mathfrak{m} = 0$ for some $a \in R$, we have $ar = 0$. For (1), since $r$ is in $I : \mathfrak{m}$, we have $r\mathfrak{m} \subseteq (I : \mathfrak{m})\mathfrak{m} = I\mathfrak{m}$, as desired. For (2), if

$a\mathfrak{m} = 0$, then the fact that $r$ is in $\mathfrak{m}$ implies that $ar = 0$, as desired. This completes the proof of the claim.

Now, the fact that $\operatorname{depth}(R/I) = 0$ implies that there is an element $x \in R \smallsetminus I$ such that $x\mathfrak{m} \subseteq I$. In other words, $x \in (I : \mathfrak{m}) \smallsetminus I$, contradicting the above claim. □

**2.8.** Let $I$ be an integrally closed ideal of $R$. If one assumes that $I$ is $\mathfrak{m}$-primary (stronger than the assumption $\operatorname{depth}(R/I) = 0$ from Lemma 2.3) then one gets the following very strong conclusion. Given a finitely generated $R$-module $M$, if $\operatorname{Tor}_i^R(R/I, M) = 0$ for some $i \geqslant 1$, then $\operatorname{pd}_R(M) < i$, by [14, (3.3)].

## 3 H-dim-test complexes

In this section, let $C$ be a semidualizing $R$-complex.

We now introduce the main object of study for this paper.

**Definition 3.1** Let $M \in \mathcal{D}_b^f(R)$, and let H-dim denote either projective dimension pd or $G_C$-dimension $G_C$-dim. Then $M$ is an H-dim-*test complex* over $R$ if the following condition holds for all $N \in \mathcal{D}_b^f(R)$: If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, i.e., if $M \otimes_R^{\mathbf{L}} N \in \mathcal{D}_b(R)$, then $\text{H-dim}_R(N) < \infty$.

**3.2.** Let $M$ be an $R$-module. A standard truncation argument shows that $M$ is a H-dim-test module if and only if it is a H-dim-test complex; see [9, Proof of (3.2)].

**3.3.** Examples of pd-test modules are given in 2.8 and Lemma 2.3. Note that this includes the standard example $k = R/\mathfrak{m}$. See also [10, Appendix A].

**3.3.1.** Given an $R$-complex $X \in \mathcal{D}_b^f(R)$, if $\operatorname{pd}_R(X)$ is finite, then so is $G_C\text{-dim}_R(X)$, by [12, (2.9)]. Thus, if $M$ is a pd-test complex, then it is also a $G_C$-dim-test, in particular, $M$ is a G-dim-test complex.

**3.3.2** If $R$ is G-regular (i.e., if every $R$-module of finite G-dimension has finite projective dimension), then the pd-test complexes and G-dim-test complexes over $R$ are the same. Examples of G-regular rings include regular rings and Cohen–Macaulay rings of minimal multiplicity; see [34].

**3.3.3.** Examples of $G_C$-dim test modules, e.g., of G-dim test modules, that are not pd test modules are more mysterious. See Example 3.11 for a non-trivial example.

**3.3.4.** Assume that $R$ has a dualizing complex $D$. A natural candidate for a $G_C$-dim-test complex is $C^\dagger = \mathbf{R}\operatorname{Hom}_R(C, D)$. Indeed, if $G_C\text{-dim}_R(X) < \infty$, then $X \in \mathcal{A}_{C^\dagger}(R)$ by 2.4.1, so by definition we have $C^\dagger \otimes_R^{\mathbf{L}} X \in \mathcal{D}_b(R)$. In particular, a natural candidate for a G-dim-test complex is $D$; see, e.g., Corollary 4.5.

However, $D$ can fail to be a G-dim-test complex. Indeed, Jorgensen and Şega [26, (1.7)] construct an artinian local ring $R$ with a finitely generated module $L$ that satisfies $\operatorname{Ext}_R^i(L, R) = 0$ for all $i \geqslant 1$ and $\text{G-dim}_R(L) = \infty$. Since $R$ is local and artinian, it has a dualizing complex, namely $D = E_R(k)$ the injective hull of the residue field $k$. We claim that $\operatorname{Tor}_i^R(D, L) = 0$ for $i \geqslant 1$. (This shows that $D$ is not G-dim-test over $R$.) To this end, recall the following for $i \geqslant 1$:

$$\operatorname{Hom}_R(\operatorname{Tor}_i^R(D, L), D) \cong \operatorname{Ext}_R^i(L, R) = 0.$$

The fact that $D$ is faithfully injective implies that $\operatorname{Tor}_i^R(D, L) = 0$ for $i \geqslant 1$.

**3.4.** The ring $R$ is regular if and only if every $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ has $\mathrm{pd}_R(X) < \infty$. Hence, the trivial complex 0 is a pd-test complex if and only if $R$ is regular, equivalently, if and only if $R$ has a pd-test complex of finite projective dimension. Similarly, if $C$ is a semidualizing $R$-complex, then 0 is a $\mathrm{G}_C$-dim-test complex if and only if $C$ is dualizing, equivalently, if and only if $R$ has a $\mathrm{G}_C$-dim-test complex of finite projective dimension; see 2.3.1. In particular, 0 is a G-dim-test complex if and only if $R$ is Gorenstein, equivalently, if and only if $R$ has a G-dim-test complex of finite projective dimension.

We continue with a discussion of ascent and descent of test complexes.

**Theorem 3.2** *Let $R \xrightarrow{\varphi} S$ be a flat local ring homomorphism, and let $M \in \mathcal{D}_b^f(R)$.*

(a) *If $S \otimes_R^{\mathbf{L}} M$ is $\mathrm{G}_{S\otimes_R^{\mathbf{L}} C}$-dim-test over $S$, then $M$ is $\mathrm{G}_C$-dim-test over $R$.*
(b) *If $S \otimes_R^{\mathbf{L}} M$ is G-dim-test over $S$, then $M$ is G-dim-test over $R$.*
(c) *If $S \otimes_R^{\mathbf{L}} M$ is pd-test over $S$, then $M$ is pd-test over $R$.*

*Proof* (a) Assume that $S \otimes_R^{\mathbf{L}} M$ is a $\mathrm{G}_{S\otimes_R^{\mathbf{L}} C}$-dim-test complex over $S$. To show that $M$ is a $\mathrm{G}_C$-dim-test complex over $R$, let $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ such that $M \otimes_R^{\mathbf{L}} X \in \mathcal{D}_{\mathrm{b}}(R)$. By flatness, the complexes $S \otimes_R^{\mathbf{L}} X$, $S \otimes_R^{\mathbf{L}} M$, and $S \otimes_R^{\mathbf{L}} (M \otimes_R^{\mathbf{L}} X)$ are all in $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$. Moreover, we have the following isomorphisms in $\mathcal{D}(S)$:

$$(S \otimes_R^{\mathbf{L}} M) \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} X) \simeq (S \otimes_R^{\mathbf{L}} M) \otimes_R^{\mathbf{L}} X \simeq S \otimes_R^{\mathbf{L}} (M \otimes_R^{\mathbf{L}} X).$$

As $S \otimes_R^{\mathbf{L}} M$ is a $\mathrm{G}_{S\otimes_R^{\mathbf{L}} C}$-dim-test complex over $S$, we have $\mathrm{G}_{S\otimes_R^{\mathbf{L}} C}\text{-}\mathrm{dim}_S(S \otimes_R^{\mathbf{L}} X) < \infty$. Using 2.3.2, we conclude that $\mathrm{G}_C\text{-}\mathrm{dim}_R(X) < \infty$, as desired.
(b) This is the special case $C = R$ of part (a).
(c) Argue as in part (a), using [6, (1.5.3)] in place of 2.3.2. □

Note that the conditions on $\varphi$ in the following three items hold for the natural maps from $R$ to its completion $\widehat{R}$ or to its henselization $R^{\mathrm{h}}$.

*Remark 3.3* Let $\varphi\colon R \to S$ be a flat local ring homomorphism, and assume the closed fibre $S/\mathfrak{m}S$ is module-finite over $k$. Let $M \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(S)$ such that $(S \otimes_R^{\mathbf{L}} M) \otimes_S^{\mathbf{L}} X \in \mathcal{D}_{\mathrm{b}}(S)$. Let $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ be a generating sequence for $\mathfrak{m}$, and set $K := K^S(\mathbf{x})$, the Koszul complex on $\mathbf{x}$ over $S$. It follows that $K \otimes_S^{\mathbf{L}} ((S \otimes_R^{\mathbf{L}} M) \otimes_S^{\mathbf{L}} X) \in \mathcal{D}_{\mathrm{b}}(S)$. From the following isomorphisms

$$K \otimes_S^{\mathbf{L}} ((S \otimes_R^{\mathbf{L}} M) \otimes_S^{\mathbf{L}} X) \simeq (K \otimes_S^{\mathbf{L}} X) \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} M) \simeq (K \otimes_S^{\mathbf{L}} X) \otimes_R^{\mathbf{L}} M$$

we conclude that $(K \otimes_S^{\mathbf{L}} X) \otimes_R^{\mathbf{L}} M \in \mathcal{D}_{\mathrm{b}}(R)$.

Note that $K \otimes_S^{\mathbf{L}} X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$. Indeed, we already have $K \otimes_S^{\mathbf{L}} X \in \mathcal{D}_{\mathrm{b}}(R)$, so it suffices to show that every homology module $\mathrm{H}_i(K \otimes_S^{\mathbf{L}} X)$ is finitely generated over $R$. We know that $\mathrm{H}_i(K \otimes_S^{\mathbf{L}} X)$ is finitely generated over $S$. Moreover, it is annihilated by $(\mathbf{x})S = \mathfrak{m}S$. Thus, it is a finitely generated $S/\mathfrak{m}S$-module; since $S/\mathfrak{m}S$ is module finite over $k$, each $\mathrm{H}_i(K \otimes_S^{\mathbf{L}} X)$ is finitely generated over $k$, so it is finitely generated over $R$.

**Theorem 3.4** *Let $\varphi\colon R \to S$ be a flat local ring homomorphism, and let $M \in \mathcal{D}_b^f(R)$. Assume that the closed fibre $S/\mathfrak{m}S$ is Gorenstein and module-finite over $k$.*

(a) *The $R$-complex $M$ is $\mathrm{G}_C$-dim-test if and only if $S \otimes_R^{\mathbf{L}} M$ is $\mathrm{G}_{S\otimes_R^{\mathbf{L}} C}$-dim-test over $S$.*
(b) *The $R$-complex $M$ is G-dim-test if and only if $S \otimes_R^{\mathbf{L}} M$ is G-dim-test over $S$.*

*Proof* (a) One implication is covered by Theorem 3.2(a). For the reverse implication, assume that $M$ is a $\mathrm{G}_C$-dim-test complex over $R$. To show that $S \otimes^{\mathbf{L}}_R M$ is a $\mathrm{G}_{S\otimes^{\mathbf{L}}_R C}$-dim-test complex over $S$, let $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ such that $(S \otimes^{\mathbf{L}}_R M) \otimes^{\mathbf{L}}_S X \in \mathcal{D}_{\mathrm{b}}(S)$. Let $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ be a generating sequence for $\mathfrak{m}$, and set $K := K^S(\mathbf{x})$. By Remark 3.3 we have $(K \otimes^{\mathbf{L}}_S X) \otimes^{\mathbf{L}}_R M \in \mathcal{D}_{\mathrm{b}}(R)$ and $K \otimes^{\mathbf{L}}_S X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. Since $M$ is an $\mathrm{G}_C$-dim-test complex over $R$, one has $\mathrm{G}_C\text{-}\dim_R(K \otimes^{\mathbf{L}}_S X) < \infty$. It follows from Lemma 2.2 that $\mathrm{G}_{S\otimes^{\mathbf{L}}_R C}\text{-}\dim_S(K \otimes^{\mathbf{L}}_S X) < \infty$. We deduce from [17, (4.4)] that $\mathrm{G}_{S\otimes^{\mathbf{L}}_R C}\text{-}\dim_S(X) < \infty$, as desired.

(b) This is the special case $C = R$ of part (a). □

Here is one of our main results; see also Corollary 3.9 and Theorem 4.4.

**Theorem 3.5** *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism, and let $M \in \mathcal{D}^f_b(R)$. Assume the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is a finite field extension, i.e., we have $\mathfrak{m}S = \mathfrak{n}$ and the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is finite. Then $M$ is a* pd-*test complex over $R$ if and only if $S \otimes^{\mathbf{L}}_R M$ is a* pd-*test complex over $S$.*

*Proof* One implication is covered by Theorem 3.2(c). For the reverse implication, assume that $M$ is a pd-test complex over $R$.

*Case 1* $S$ is complete. To show that $S \otimes^{\mathbf{L}}_R M$ is a pd-test complex over $S$, let $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(S)$ such that $(S \otimes^{\mathbf{L}}_R M) \otimes^{\mathbf{L}}_S X \in \mathcal{D}_{\mathrm{b}}(S)$. Let $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ be a generating sequence for $\mathfrak{m}$, and set $K := K^S(\mathbf{x})$. It follows that $K \otimes^{\mathbf{L}}_S ((S \otimes^{\mathbf{L}}_R M) \otimes^{\mathbf{L}}_S X) \in \mathcal{D}_{\mathrm{b}}(S)$. By Remark 3.3 we have $(K \otimes^{\mathbf{L}}_S X) \otimes^{\mathbf{L}}_R M \in \mathcal{D}_{\mathrm{b}}(R)$ and $K \otimes^{\mathbf{L}}_S X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. As $M$ is an pd-test complex over $R$, we have $\mathrm{pd}_R(K \otimes^{\mathbf{L}}_S X) < \infty$. It follows from [7, (2.5)] that $\mathrm{pd}_S(K \otimes^{\mathbf{L}}_S X) < \infty$, and [6, (1.5.3)] implies that $\mathrm{pd}_S(X) < \infty$.

*Case 2* The general case. Case 1 implies that $\widehat{S} \otimes^{\mathbf{L}}_R M$ is a pd-test complex over $\widehat{S}$, i.e., $\widehat{S} \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R M)$ is a pd-test complex over $\widehat{S}$. So, $S \otimes^{\mathbf{L}}_R M$ is a pd-test complex over $S$, by Theorem 3.2(c), as desired. □

The next example, from discussions with Ryo Takahashi, shows that the hypothesis $\mathfrak{m}S = \mathfrak{n}$ is necessary for the conclusion of Theorem 3.5.

*Example 3.6* Let $k$ be a field, $R = k$ and $S = k[\![y]\!]/(y^2)$. Then the natural map $R \to S$ is a finite free map since $S$ is free over $R$ with $R$-basis $\{1, y\}$. Let $M = R$. Then, since $R$ is regular, $M$ is a pd-test module over $R$. However $N = M \otimes_R S = S$ is not a pd-test module over $S$ since $S$ is not regular; see 3.4.

On the other hand, we do not know whether or not having a regular closed fibre in Theorem 3.5 is sufficient, as we note next.

**Question 3.7** Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism, and let $M \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. Assume that $S/\mathfrak{m}S$ is regular. If $M$ is a pd-test complex over $R$, then must $S \otimes^{\mathbf{L}}_R M$ be a pd-test complex over $S$?

The next corollary answers Question 1.3(a).

**Corollary 3.8** *Let $M$ be an $R$-module, and set $\widehat{C} := \widehat{R} \otimes^{\mathbf{L}}_R C$.*

(a) *The module $M$ is* $\mathrm{G}_C$-dim-*test over $R$ if and only if $\widehat{M}$ is* $\mathrm{G}_{\widehat{C}}$-dim-*test over $\widehat{R}$.*

(b) *The module $M$ is* G-dim-*test over $R$ if and only if $\widehat{M}$ is* G-dim-*test over $\widehat{R}$.*
(c) *The module $M$ is* pd-*test over $R$ if and only if $\widehat{M}$ is* pd-*test over $\widehat{R}$.*

*Proof* Since $\widehat{R} \otimes^{\mathbf{L}}_R M \simeq \widehat{M}$ in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(\widehat{R})$, the desired conclusions follow from Theorems 3.4 and 3.5. □

The next corollary answers Question 1.3(b). We are able to improve this result significantly in the next section; see Theorem 4.4 and the subsequent paragraph.

**Corollary 3.9** *Let $M$ be a* pd-*test module over $R$. If* G-dim$_R(M) < \infty$*, then $R$ is Gorenstein.*

*Proof* Corollary 3.8(c) says that $\widehat{M}$ is a pd-test module for $\widehat{R}$, with G-dim$_{\widehat{R}}(\widehat{M}) < \infty$ by 2.3.2. Using [9, (1.3)], we conclude that $\widehat{R}$ is Gorenstein, hence so is $R$. □

We end this section by building a module that is G-dim-test but not pd-test; see Example 3.11.

**Proposition 3.10** *Let $\varphi\colon (A, \mathfrak{n}, F) \to R$ be a flat local ring homomorphism, and set $N := R/\mathfrak{n}R \simeq R \otimes^{\mathbf{L}}_A F$.*

(a) *The $R$-module $N$ is Tor-rigid, i.e., for any finitely generated $R$-module $M$, if $\mathrm{Tor}^R_i(M, N) = 0$ for some $i \geqslant 1$, then $\mathrm{Tor}^R_j(M, N) = 0$ for all $j \geqslant i$.*
(b) *Let $B$ be a semidualizing $A$-complex, and set $C := R \otimes^{\mathbf{L}}_A B$. If $R/\mathfrak{n}R$ is Gorenstein, then $N$ is a* G$_C$-dim-*test complex over $R$.*
(c) *If $R/\mathfrak{n}R$ is Gorenstein, then $N$ is a* G-dim-*test complex over $R$.*

*Proof* First, note that we have $N := R/\mathfrak{n}R \cong R \otimes_A (A/\mathfrak{n}) = R \otimes_A F \simeq R \otimes^{\mathbf{L}}_A F$ since $R$ is flat over $A$. Furthermore, for every $R$-complex $M$, there are isomorphisms

$$M \otimes^{\mathbf{L}}_R N \simeq M \otimes^{\mathbf{L}}_R (R \otimes^{\mathbf{L}}_A F) \simeq M \otimes^{\mathbf{L}}_A F. \tag{3.1}$$

In particular, one has $\mathrm{Tor}^R_i(M, N) \cong \mathrm{Tor}^A_i(M, F)$ for all $i$.

(a) Let $M$ be a finitely generated $R$-module. From [29, (22.3)], if $\mathrm{Tor}^A_1(M, F) = 0$, then $M$ is flat over $A$, so we have $\mathrm{Tor}^A_j(M, F) = 0$ for all $j \geqslant 1$. More generally, by dimension-shifting, if $\mathrm{Tor}^A_i(M, F) = 0$ for some $i \geqslant 1$, then we have $\mathrm{Tor}^A_j(M, F) = 0$ for all $j \geqslant i$. Thus, the isomorphism from the previous paragraph implies the desired Tor-rigidity.

(b) Corollary 3.8(a) shows that it suffices to show that $\widehat{N}$ is a $\mathrm{G}_{\widehat{R} \otimes^{\mathbf{L}}_R C}$-dim-test complex over $\widehat{R}$. Note that the induced map $\widehat{\varphi}\colon \widehat{A} \to \widehat{R}$ is flat and local with Gorenstein closed fibre $\widehat{R/\mathfrak{n}R}$. Also, there are isomorphisms in $\mathcal{D}(\widehat{R})$:

$$\widehat{R} \otimes^{\mathbf{L}}_{\widehat{A}} (\widehat{A} \otimes^{\mathbf{L}}_A B) \simeq \widehat{R} \otimes^{\mathbf{L}}_A B \simeq \widehat{R} \otimes^{\mathbf{L}}_R (R \otimes^{\mathbf{L}}_A B) \simeq \widehat{R} \otimes^{\mathbf{L}}_R C.$$

Thus, we may replace $\varphi$ with the induced map $\widehat{\varphi}$ to assume for the rest of the proof that $A$ and $R$ are complete. Let $D^A$ be a dualizing $A$-complex; see 2.2.1. Then the $R$-complex $D^R := R \otimes^{\mathbf{L}}_A D^A$ is dualizing for $R$, by 2.2.2. Set $B^\dagger := \mathbf{R}\mathrm{Hom}_A(B, D^A)$ and $C^\dagger := \mathbf{R}\mathrm{Hom}_R(C, D^R)$, noting that $C^\dagger \simeq R \otimes^{\mathbf{L}}_A B^\dagger$ by [12, proof of (5.10)(*)].

Let $M \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ such that $M \otimes^{\mathbf{L}}_R N \in \mathcal{D}_{\mathrm{b}}(R)$. We need to show $\mathrm{G}_C\text{-dim}_R(M) < \infty$ i.e., that $M \in \mathcal{A}_{C^\dagger}(R)$; see 2.4.1. By (3.1), the complex $M \otimes^{\mathbf{L}}_A F$ is in $\mathcal{D}_{\mathrm{b}}(R)$. Since $M$ is in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, we conclude that $\mathrm{fd}_A(M) < \infty$ by [4, (5.5.F)]. It follows from [12, (4.4)] that $M \in \mathcal{A}_{B^\dagger}(A)$, so $M \in \mathcal{A}_{R \otimes^{\mathbf{L}}_A B^\dagger}(R) = \mathcal{A}_{C^\dagger}(R)$ by [12, (5.3.a)].

(c) This is the special case $B = A$, hence $C = R$, of part (b). □

*Example 3.11* Let $k$ be a field. Consider the finite-dimensional local $k$-algebras $A := k[y,z]/(y,z)^2$ and

$$R := k[x,y,z]/(x^2,y^2,z^2,yz) \cong A[x]/(x^2).$$

Notice that $R$ is free over $A$, hence flat. Also, the natural map $A \to R$ is local with Gorenstein closed fibre $R/\mathfrak{n}R \cong k[x]/(x^2)$; here, as in Proposition 3.10, we let $\mathfrak{n}$ denote the maximal ideal of $A$. Since the assumptions of Proposition 3.10 are satisfied, the $R$-module

$$N = R \otimes_A k = R \otimes_A (A/(y,z)A) \cong R/(y,z)R$$

is Tor-rigid and G-dim-test over $R$. Furthermore, from [9, (4.1)], we know that $N$ is not pd-test.

Since $A$ is artinian and local, the injective hull $D^A = E_R(k)$ is a dualizing $A$-module. Thus, the $R$-module $D^R := R \otimes_A D^A \simeq R \otimes^{\mathbf{L}}_A D^A$ is dualizing for $R$, by 2.2.2. We conclude by showing that $D^R$ is also a G-dim-test module that is not pd-test and, moreover, is not Tor-rigid.

Note that $A$ has length 3 and type 2. From this, we construct an exact sequence over $A$ of the following form:

$$0 \to k^3 \to A^2 \to D^A \to 0. \tag{3.2}$$

Indeed, the condition $\operatorname{type}(A) = 2$ says that $D^A$ is minimally generated by 2 elements. Let $A^2 \xrightarrow{p} D^A \to 0$ be a minimal presentation, and consider the corresponding short exact sequence

$$0 \to \operatorname{Ker}(p) \to A^2 \xrightarrow{p} D^A \to 0.$$

From the minimality of the presentation, it follows that $\operatorname{Ker}(p) \subseteq \mathfrak{n}A^2$. Since $\mathfrak{n}^2 = 0$, we conclude that $\operatorname{Ker}(p)$ is a $k$-vector space, so we need only verify that $\operatorname{len}_A(\operatorname{Ker}(p)) = 3$. This equality follows from the additivity of length, via the condition $\operatorname{len}_A(D^A) = \operatorname{len}(A) = 3$.

Since $R$ is flat over $A$, we apply the base-change functor $R \otimes_A -$ to the sequence (3.2) to obtain the next exact sequence over $R$:

$$0 \to N^3 \to R^2 \to D^R \to 0.$$

For any $R$-module $M$, the associate long exact in $\operatorname{Tor}_i^R(M,-)$ shows that we have $\operatorname{Tor}_i^R(M,D^R) \cong \operatorname{Tor}_{i-1}^R(M,N)^3$ for all $i \geqslant 2$. In particular, we have

$$\operatorname{Tor}_i^R(M,D^R) = 0 \text{ if and only if } \operatorname{Tor}_{i-1}^R(M,N) = 0, \text{ for } i \geqslant 2. \tag{3.3}$$

**Claim** $D^R$ is G-dim-test over $R$. To show this, let $M$ be a finitely generated $R$-module such that $\operatorname{Tor}_i^R(M,D^R) = 0$ for $i \gg 0$. The display (3.3) implies that $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$. Since $N$ is G-dim-test over $R$, we have $\operatorname{G-dim}_R(M) < \infty$, as desired.

**Claim** $D^R$ is not pd-test over $R$. To show this, suppose by way of contradiction that $D^R$ is pd-test over $R$. We show that $N$ is pd-test, contradicting [9, (4.1)]. Let $M$ be a finitely generated $R$-module such that $\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$. The display (3.3) implies that $\operatorname{Tor}_i^R(M,D^R) = 0$ for $i \gg 0$. Since $D^R$ is pd-test, we have $\operatorname{pd}_R(M) < \infty$. Thus, $N$ is pd-test, giving the advertised contradiction, and establishing the claim.

**Claim** If $M$ is a finitely generated $R$-module such that $\mathrm{Tor}_i^R(M, D^R) = 0$ for some $i \geqslant 2$, then $\mathrm{Tor}_j^R(M, D^R) = 0$ for all $j \geqslant i$. (This shows that $D^R$ is almost Tor-rigid.) Since $N$ is Tor-rigid by Proposition 3.10(a), this follows from (3.3).

**Claim** $D^R$ is not Tor-rigid over $R$. To this end, we follow a construction of [15, Chapter 3] and build an $R$-module $L$ such that $\mathrm{Tor}_1^R(L, D^R) = 0$ and $\mathrm{Tor}_i^R(L, D^R) \neq 0$ for all $i \geqslant 2$. Let $f_1, f_2$ be a minimal generating sequence for $\mathrm{Hom}_A(k, A) \cong k^2$. For instance, $f_1(1) = y$ and $f_2(1) = z$ will work here. Define $u\colon k \to A^2$ by the formula $a \mapsto (f_1(a), f_2(a))$. Since $k$ is simple, the non-zero map $u$ is a monomorphism. Let $M := \mathrm{Coker}(u)$. The long exact sequence in $\mathrm{Ext}_A(-, A)$ associated to the sequence

$$0 \to k \xrightarrow{u} A^2 \to M \to 0 \tag{3.4}$$

shows that $\mathrm{Ext}_A^1(M, A) = 0$.

Set $L := R \otimes_A M$. To make things concrete, if one uses the specific functions $f_1, f_2$ suggested in the previous paragraph, then $M$ has the following minimal free presentation over $A$

$$A \xrightarrow{\binom{y}{z}} A^2 \to M \to 0$$

so $L$ has the following minimal free presentation over $R$

$$R \xrightarrow{\binom{y}{z}} R^2 \to L \to 0.$$

Now, flat base-change implies that

$$\mathrm{Ext}_R^1(L, R) \cong R \otimes_A \mathrm{Ext}_A^1(M, A) = 0.$$

Thus, we have

$$\mathrm{Hom}_R(\mathrm{Tor}_1^R(D^R, L), D^R) \cong \mathrm{Ext}_R^1(L, R) = 0.$$

The fact that $D^R$ is faithfully injective over $R$ implies that $\mathrm{Tor}_1^R(D^R, L) = 0$.

Since $A$ is not regular, we have $\mathrm{pd}_A(k) = \infty$ and $\mathrm{Tor}_i^A(k, k) \neq 0$ for all $i > 0$. Therefore, by (3.4), we conclude that $\mathrm{pd}_A(M) = \infty$. In particular, this implies that $\mathrm{Tor}_1^A(k, M) \neq 0$. So it follows from (3.2) and (3.4) that

$$\mathrm{Tor}_i^A(D^A, M) \cong \mathrm{Tor}_{i-1}^A(k^3, M) \cong \mathrm{Tor}_{i-2}^A(k^3, k) \cong (\mathrm{Tor}_{i-2}^A(k, k))^3 \neq 0 \quad \text{for all } i > 2$$

and

$$\mathrm{Tor}_2^A(D^A, M) \cong \mathrm{Tor}_1^A(k^3, M) \cong (\mathrm{Tor}_1^A(k, M))^3 \neq 0.$$

Thus flat base-change implies that

$$\mathrm{Tor}_i^R(D^R, L) \cong R \otimes_A \mathrm{Tor}_i^A(D^A, M) \neq 0 \text{ for all } i \geqslant 2.$$

This completes the claim and the example.

## 4 Detecting the dualizing and Gorenstein properties

Our next result yields both Theorems 4.4 and 4.8 highlighted in the introduction. Note that condition (2) in this result does not assume *a priori* that $R$ has a dualizing complex; however, the result shows that this condition implies that $R$ has a dualizing complex.

**Theorem 4.1** *Let $B$, $C$ be semidualizing $R$-complexes. Let $M \not\simeq 0$ be a $\mathrm{G}_B$-dim-test $R$-complex such that $\mathbf{R}\mathrm{Hom}_R(M, C) \in \mathcal{D}_b(R)$. Assume that one of the following conditions holds:*

(1) *The ring $R$ has a dualizing complex $D$, and $\mathrm{G}_{B^\dagger}\text{-}\mathrm{dim}_R(C) < \infty$ where $B^\dagger := \mathbf{R}\mathrm{Hom}_R(B, D)$, or*
(2) *One has $C \in \mathcal{A}_B(R)$.*

*Then $B \otimes^{\mathbf{L}}_R C$ is dualizing for $R$.*

*Proof* (1) Assume that $R$ has a dualizing complex $D$, and $\mathrm{G}_{B^\dagger}\text{-}\mathrm{dim}_R(C) < \infty$. Set $C^\dagger := \mathbf{R}\mathrm{Hom}_R(C, D)$ which is semidualizing by 2.2.1. Let $\mathbf{x} = x_1, \dots, x_n \in \mathfrak{m}$ be a generating sequence for $\mathfrak{m}$, and consider the Koszul complex $K := K^R(\mathbf{x})$. Set $X := K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, E)$ where $E = E_R(k)$ is the injective hull of $k$.

We note that $X$ is in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. Indeed, the complex $\mathbf{R}\mathrm{Hom}_R(K, C)$ is homologically finite since $C$ is. The total homology module $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(K, C))$ is annihilated by $(\mathbf{x})R = \mathfrak{m}$, so it is a finite dimensional vector space over $k$. By Matlis duality, the total homology module of $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K, C), E)$ is also a finite dimensional vector space over $k$, so we have

$$X = K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, E) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K, C), E) \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R) \tag{4.1}$$

by Hom-evaluation [4, (4.4)].

The assumption $\mathbf{R}\mathrm{Hom}_R(M, C) \in \mathcal{D}_{\mathrm{b}}(R)$ implies that

$$M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, E) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, C), E) \in \mathcal{D}_{\mathrm{b}}(R)$$

again by Hom-evaluation [4, (4.4)]. From this, we conclude that

$$M \otimes^{\mathbf{L}}_R X = M \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, E)) \simeq K \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, E)) \in \mathcal{D}_{\mathrm{b}}(R).$$

Since $M$ is a $\mathrm{G}_B$-dim-test complex, this implies that $\mathrm{G}_B\text{-}\mathrm{dim}_R(X) < \infty$. From 2.4.1 we conclude that $X \in \mathcal{A}_{B^\dagger}(R)$, i.e., we have $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K, C), E) \in \mathcal{A}_{B^\dagger}(R)$ by (4.1). Since $E$ is faithfully injective, argue as in the proof of [11, (3.2.9)] to conclude that $\mathbf{R}\mathrm{Hom}_R(K, C) \in \mathcal{B}_{B^\dagger}(R)$; see also [32, 4.14]. By assumption, we have $C \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, so Lemma 2.1(b) shows that $C \in \mathcal{B}_{B^\dagger}(R)$.

By assumption, we have $\mathrm{G}_{B^\dagger}\text{-}\mathrm{dim}_R(C) < \infty$, so [18, (1.3)] implies that $B^\dagger \in \mathcal{B}_C(R)$. We conclude from [18, (1.4) and (4.10.4)] that $C$ and $B^\dagger$ are isomorphic up to a shift in $\mathcal{D}(R)$. Apply a shift to $C$ to assume that $C \simeq B^\dagger$. From [12, (4.4)] we have $D \simeq B \otimes^{\mathbf{L}}_R B^\dagger \simeq B \otimes^{\mathbf{L}}_R C$, as desired.

(2) Assume now that $C \in \mathcal{A}_B(R)$. The completion $\widehat{R}$ has a dualizing complex $D^{\widehat{R}}$, by 2.2.1, and the complexes $\widehat{R} \otimes^{\mathbf{L}}_R B$ and $\widehat{R} \otimes^{\mathbf{L}}_R C$ are semidualizing over $\widehat{R}$, by 2.2.2. We have $\widehat{R} \otimes^{\mathbf{L}}_R M \not\simeq 0$ by faithful flatness of $\widehat{R}$, and the complex $\widehat{R} \otimes^{\mathbf{L}}_R M$ is $\mathrm{G}_{\widehat{R} \otimes^{\mathbf{L}}_R B}$-dim-test by Theorem 3.4(a). Also, by faithful flatness, the condition $\mathbf{R}\mathrm{Hom}_R(M, C) \in \mathcal{D}_{\mathrm{b}}(R)$ implies that

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{R} \otimes^{\mathbf{L}}_R M, \widehat{R} \otimes^{\mathbf{L}}_R C) \simeq \widehat{R} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, C) \in \mathcal{D}_{\mathrm{b}}(\widehat{R})$$

by [17, (1.9.a)].

With $(-)^{\dagger} := \mathbf{R}\mathrm{Hom}_{\widehat{R}}(-, D^{\widehat{R}})$, we have an isomorphism $\widehat{R}\otimes^{\mathbf{L}}_{R} B \simeq (\widehat{R}\otimes^{\mathbf{L}}_{R} B)^{\dagger\dagger}$ by 2.3.1. In addition, from [12, (5.8)], the assumption $C \in \mathcal{A}_B(R)$ implies that we have $\widehat{R} \otimes^{\mathbf{L}}_{R} C \in \mathcal{A}_{\widehat{R}\otimes^{\mathbf{L}}_{R} B}(\widehat{R}) = \mathcal{A}_{(\widehat{R}\otimes^{\mathbf{L}}_{R} B)^{\dagger\dagger}}(\widehat{R})$. We conclude from 2.4.1 that $\mathrm{G}_{(\widehat{R}\otimes^{\mathbf{L}}_{R} B)^{\dagger}}\text{-}\dim_{\widehat{R}}(\widehat{R} \otimes^{\mathbf{L}}_{R} C) < \infty$.

It follows that condition (1) is satisfied over $\widehat{R}$. Thus, the $\widehat{R}$-complex

$$(\widehat{R} \otimes^{\mathbf{L}}_{R} B) \otimes^{\mathbf{L}}_{\widehat{R}} (\widehat{R} \otimes^{\mathbf{L}}_{R} C) \simeq (\widehat{R} \otimes^{\mathbf{L}}_{R} B) \otimes^{\mathbf{L}}_{R} C \simeq \widehat{R} \otimes^{\mathbf{L}}_{R} (B \otimes^{\mathbf{L}}_{R} C)$$

is dualizing for $\widehat{R}$. Note that the condition $C \in \mathcal{A}_B(R)$ implies by definition that $B \otimes^{\mathbf{L}}_{R} C \in \mathcal{D}_{\mathrm{b}}(R)$. Thus, the condition $B, C \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ implies that $B \otimes^{\mathbf{L}}_{R} C \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. From this, the fact that $\widehat{R} \otimes^{\mathbf{L}}_{R} (B \otimes^{\mathbf{L}}_{R} C)$ is dualizing for $\widehat{R}$ implies that $B \otimes^{\mathbf{L}}_{R} C$ is dualizing for $R$, by [5, (5.1)], as desired. □

We now give several consequences of Theorem 4.1. Compare the next result to [9, (3.4)].

**Corollary 4.2** *Let $M$ be a* G-dim*-test $R$-complex. Let $C$ be a semidualizing $R$-complex such that* $\mathbf{R}\mathrm{Hom}_R(M, C) \in \mathcal{D}_b(R)$*. Then $C$ is dualizing for $R$.*

*Proof* By 3.4, we assume that $M \not\simeq 0$. The desired conclusion follows from Theorem 4.1 with $B = R$, once we note that $C \in \mathcal{D}_{\mathrm{b}}(R) = \mathcal{A}_R(R)$. □

**Corollary 4.3** *Let $M$ be a* G-dim*-test $R$-complex such that* $\mathbf{R}\mathrm{Hom}_R(M, R) \in \mathcal{D}_b(R)$*. Then $R$ is Gorenstein.*

*Proof* Use $C = R$ in Corollary 4.2. □

**Theorem 4.4** *Let $M$ be a* G-dim*-test $R$-module such that* $\mathrm{Ext}^i_R(M, R) = 0$ *for $i \gg 0$, e.g., with* $\mathrm{G}\text{-}\dim_R(M) < \infty$*. Then $R$ is Gorenstein.*

*Proof* This is immediate from Corollary 4.3. □

We note that the hypotheses of Theorem 4.4 are weaker than those in Corollary 3.9. Indeed, Example 3.11 above exhibits a G-dim-test module that is not a pd-test module. Furthermore, as we noted in 3.3.4, there exist examples of finitely generated modules $L$ such that $\mathrm{Ext}^i_R(L, R) = 0$ for all $i \geqslant 1$ and $\mathrm{G}\text{-}\dim_R(L) = \infty$.

**Corollary 4.5** *Let $C$ be a semidualizing $R$-complex. If $C$ is* G-dim*-test over $R$, then $C$ is dualizing for $R$.*

*Proof* This follows from Corollary 4.2 since $\mathbf{R}\mathrm{Hom}_R(C, C) \simeq R \in \mathcal{D}_{\mathrm{b}}(R)$. □

*Remark 4.6* In light of Corollary 4.5, it is worth noting that there are rings with semidualizing complexes that are not dualizing and that have infinite projective dimension. (In particular, these complexes are neither pd-test nor G-dim-test by 3.3.1 and Corollary 4.5.) The first examples were constructed (though not published) by Foxby. See also [12, (7.8)] and [30,31].

It is also worth noting that the converse of Corollary 4.5 fails in general by 3.3.4.

**Corollary 4.7** *Let $I \subseteq R$ be an integrally closed ideal such that* $\mathrm{depth}(R/I) = 0$*, e.g., such that $I$ is $\mathfrak{m}$-primary, and let $C$ be a semidualizing $R$-complex. Then $C$ is dualizing for $R$ if and only if* $\mathrm{G}_C\text{-}\dim_R(R/I) < \infty$.

*Proof* Note that $I$ is a pd-test by Lemma 2.3, and apply 2.3.1 and Corollary 4.2. □

Recall that the next result has been initially obtained by Goto and Hayasaka [21, (1.1)] under some extra conditions; see also [22, (2.2)].

**Theorem 4.8** *Let $I \subseteq R$ be an integrally closed ideal with* $\operatorname{depth}(R/I) = 0$*, e.g., such that $I$ is $\mathfrak{m}$-primary. Then $R$ is Gorenstein if and only if* $\text{G-dim}_R(R/I) < \infty$.

*Proof* Apply Corollary 4.7 with $C = R$, or use Theorem 4.4 with Lemma 2.3. □

We finish this section by giving two examples that show the integrally closed and depth hypotheses of Theorem 4.8 are necessary:

*Example 4.9* Let $k$ be a field and let $R = k[\![x, y, z]\!]/(x^2, y^2, z^2, yz)$, as in Example 3.11. Then, since $R$ is Artinian, each proper ideal of $R$ is $\mathfrak{m}$-primary but is not integrally closed; see [24, (1.1.3)(3)]. In particular the principal ideal $I$ of $R$ generated by $x$ is $\mathfrak{m}$-primary but not integrally closed. Note that $R$ is not Gorenstein. Also, we have $\text{G-dim}(I) = 0$ because the fact that $R$ is of the form $A[x]/(x^2)$ implies that $I$ has a complete resolution $\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots$.

*Example 4.10* Let $k$ be a field, set $S := k[\![x_1, x_2, x_3, y_1, y_2, y_3]\!]$, and let $I$ be the ideal of $S$ generated by $x_1y_2 - x_2y_1$, $x_1y_3 - x_3y_1$, and $x_2y_3 - y_2x_3$. Set $R := S/I$. Then $R$ is a four-dimensional normal Cohen-Macaulay domain that is not Gorenstein; see [20, Theorem (a)]. Let $0 \neq f \in \mathfrak{m}$. Then the ideal $fR$ of $R$ generated by $f$ is integrally closed; see [24, (1.5.2)]. Furthermore $\operatorname{pd}_R(R/fR) = \text{G-dim}_R(R/fR) = 1$ and $\operatorname{depth}(R/fR) = 3$.

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