

A DIMENSION INEQUALITY FOR COHEN-MACAULAY RINGS

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ABSTRACT. The recent work of Kurano and Roberts on Serre's positivity conjecture suggests the following dimension inequality: for prime ideals \mathfrak{p} and \mathfrak{q} in a local, Cohen-Macaulay ring (A, \mathfrak{n}) such that $e(A_{\mathfrak{p}}) = e(A)$ we have $\dim(A/\mathfrak{p}) + \dim(A/\mathfrak{q}) \leq \dim(A)$. We establish this dimension inequality for excellent, local, Cohen-Macaulay rings which contain a field, for certain low-dimensional cases and when R/\mathfrak{p} is regular.

1. INTRODUCTION

Let (R, \mathfrak{m}) be a local Noetherian ring of dimension d , and let M and N be finitely generated R -modules such that M has finite projective dimension and $M \otimes_R N$ is a module of finite length. Serre [13] defined the *intersection multiplicity* of M and N to be

$$\chi(M, N) = \sum_{i=0}^n (-1)^i \text{length}(\text{Tor}_i^R(M, N))$$

and conjectured that $\chi(M, N)$ satisfies the following properties when R is regular:

1. $\dim(M) + \dim(N) \leq \dim(R)$.
2. (Nonnegativity) $\chi(M, N) \geq 0$.
3. (Vanishing) If $\dim(M) + \dim(N) < \dim(R)$, then $\chi(M, N) = 0$.
4. (Positivity) If $\dim(M) + \dim(N) = \dim(R)$, then $\chi(M, N) > 0$.

Serre was able to verify the first statement for any regular local ring and the others in the case when R is unramified. Since $\chi(M, N)$ has many of the characteristics we desire from an intersection multiplicity (for example, Bézout's Theorem holds), it was reasonable to suppose that these further properties are satisfied over an arbitrary regular local ring. The results were left unproved for ramified rings.

The vanishing conjecture was proved about ten years ago by Gillet and Soulé [3] and independently by Roberts [11] using K -theoretic methods. Gabber proved the nonnegativity conjecture recently [1, 6, 12] using a theorem of de Jong [2]. Kurano and Roberts have proved the following using methods introduced by Gabber.

Theorem 1.1 ([7] Theorem 3.2). *Assume that (R, \mathfrak{m}) is a regular local ring which either contains a field or is ramified. Also, assume that \mathfrak{p} and \mathfrak{q} are prime ideals in R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$. If $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$ then*

$$(1.1) \quad \mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1} \quad \text{for all } n > 0.$$

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As a result, they conjectured that (1.1) should hold for all regular local rings. More specifically,

Conjecture 1.2. *Assume that (R, \mathfrak{m}) is a regular local ring and that \mathfrak{p} and \mathfrak{q} are prime ideals in R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$. Then $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$ for all $n > 0$.*

We study Conjecture 1.2, as a verification of this conjecture could introduce new tools to apply to the positivity conjecture.

For any local ring (A, \mathfrak{n}) let $e(A)$ denote the Samuel multiplicity of A with respect to \mathfrak{n} (see Section 2 for the precise definition). It is straightforward to verify that, if R is a regular local ring with prime ideal \mathfrak{p} and $0 \neq f \in \mathfrak{p}$, then $e(R_{\mathfrak{p}}/(f)) = m$ if and only if $f \in \mathfrak{p}^{(m)} \setminus \mathfrak{p}^{(m+1)}$. Thus, Conjecture 1.2 may be rephrased as the following.

Conjecture 1.2'. *Assume that (R, \mathfrak{m}) is a regular local ring and that \mathfrak{p} and \mathfrak{q} are prime ideals in R such that $\mathfrak{p} + \mathfrak{q}$ is \mathfrak{m} -primary. If there exists $0 \neq f \in \mathfrak{p} \cap \mathfrak{q}$ such that $e(R_{\mathfrak{p}}/(f)) = e(R/(f))$, then $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R) - 1$.*

Conjecture 1.2' motivates the following generalization.

Conjecture 1.3. *Assume that (R, \mathfrak{m}) is a quasi-unmixed local ring and that \mathfrak{p} and \mathfrak{q} are prime ideals in R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R)$. Then $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$.*

Below, we present examples to show that our assumptions in Conjectures 1.2 and 1.3 are necessary.

As we noted above, Serre proved this conjecture in the case where R is regular (where the condition $e(R_{\mathfrak{p}}) = e(R)$ holds automatically). The following is a famous conjecture which is rather similar to Conjecture 1.3.

Conjecture 1.4 (Peskin and Szpiro [10]). *Assume that (R, \mathfrak{m}) is a local ring and \mathfrak{p} and \mathfrak{q} are prime ideals in R such that \mathfrak{p} has finite projective dimension and $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$. Then $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$.*

We discuss connections between Conjectures 1.3 and 1.4 below.

In Section 2 we prove Conjecture 1.3 in the case where R is excellent and contains a field. In Section 3 we establish the Conjecture 1.3 for some low-dimensional cases and for the case where R/\mathfrak{p} is regular. In Section 4 we present a number of examples.

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2. THE EQUICARACTERISTIC CASE

All rings are assumed to be commutative and Noetherian with identity, and all modules are assumed to be unital.

The goal of this section is to prove the following theorem.

Theorem 2.1. *Assume that (R, \mathfrak{m}) is an excellent local Cohen-Macaulay ring which contains a field. Also, assume that P and Q are prime ideals of R such that $\sqrt{P + Q} = \mathfrak{m}$ and $e(R_P) = e(R)$. Then $\dim(R/P) + \dim(R/Q) \leq \dim(R)$.*

We accomplish this in steps. First, we fix some notation and definitions and prove Theorem 2.1 in the case where R is complete with infinite residue field. Then, we

establish some results on the behavior of the Samuel multiplicity and reduce the case where R is excellent to the the case where our ring is complete.

The Samuel multiplicity shall play a central role in our work. For the sake of clarity we specify which multiplicity we are considering. Assume that (R, \mathfrak{m}) is a local ring of dimension d with an ideal \mathfrak{a} such that $\sqrt{\mathfrak{a}} = \mathfrak{m}$. For $n \gg 0$ the Hilbert function $H_{\mathfrak{a}}(n) = \text{len}(R/\mathfrak{a}^{n+1})$ is a polynomial in n of degree d with rational coefficients. If e_d is the leading coefficient of this polynomial, then the Samuel multiplicity of \mathfrak{a} on R is $e(\mathfrak{a}, R) = d!e_d$. We denote $e(\mathfrak{m}, R)$ by $e(R)$. Recall that the Samuel multiplicity satisfies the *Associativity Formula*

$$e(R) = \sum_{\mathfrak{p}} \text{len}(R_{\mathfrak{p}})e(R/\mathfrak{p})$$

where the sum is taken over all prime ideals \mathfrak{p} of R such that $\dim(R/\mathfrak{p}) = d$.

We first prove Theorem 2.1 for the case where R is a complete, Cohen-Macaulay ring with infinite residue field K . In this case we can find a system of parameters y_1, \dots, y_n which form part of a minimal set of generators of \mathfrak{m} such that $\text{len}(R/(y_1, \dots, y_n)) = e(R)$ (see Matsumura [9] Theorem 14.14). Since R is assumed complete, R is a finite extension of the power series ring $K[[Y_1, \dots, Y_n]]$, and since R is Cohen-Macaulay, it is a free module over this subring of rank $e(R)$.

The main result used in proving Theorem 2.1 is the following:

Theorem 2.2. *Assume that B is a Cohen-Macaulay ring and (A, \mathfrak{m}) a regular local subring such that B is a finite free A -module. Assume that P is a prime ideal of B with $P \cap A = \mathfrak{p}$ and $e(B_P) = \text{rank}_A(B)$. Then P is the unique prime ideal of B which contracts to \mathfrak{p} in A , and $B_P/P_P \cong A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$.*

Proof. First, we reduce to the case where \mathfrak{p} is the maximal ideal of A . Any localization of a Cohen-Macaulay ring is Cohen-Macaulay, so $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}$ is Cohen-Macaulay. The ring $A_{\mathfrak{p}}$ is regular and $A_{\mathfrak{p}}$ is a subring of $B_{\mathfrak{p}}$. If $r = \text{rank}_A(B)$ then $B \cong A^r$ and so $B_{\mathfrak{p}} \cong A_{\mathfrak{p}}^r$ is finite and free over $A_{\mathfrak{p}}$. Furthermore, $P_{\mathfrak{p}}$ is a prime ideal of $B_{\mathfrak{p}}$ which contracts to the maximal ideal of $A_{\mathfrak{p}}$ and $e((B_{\mathfrak{p}})_{P_{\mathfrak{p}}}) = e(B_P) = \text{rank}_A(B) = r = \text{rank}_{A_{\mathfrak{p}}}(B_{\mathfrak{p}})$. Thus, the rings $B_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ satisfy the hypotheses of the theorem and we may assume that $P \cap A = \mathfrak{m}$.

By the finiteness of the extension $A \rightarrow B$, the prime ideal P is maximal. The ring $B/\mathfrak{m}B$ is Artinian because any prime ideal of B which contains $\mathfrak{m}B$ must contract to \mathfrak{m} in A and therefore must be maximal. In particular, a regular system of parameters of A passes to a system of parameters of B_P (and since B_P is Cohen-Macaulay, a maximal B_P -regular sequence). Let $K = A/\mathfrak{m}$, $L = B/P$ and $\hat{P} = P/\mathfrak{m}B$. By the finiteness of the extension $A \rightarrow B$, the extension $K \rightarrow L$ is finite. Since $B/\mathfrak{m}B$ is a finite-dimensional vector space over K , $B/\mathfrak{m}B$ has finite length as an A -module. By the structure theorem for Artinian rings, $B/\mathfrak{m}B$ has a finite number of maximal ideals, $\hat{P} = \hat{P}_1, \hat{P}_2, \dots, \hat{P}_l$, and $B/\mathfrak{m}B \cong \prod_i (B/\mathfrak{m}B)_{\hat{P}_i}$. In particular, each $(B/\mathfrak{m}B)_{\hat{P}_i}$ is finitely generated over A and has finite length as an A -module. It is straightforward to verify that

$$\text{len}_{B_P}(B_P/\mathfrak{m}B_P) = \text{len}_A(B_P/\mathfrak{m}B_P) / \dim_K(L).$$

By [9] Theorem 14.9, $e(B_P) \leq e(B_P/\mathfrak{m}B_P)$ since \mathfrak{m} is generated by a B_P -regular sequence. Our assumptions imply that

$$\begin{aligned} \sum_i \dim_K((B/\mathfrak{m}B)_{\hat{P}_i}) &= \dim_K(B/\mathfrak{m}B) = \text{rank}_A(B) = e(B_P) \leq e(B_P/\mathfrak{m}B_P) \\ &= \text{len}_{B_P}(B_P/\mathfrak{m}B_P) = \text{len}_A(B_P/\mathfrak{m}B_P) / \dim_K(L) \\ &= \dim_K(B_P/\mathfrak{m}B_P) / \dim_K(L) \leq \dim_K(B_P/\mathfrak{m}B_P) \\ &\leq \sum_i \dim_K((B/\mathfrak{m}B)_{\hat{P}_i}) \end{aligned}$$

so we must have equality. This can only happen if (i) \hat{P} is the unique prime ideal of $B/\mathfrak{m}B$ and (ii) $\dim_K(L) = 1$. These are the desired results. \square

Proof of Theorem 2.1 when R is complete with infinite residue field. As noted previously, we may choose a system of parameters y_1, \dots, y_n of R such that (i) the y_i form part of a minimal generating set of \mathfrak{m} and (ii) $e(R) = \text{len}(R/(y_1, \dots, y_n))$. Fix $z_1, \dots, z_q \in \mathfrak{m}$ such that $y_1, \dots, y_n, z_1, \dots, z_q$ form a minimal generating set for \mathfrak{m} . The natural map $K[[Y_1, \dots, Y_n]] \rightarrow R$ given by $Y_i \mapsto y_i$ is injective since R is complete, and R is module finite over $A = K[[Y_1, \dots, Y_n]]$ (cf., [9] §29), and the fact that R is local Cohen-Macaulay implies that R is free over A of finite rank r . Furthermore, the natural map $\rho : K[[Y_1, \dots, Y_n, Z_1, \dots, Z_q]] \rightarrow R$ given by $Y_i \mapsto y_i$ and $Z_j \mapsto z_j$ is surjective. Let $A' = K[[Y_1, \dots, Y_n, Z_1, \dots, Z_q]]$ with maximal ideal \mathfrak{m}' and $I = \ker(\rho)$. The constructions show that we have a natural commuting diagram:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ & \searrow \psi & \downarrow \rho \\ & & R \end{array}$$

Let $\mathfrak{p} = P \cap A$. Since the extension $A \rightarrow R$ is finite and free, both the going-up and going-down properties hold so that $\text{ht}(\mathfrak{p}) = \text{ht}(P)$ and $\dim(A/\mathfrak{p}) = \dim(R/P)$. If we can show that $\sqrt{\mathfrak{p}A' + \rho^{-1}(Q)} = \mathfrak{m}'$, then it follows that

$$\begin{aligned} \dim(R/P) + \dim(R/Q) &= \dim(A/\mathfrak{p}) + \dim(A'/\rho^{-1}(Q)) \\ &= \dim(A'/\mathfrak{p}A') - q + \dim(A'/\rho^{-1}(Q)) \\ &\leq \dim(A') - q = n = \dim(R) \end{aligned}$$

where the final inequality follows from the fact that Theorem 2.1 holds for regular local rings.

For an ideal \mathfrak{a} of A' , let $Z(\mathfrak{a}) \subseteq \text{Spec}(A')$ denote the closed subscheme determined by \mathfrak{a} . In order to show that $\sqrt{\mathfrak{p}A' + \rho^{-1}(Q)} = \mathfrak{m}'$, it suffices to show that

$$Z(\mathfrak{p}A' + \rho^{-1}(Q)) = Z(\rho^{-1}(P) + \rho^{-1}(Q))$$

as the surjectivity of ρ implies that $\sqrt{\rho^{-1}(P) + \rho^{-1}(Q)} = \mathfrak{m}'$. Since $\mathfrak{p}A' + \rho^{-1}(Q) \subseteq \rho^{-1}(P) + \rho^{-1}(Q)$, the inclusion $Z(\mathfrak{p}A' + \rho^{-1}(Q)) \supseteq Z(\rho^{-1}(P) + \rho^{-1}(Q))$ is clear. For the other inclusion we note that, since $I \subseteq \rho^{-1}(Q)$,

$$\mathfrak{p}A' + \rho^{-1}(Q) = (\mathfrak{p}A' + I) + \rho^{-1}(Q)$$

so that

$$Z(\mathfrak{p}A' + \rho^{-1}(Q)) = Z((\mathfrak{p}A' + I) + \rho^{-1}(Q)) = Z(\mathfrak{p}A' + I) \cap Z(\rho^{-1}(Q)).$$

Since $Z(\rho^{-1}(P) + \rho^{-1}(Q)) = Z(\rho^{-1}(P)) \cap Z(\rho^{-1}(Q))$ it then suffices to show that $Z(\mathfrak{p}A' + I) \subseteq Z(\rho^{-1}(P))$. It suffices to show that $\rho^{-1}(P)$ determines the unique minimal prime ideal of $A'/(\mathfrak{p}A' + I)$. By our commuting diagram, the (minimal) primes of $\mathfrak{p}A' + I$ are in bijection with the (minimal) primes of $\mathfrak{p}R = \rho(\mathfrak{p}A' + I)$. Thus, it suffices to show that P is the unique minimal prime of $\mathfrak{p}R$. By Lemma 2.3 below, every minimal prime P' of $\mathfrak{p}R$ contracts to \mathfrak{p} in A , so it suffices to show that P is the unique prime ideal of R such that $P \cap A = \mathfrak{p}$. By assumption $P \cap A = \mathfrak{p}$ and $e(R_P) = e(R)$. So, if we can show that $e(R) = \text{rank}_A(R)$, then Theorem 2.2 supplies the necessary uniqueness. Our assumptions imply that

$$\begin{aligned} e(R) &= \text{len}(R/(\mathfrak{y})R) = \dim_K(R/(\mathfrak{y})R) = \dim_K(R \otimes_A A/(\mathfrak{y})A) \\ &= \dim_K(R \otimes_A K) = \dim_K(A^r \otimes_A K) = \dim_K(K^r) = r = \text{rank}_A(R) \end{aligned}$$

and this completes the proof. □

Lemma 2.3. *Assume that $R \rightarrow \tilde{R}$ is a flat local homomorphism of local rings (R, \mathfrak{m}) and $(\tilde{R}, \tilde{\mathfrak{m}})$.*

1. *If $\mathfrak{m}\tilde{R}$ is $\tilde{\mathfrak{m}}$ -primary, then $\dim(\tilde{R}) = \dim(R)$.*
2. *If $\mathfrak{m}\tilde{R}$ is $\tilde{\mathfrak{m}}$ -primary and R is Cohen-Macaulay, then \tilde{R} is Cohen-Macaulay.*
3. *If \mathfrak{p} is a prime ideal of R and P is a prime ideal of \tilde{R} which is minimal over $\mathfrak{p}\tilde{R}$, then $P \cap R = \mathfrak{p}$ and $\text{ht}(P) = \text{ht}(\mathfrak{p})$.*

Proof. 1. Let $d = \dim(R)$. By flatness, the extension $R \rightarrow \tilde{R}$ satisfies the going-down property. In particular, $d \leq \dim(\tilde{R})$. Let $x_1, \dots, x_d \in \mathfrak{m}$ be a system of parameters for R and let $\mathfrak{a} = (\mathbf{x})R$. Then $\sqrt{\mathfrak{a}} = \mathfrak{m}$ so that $\sqrt{\mathfrak{a}\tilde{R}} = \tilde{\mathfrak{m}}$. Thus, $\mathfrak{a}\tilde{R}$ is $\tilde{\mathfrak{m}}$ -primary and generated by d elements, so that $d \geq \dim(\tilde{R})$.

2. The system of parameters \mathbf{x} is a regular sequence on R of length d . By flatness \mathbf{x} is a regular sequence on \tilde{R} of length $\dim(\tilde{R})$.

3. Suppose that $P \cap R \neq \mathfrak{p}$. Since $P \cap R \supseteq \mathfrak{p}$, the going-down property implies that P contains a prime ideal Q which contracts to \mathfrak{p} . But then $\mathfrak{p}\tilde{R} \subseteq Q \subsetneq P$, contradicting the minimality of P . The fact that $\text{ht}(P) = \text{ht}(\mathfrak{p})$ is proved in [4] Lemma 18.10. □

The remainder of this section is devoted to proving that we can reduce Theorem 2.1 to the case where R is complete.

For technical reasons, we must restrict our attention to certain classes of rings. We recall the definitions here. A ring R is *equidimensional* if $\dim(R/\mathfrak{p}) = \dim(R)$ for all minimal prime ideals \mathfrak{p} of R . A local ring R is *quasi-unmixed* if its completion is equidimensional. A ring R is *Nagata* if, for every prime ideal \mathfrak{p} of R and every finite extension field L of the quotient field of R/\mathfrak{p} , the integral closure of R/\mathfrak{p} in L is module-finite over R/\mathfrak{p} .

Every Cohen-Macaulay ring is equidimensional. Every excellent ring¹ is Nagata. Furthermore, the class of Nagata rings is closed under finite ring extensions, localizations and quotients. Also, every localization of a ring of polynomials over a

¹For a complete definition of “excellent”, see Matsumura [9] Section 32. It is important to note that the class of excellent rings contains all fields, complete local rings and the ring of integers, and is closed under finite ring extensions, localizations and quotients.

quasi-unmixed ring is quasi-unmixed. Furthermore, a quasi-unmixed local ring is equidimensional and universally catenary (see Hermann, Ikeda and Orbanz [4] Section 18). One of the advantages of working with Nagata rings is that multiplicities are well-behaved with respect to localization. The following is a special case of a theorem of Lech [8].

Lemma 2.4. *Assume that (R, \mathfrak{m}) is a local, quasi-unmixed, Nagata ring with prime ideal \mathfrak{p} . Then $e(R_{\mathfrak{p}}) \leq e(R)$.*

Proof. Because R is equidimensional and catenary, $\text{ht } \mathfrak{p} + \dim(R/\mathfrak{p}) = \dim(R)$. Because R is Nagata, for every pair of primes $\mathfrak{q} \subset \mathfrak{q}'$, the integral closure of $R_{\mathfrak{q}'}/\mathfrak{q}R_{\mathfrak{q}'}$ is module-finite over $R_{\mathfrak{q}'}/\mathfrak{q}R_{\mathfrak{q}'}$. As noted by Lech ([8] comments following Theorem 1) this now implies that $e(R_{\mathfrak{p}}) \leq e(R)$. \square

Lech conjectured that Lemma 2.4 should hold for any local, equidimensional ring. Below, we indicate an example which shows that we must assume that R is at least equidimensional in order for the inequality $e(R_{\mathfrak{p}}) \leq e(R)$ to hold for all prime ideals \mathfrak{p} of R .

In the following two lemmas, we collect facts that show that multiplicities do not change under certain ring extensions. We will use these facts to reduce our general question to the situation where R is complete with infinite residue field and R/\mathfrak{p} is normal. For a finitely generated module M , let $\mu(M)$ denote the minimal number of generators of M .

Lemma 2.5. *Assume that $R \rightarrow \tilde{R}$ is a flat local homomorphism of Noetherian local rings (R, \mathfrak{m}) and $(\tilde{R}, \tilde{\mathfrak{m}})$ and that \mathfrak{a} is an ideal of R .*

1. $\mu(\mathfrak{a}) = \mu(\mathfrak{a}\tilde{R})$.
2. *If, in addition, $\mathfrak{m}\tilde{R} = \tilde{\mathfrak{m}}$ then $\text{len}_R(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \text{len}_{\tilde{R}}(\tilde{\mathfrak{m}}^n/\tilde{\mathfrak{m}}^{n+1})$. In particular, $e(R) = e(\tilde{R})$.*

Proof. Part 1 is proved by Herzog [5] Lemma 2.3. Part 2 follows from part 1 by the following computation

$$\text{len}_R(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \mu(\mathfrak{m}^n) = \mu(\mathfrak{m}^n\tilde{R}) = \mu(\tilde{\mathfrak{m}}^n) = \text{len}_{\tilde{R}}(\tilde{\mathfrak{m}}^n/\tilde{\mathfrak{m}}^{n+1})$$

and the definitions of $e(R)$ and $e(\tilde{R})$. \square

Lemma 2.6. *Let (R, \mathfrak{m}) be a quasi-unmixed local ring and $S = R[X_1, \dots, X_n]$ a polynomial ring over R . Let P be a prime ideal of S and $\mathfrak{p} = P \cap R$. Then $e(R_{\mathfrak{p}}) = e(S_P)$.*

Proof. By induction, we may assume that $n = 1$, and the result follows in this case by [8] Lemma 2. \square

The following theorem allows us to reduce Conjecture 1.3 to the case where the quotient R/\mathfrak{p} is a normal domain. This will be a key step in our reduction of Theorem 2.1 to the case where R is complete.

Theorem 2.7. *Let (R, \mathfrak{m}) be a Nagata, Cohen-Macaulay local ring, and suppose that, for every ring (S, M) which is a localization at a maximal ideal of a polynomial ring over R , the following holds: for all prime ideals P and Q such that $\sqrt{P+Q} = M$, $e(S_P) = e(S)$ and S/P is a normal domain,*

$$\dim(S/P) + \dim(S/Q) \leq \dim(S).$$

Then, for all prime ideals \mathfrak{p} and \mathfrak{q} of R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R)$,

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R).$$

Proof. Fix prime ideals \mathfrak{p} and \mathfrak{q} of R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R)$. Let B be the integral closure of R/\mathfrak{p} . Since R is Nagata, B is module-finite over R/\mathfrak{p} and in particular, $\dim(B) = \dim(R/\mathfrak{p})$. Because B is module finite over R/\mathfrak{p} , there is a surjection $T = R[X_1, \dots, X_n] \rightarrow B$. Let K denote the kernel of this map so that $T/K = B$. The commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\beta} & R/\mathfrak{p} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & B \end{array}$$

shows that $K \cap R = \mathfrak{p}$. Let $\mathfrak{n} \subset B$ be a maximal ideal. Since the extension $R/\mathfrak{p} \rightarrow B$ is finite, we know that $\mathfrak{n} \cap R/\mathfrak{p}$ is maximal in R/\mathfrak{p} , that is, $\mathfrak{n} \cap R/\mathfrak{p} = \mathfrak{m}/\mathfrak{p}$. Also, there are no prime ideals of B which are properly contained in \mathfrak{n} and contract to $\mathfrak{m}/\mathfrak{p}$ in R/\mathfrak{p} . It follows that $\sqrt{\mathfrak{q}B_{\mathfrak{n}}} = \sqrt{(\mathfrak{m}/\mathfrak{p})B_{\mathfrak{n}}} = \mathfrak{n}_{\mathfrak{n}}$.

Let $N = \phi^{-1}(\mathfrak{n})$, so that $T/N \cong B/\mathfrak{n}$. Then, $K \subseteq N$ and since $\mathfrak{m} = \beta^{-1}(\mathfrak{n}) = R \cap \phi^{-1}(\mathfrak{n}) = R \cap N$, we see that $L = \mathfrak{q}T \subseteq N$. We claim that $\sqrt{K_N + L_N} = N_N$. Since $K + L \supseteq K = \ker(\phi)$ we see that $K + L = \phi^{-1}(\phi(K + L)) = \phi^{-1}(\mathfrak{q}B)$. Let ϕ denote the map $T_N \rightarrow B_{\mathfrak{n}}$. If $x \in N_N$, then $\phi(x) \in \mathfrak{n}$ so that for some n , $\phi(x^n) = \phi(x)^n \in \mathfrak{q}B_{\mathfrak{n}}$. Then $x^n \in \phi^{-1}(\mathfrak{q}B_{\mathfrak{n}}) = (K + L)_N$ so that $N_N \subseteq \sqrt{K_N + L_N}$ as desired.

By Lemma 2.6, we see that $e(T_N) = e(R) = e(R_{\mathfrak{p}}) = e(T_K)$. By construction, T_N/K_N is a normal domain. Thus, if we know that

$$\dim(T_N/K_N) + \dim(T_N/L_N) \leq \dim(T_N),$$

then it follows that

$$\begin{aligned} \dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) &= \dim(T_N/K_N) + \dim(T_N/L_N) - n \\ &\leq \dim(T_N) - n = \dim(R) \end{aligned}$$

as desired. This gives the desired reduction. □

It is clear that, in the statement of Theorem 2.7, ‘‘Cohen-Macaulay’’ may be replaced by either ‘‘quasi-unmixed’’, ‘‘Gorenstein’’ or ‘‘complete intersection (of codimension c)’’. The same is true of the statement of Theorem 2.8 below. Below, we indicate an example which shows that we must assume that our ring is at least equidimensional for Conjecture 1.3 to hold.

The following theorem is the result which will allow us to reduce Conjecture 1.3 to the case where R is complete.

Theorem 2.8. *Let (R, \mathfrak{m}) be an excellent, Cohen-Macaulay local ring. Assume the following for every ring (S, M) which is the localization of a polynomial ring over R at a maximal ideal: for all prime ideals \hat{P}, \hat{Q} of the completion \hat{S} such that $\sqrt{\hat{P} + \hat{Q}} = \hat{M}$, $e(\hat{S}_{\hat{P}}) = e(\hat{S})$ and \hat{S}/\hat{P} is a normal domain, we have*

$$\dim(\hat{S}/\hat{P}) + \dim(\hat{S}/\hat{Q}) \leq \dim(\hat{S}).$$

Then, for all prime ideals \mathfrak{p} and \mathfrak{q} of R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R)$,

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R).$$

Proof. By Theorem 2.7 it suffices to show that for every ring (S, M) which is a localization at a maximal ideal of a polynomial ring over R , the following holds: for all prime ideals P and Q such that $\sqrt{P+Q} = M$, $e(S_P) = e(S)$ and S/P is a normal domain,

$$\dim(S/P) + \dim(S/Q) \leq \dim(S).$$

Let S, M, P, Q satisfy these hypotheses. The ring S is excellent, which implies that S/P is also excellent (cf., [9] §32). By [9] Theorem 32.2, $\widehat{S/P} = \hat{S}/P\hat{S}$ is normal, in particular, $\hat{P} = P\hat{S}$ is a prime ideal of \hat{S} such that \hat{S}/\hat{P} is a normal domain. Since the map $S \rightarrow \hat{S}$ is faithfully flat, $\hat{P} \cap S = P$. Thus, by Lemma 2.3, $\text{ht}(\hat{P}) = \text{ht}(P)$. By [4] Theorem 18.13 (d), $\dim(\hat{S}/\hat{P}) = \dim(S/P)$. Let \hat{Q} be a minimal prime ideal containing $Q\hat{S}$ such that $\dim(\hat{S}/\hat{Q}) = \dim(\hat{S}/Q\hat{S})$. The extension $S_P \rightarrow \hat{S}_{\hat{P}}$ is faithfully flat and $P_P\hat{S}_{\hat{P}} = \hat{P}_{\hat{P}}$ so that, by Lemma 2.5

$$e(\hat{S}_{\hat{P}}) = e(S_P) = e(S) = e(\hat{S}).$$

Thus, by assumption

$$\dim(S/P) + \dim(S/Q) = \dim(\hat{S}/\hat{P}) + \dim(\hat{S}/\hat{Q}) \leq \dim(\hat{S}) = \dim(S)$$

as desired. □

Now, we complete the proof of Theorem 2.1.

Proof of Theorem 2.1. If the residue field of R is finite, let X be an indeterminate over R and $R(X) = R[X]_{\mathfrak{m}[X]}$ which is an excellent, Cohen-Macaulay local ring that contains a field. The prime ideals $P = \mathfrak{p}R(X)$ and $Q = \mathfrak{q}R(X)$ satisfy the following properties: $\sqrt{P+Q} = \mathfrak{m}R(X)$, $R(X)/P = R/\mathfrak{p}(X)$, $R(X)/Q = R/\mathfrak{q}(X)$ and (by Lemma 2.6) $e(R(X)_P) = e(R_{\mathfrak{p}}) = e(R) = e(R(X))$. Thus, if the theorem holds for $R(X)$, then

$$\begin{aligned} \dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) &= \dim(R(X)/P) + \dim(R(X)/Q) \\ &\leq \dim(R(X)) = \dim(R) \end{aligned}$$

and the theorem holds for R . Since the residue field of $R(X)$ is $K(X)$, we may assume that the residue field of R is infinite. By Theorems 2.7 and 2.8 we may pass to the completion of the localization of a polynomial ring over R . In particular, we may assume that R is a complete local Cohen-Macaulay ring which contains a field, and by our previous remarks this completes the proof. □

3. OTHER CASES

A number of low dimensional cases of Conjecture 1.3 can be dealt with rather easily using Lemma 2.4 and the Associativity Formula for multiplicities.

Proposition 3.1. *Assume that (R, \mathfrak{m}) is an equidimensional local ring of dimension d with prime ideals \mathfrak{p} and \mathfrak{q} such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $e(R_{\mathfrak{p}}) = e(R)$.*

1. *If \mathfrak{p} is a minimal prime ideal of R then $\mathfrak{q} = \mathfrak{m}$.*
2. *If $\dim(R/\mathfrak{q}) = 1$ then $\dim(R/\mathfrak{p}) < d$.*
3. *If R is Nagata and \mathfrak{q} is a minimal prime ideal of R then $\mathfrak{p} = \mathfrak{m}$.*
4. *If R is Nagata and $\dim(R/\mathfrak{p}) = 1$ then $\dim(R/\mathfrak{q}) < d$.*

In particular, Conjecture 1.3 holds when R is Nagata and one of the following holds: $\dim(R/\mathfrak{p}) = 1$, $\dim(R/\mathfrak{p}) = d$, $\dim(R/\mathfrak{q}) = 1$, or $\dim(R/\mathfrak{q}) = d$.

Proof. We prove part 1 here. The remaining parts are proved similarly. Our assumptions imply that

$$\text{len}(R_{\mathfrak{p}}) = e(R_{\mathfrak{p}}) = e(R)$$

and the Associativity Formula implies that

$$\text{len}(R_{\mathfrak{p}}) = e(R) = \sum_{\mathfrak{r}} \text{len}(R_{\mathfrak{r}})e(R/\mathfrak{r})$$

where the sum is taken over all prime ideals \mathfrak{r} of R such that $\dim(R/\mathfrak{r}) = d$. Since each $e(R/\mathfrak{r}) > 0$, it follows that $\text{len}(R_{\mathfrak{p}})e(R/\mathfrak{p})$ is the only term in the sum. Since R is equidimensional, this implies that \mathfrak{p} is the unique minimal prime ideal of R . In particular, $\mathfrak{q} \supseteq \sqrt{(0)} = \mathfrak{p}$. The fact that \mathfrak{q} is prime implies that $\mathfrak{m} = \sqrt{\mathfrak{p} + \mathfrak{q}} = \sqrt{\mathfrak{q}} = \mathfrak{q}$, as desired. \square

The proof of Proposition 3.1 shows that, for minimal prime ideals \mathfrak{p} , the assumption $e(R_{\mathfrak{p}}) = e(R)$ is quite strong. In fact, we have the following.

Proposition 3.2. *Assume that (R, \mathfrak{m}) is an equidimensional local ring with minimal prime ideal \mathfrak{p} , and consider the following statements.*

1. $e(R/\mathfrak{p}) = 1$.
2. $e(R_{\mathfrak{p}}) = e(R)$.
3. \mathfrak{p} is the unique minimal prime of R .

(a) *Any two of these conditions imply the third.*

(b) *If R/\mathfrak{p} is unmixed (that is, if every associated prime \mathfrak{p}' of the completion R' of R/\mathfrak{p} satisfies the condition $\dim(R'/\mathfrak{p}') = \dim(R')$) then condition 1 may be replaced by “ R/\mathfrak{p} is regular”.*

Proof. (a) If $e(R/\mathfrak{p}) = 1$ and $e(R_{\mathfrak{p}}) = e(R)$, then

$$\begin{aligned} \text{len}(R_{\mathfrak{p}}) = e(R_{\mathfrak{p}}) = e(R) &= \sum_{\mathfrak{r}} \text{len}(R_{\mathfrak{r}})e(R/\mathfrak{r}) \\ &\geq \text{len}(R_{\mathfrak{p}})e(R/\mathfrak{p}) = \text{len}(R_{\mathfrak{p}}) \end{aligned}$$

where the sum is taken over all prime ideals \mathfrak{r} of R such that $\dim(R/\mathfrak{r}) = \dim(R)$. It follows that the only term in the sum is $\text{len}(R_{\mathfrak{p}})e(R/\mathfrak{p})$. Since R is equidimensional, this implies that \mathfrak{p} is the unique minimal prime of R .

If $e(R/\mathfrak{p}) = 1$ and \mathfrak{p} is the unique minimal prime of R

$$e(R) = \sum_{\mathfrak{r}} \text{len}(R_{\mathfrak{r}})e(R/\mathfrak{r}) = \text{len}(R_{\mathfrak{p}})e(R/\mathfrak{p}) = \text{len}(R_{\mathfrak{p}}) = e(R_{\mathfrak{p}})$$

where the sum is taken over all prime ideals \mathfrak{r} of R such that $\dim(R/\mathfrak{r}) = \dim(R)$. If $e(R_{\mathfrak{p}}) = e(R)$ and \mathfrak{p} is the unique minimal prime of R , then a similar computation shows that $e(R/\mathfrak{p}) = 1$.

(b) If R/\mathfrak{p} is unmixed, then [4] Theorem 6.8 implies that $e(R/\mathfrak{p}) = 1$ if and only if R/\mathfrak{p} is regular. \square

A second indication that the condition $e(R_{\mathfrak{p}}) = e(R)$ is rather strong is supplied by the following lemma. First, we recall several notions. If \mathfrak{a} and \mathfrak{b} are ideals such that $\mathfrak{b} \subseteq \mathfrak{a}$, then \mathfrak{b} is a *reduction* of \mathfrak{a} if there exists a positive integer n such that $\mathfrak{a}^{n+1} = \mathfrak{b}\mathfrak{a}^n$. In this case, \mathfrak{b} is a *minimal reduction* of \mathfrak{a} if it is minimal among all

reductions of \mathfrak{a} with respect to inclusion. If (R, \mathfrak{m}) is a Noetherian local ring with an ideal \mathfrak{a} , the *Rees algebra* of \mathfrak{a} is the graded ring

$$R[\mathfrak{a}t] = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n t.$$

The *special fibre* of the Rees algebra $R[\mathfrak{a}t]$ is the ring

$$F(\mathfrak{a}) = R[\mathfrak{a}t] \otimes_R R/\mathfrak{m}.$$

The Krull dimension of $F(\mathfrak{a})$ is the *analytic spread* of \mathfrak{a} and is denoted $s(\mathfrak{a})$. If the residue field of R is infinite, then $s(\mathfrak{a})$ is the least number of elements required to generate a reduction ideal of \mathfrak{a} .

Lemma 3.3 ([4] Theorem 20.9). *Let R be a quasi-unmixed local ring and let \mathfrak{p} be a prime ideal of R for which R/\mathfrak{p} is regular. Then the following conditions are equivalent.*

1. $e(R) = e(R_{\mathfrak{p}})$,
2. $\text{ht}(\mathfrak{p}) = s(\mathfrak{p})$.

Below, we indicate examples which show that the assumption “ R/\mathfrak{p} is regular” is essential for both implications.

Proposition 3.4. *Assume that (R, \mathfrak{m}) is a quasi-unmixed local ring of dimension d with prime ideals \mathfrak{p} and \mathfrak{q} such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$, R/\mathfrak{p} is regular and $e(R_{\mathfrak{p}}) = e(R)$. Then $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq d$.*

Proof. As in the proof of Theorem 2.1, we may assume that R has infinite residue field by passing to the ring $R(X) = R[X]_{\mathfrak{m}[X]}$. We note that $R(X)$ is quasi-unmixed and $R(X)/\mathfrak{p}R(X)$ is regular.

By Lemma 3.3, our assumptions imply that $s(\mathfrak{p}) = \text{ht}(\mathfrak{p})$. Since the residue field of R is infinite, \mathfrak{p} contains a sequence y_1, \dots, y_i which generate a minimal reduction of \mathfrak{p} where $i = \text{ht}(\mathfrak{p})$. Since $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$, we see that \mathfrak{q} is an ideal of definition for R/\mathfrak{p} and therefore \mathfrak{q} contains a system of parameters z_1, \dots, z_j for R/\mathfrak{p} . In particular $j = \dim(R/\mathfrak{p})$. We claim that $y_1, \dots, y_i, z_1, \dots, z_j$ is a system of parameters for R . Since $i + j = \text{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$, the sequence has the correct length and we need only check that the sequence generates an \mathfrak{m} -primary ideal. We compute

$$\sqrt{(\mathbf{y}, \mathbf{z})R} = \sqrt{\sqrt{(\mathbf{y})R} + \sqrt{(\mathbf{z})R}} = \sqrt{\mathfrak{p} + \sqrt{(\mathbf{z})R}}.$$

The fact that \mathbf{z} is a system of parameters for R/\mathfrak{p} implies that the only prime ideal of R containing \mathfrak{p} and \mathbf{z} is \mathfrak{m} , as desired.

To prove the result, it suffices to show that $i \geq \dim(R/\mathfrak{q})$, as this will show that $\dim(R) = i + j \geq \dim(R/\mathfrak{q}) + \dim(R/\mathfrak{p})$. In the ring R/\mathfrak{q} , the images of \mathbf{y} generate an ideal which is primary to $\mathfrak{m}/\mathfrak{q}$ since $\sqrt{(\mathbf{y})R} = \mathfrak{p}$. Since $\dim(R/\mathfrak{q})$ is the least integer l such that an ideal primary to the maximal ideal of R/\mathfrak{q} can be generated by l elements, we are done. □

This result may be more interesting than it appears, as we hope to use de Jong’s theory of regular alterations to reduce a number of general cases to the case where R/\mathfrak{p} is regular.

4. EXAMPLES

The following examples show that many of the assumptions we place on the local ring R and the prime ideals \mathfrak{p} and \mathfrak{q} in Conjectures 1.2 and 1.3 may not be relaxed.

Example 4.1. If we do not require R to be regular, then Conjecture 1.2 does not hold. Let k be a field and let $R = k[[X, Y, Z]]/(X^2 - YZ) = k[[x, y, z]]$. Then R is a complete intersection of dimension 2, but is not regular. Let $\mathfrak{p} = (x, z)$ and $\mathfrak{q} = (x, y)$ which are prime ideals in R such that $\mathfrak{p} + \mathfrak{q} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 2 = \dim(R)$. However, $X \in \mathfrak{p} \cap \mathfrak{q}$ so that $\mathfrak{p} \cap \mathfrak{q} \not\subseteq \mathfrak{m}^2$. It is straightforward to verify that $\mathfrak{p}^{(2)} = (z)$ so that $\mathfrak{p}^{(2)} \cap \mathfrak{q} = (xz, yz)$.

Example 4.2. If we do not require $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$, then Conjecture 1.2 does not hold. Let $R = k[[X, Y]]$ and $\mathfrak{p} = \mathfrak{q} = (X)$. Then R is regular and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 2 = \dim(R)$. However, $\mathfrak{p} \cap \mathfrak{q} = (X) \not\subseteq \mathfrak{m}^2$.

Example 4.3. If we do not require that $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$, then Conjecture 1.2 does not hold. Let $R = k[[X]]$ and $\mathfrak{p} = \mathfrak{q} = (X)$. Then R is regular and $\mathfrak{p} + \mathfrak{q} = \mathfrak{m}$. However, $\mathfrak{p} \cap \mathfrak{q} = (X) \not\subseteq \mathfrak{m}^2$.

Example 4.4. If we do not require that $e(R_{\mathfrak{p}}) = e(R)$, then Conjecture 1.3 does not hold. Let k be a field, $R = k[[X, Y, Z, W]]/(XY - ZW) = k[[x, y, z, w]]$ with $\mathfrak{p} = (x, z)R$ and $\mathfrak{q} = (y, w)R$. Then $e(R) = 2 > 1 = e(R_{\mathfrak{p}})$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 4 > 3 = \dim(R)$.

Example 4.5. If we do not require that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$, then Conjecture 1.3 does not hold. Let $R = k[[X]]$ and $\mathfrak{p} = \mathfrak{q} = (0)$. Then $e(R_{\mathfrak{p}}) = e(R)$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 2 > \dim(R)$.

Example 4.6. If we do not require that R is at least equidimensional, then Conjecture 1.3 does not hold. Let $R = k[[X]] \times_k k[[Y, Z]]$. That is, in the diagram

$$\begin{array}{ccc} k[[X]] \times k[[Y, Z]] & \longrightarrow & k[[X]] \\ \downarrow & & \downarrow \beta \\ k[[Y, Z]] & \xrightarrow{\alpha} & k \end{array}$$

$R = \{(a, b) \in A \times B : \alpha(a) = \beta(b)\}$. Let $\mathfrak{p} = ((X, 0))R$ and $\mathfrak{q} = ((0, Y), (0, Z))R$. It is straightforward to verify the following facts: (i) R is a local ring with maximal ideal $\mathfrak{m} = (X) \times (Y, Z)$; (ii) $\dim(R) = 2$; (iii) $e(R_{\mathfrak{p}}) = e(R_{\mathfrak{q}}) = e(R) = 1$; and (iv) $\mathfrak{p} + \mathfrak{q} = \mathfrak{m}$. However, $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 3 > 2 = \dim(R)$. We should note, as this pertains to Conjecture 1.4, that both \mathfrak{p} and \mathfrak{q} have infinite projective dimension.

With Conjectures 1.3 and 1.4 in mind, one might hope that there is a nice relation between the conditions “ $e(A_{\mathfrak{p}}) = e(A)$ ” and “ \mathfrak{p} has finite projective dimension”. The following examples show that neither of these conditions implies the other.

Example 4.7. Let $R = k[[X, Y, Z, W]]/(X^2 - YZ) = k[[x, y, z, w]]$ with $\mathfrak{p} = (x, y, z)$. Then \mathfrak{p} has infinite projective dimension and $e(R) = 2 = e(R_{\mathfrak{p}})$. Notice that R is a domain and R/\mathfrak{p} is regular.

Example 4.8. Let $R = k[[X, Y, Z, W]]/(X^2 - YZ)$ with $\mathfrak{p} = (0)$. Then R/\mathfrak{p} is free (and therefore has finite projective dimension) and $e(R) = 2 > 1 = e(R_{\mathfrak{p}})$.

We notice that Example 4.8 is not as satisfying as Example 4.7, in the sense that R/\mathfrak{p} is not regular. We were attempting to find a complete local domain R with prime ideal \mathfrak{p} satisfying the following conditions: (i) R/\mathfrak{p} is regular, (ii) \mathfrak{p} has finite projective dimension, and (iii) $e(R) > e(R_{\mathfrak{p}})$. We have already seen that these conditions are quite restrictive. In fact, such an example does not exist, as we see in the following.

Lemma 4.9. *Let (R, \mathfrak{m}) be a local ring and M a finitely generated R -module. Let $F_{\bullet} \rightarrow M$ be an R -free resolution of M and assume that $\mathbf{x} = x_1, \dots, x_i \in R$ is a sequence which is regular on M . Let $K_{\bullet} = K_{\bullet}(\mathbf{x})$ denote the Koszul complex of \mathbf{x} on R . Then $F_{\bullet} \otimes_R K_{\bullet}$ is a free resolution of $M/\mathbf{x}M$.*

Proof. The fact that $K_{\bullet}(x_1, \dots, x_i) = K_{\bullet}(x_1) \otimes K_{\bullet}(x_2, \dots, x_i)$ allows us to assume that $i = 1$. In this case, K_{\bullet} is exactly

$$0 \rightarrow R \xrightarrow{x} R \rightarrow 0$$

We apply the mapping cone and the long exact sequence of homology to give the desired result. \square

Lemma 4.10. *Assume that (R, \mathfrak{m}) is a local Noetherian ring with prime ideal \mathfrak{p} of finite projective dimension. Then $R_{\mathfrak{p}}$ is a regular local ring. In particular, \mathfrak{p} is locally a complete intersection.*

Proof. A finite free resolution of R/\mathfrak{p} over R localizes to a finite free resolution of $(R/\mathfrak{p})_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. Now apply the theorem of Auslander, Buchsbaum and Serre to see that $R_{\mathfrak{p}}$ is regular. \square

Proposition 4.11. *Let (R, \mathfrak{m}) be a local ring of dimension d with prime ideal \mathfrak{p} such that \mathfrak{p} has finite projective dimension and let \hat{R} denote the completion of R . Assume that one of the following holds:*

1. R/\mathfrak{p} is regular.
2. R is unmixed and $e(R_{\mathfrak{p}}) = e(R)$.

Then R is regular. (In particular, $e(R_{\mathfrak{p}}) = 1 = e(R)$ and every prime ideal has finite projective dimension.)

Proof. 1. Assume that (R, \mathfrak{m}) is a local ring with prime ideal \mathfrak{p} such that R/\mathfrak{p} is regular and \mathfrak{p} has finite projective dimension. Let $F_{\bullet} \rightarrow R/\mathfrak{p}$ be a minimal, finite R -free resolution of R/\mathfrak{p} . Since R/\mathfrak{p} is regular, let $\mathbf{x} = x_1, \dots, x_i \in R$ be a sequence which forms a regular system of parameters of R/\mathfrak{p} . Let $K_{\bullet} = K_{\bullet}(\mathbf{x})$ denote the Koszul complex of \mathbf{x} on R , so that, by Lemma 4.9, $F_{\bullet} \otimes_R K_{\bullet}$ is a finite, R -free resolution of $(R/\mathfrak{p})/(\mathbf{x}R/\mathfrak{p}) = R/\mathfrak{m}$. The theorem of Auslander, Buchsbaum and Serre implies that R is regular.

2. Under our assumptions, R is regular if and only if $e(R) = 1$. By Lemma 4.10, $R_{\mathfrak{p}}$ is regular so that $e(R) = e(R_{\mathfrak{p}}) = 1$, as desired. \square

The following example shows that the requirement of equidimensionality is necessary for the inequality $e(R_{\mathfrak{p}}) \leq e(R)$ to hold.

Example 4.12. Let k be a field and $R = k[[X]] \times_k k[[Y]]/(Y^n)$ where $n > 1$, as in Example 4.6. Then R is a local ring with maximal ideal $\mathfrak{m} = (X) \times (Y)$, $\dim(R) = 1$, $e(R) = 1$, and the ideal $\mathfrak{p} = (0) \times (Y)$ is a prime such that $e(R_{\mathfrak{p}}) = n > 1 = e(R)$.

The following examples show that if R/\mathfrak{p} is not regular, neither implication of Lemma 3.3 holds. Note that each example is quasi-unmixed.

Example 4.13. Let $R = k[[X, Y, Z, U, V, W]]$ and \mathfrak{p} be the prime ideal generated by the size two minors of the generic matrix

$$\begin{pmatrix} X & Y & Z \\ U & V & W \end{pmatrix}.$$

Since R is regular, $e(R) = 1 = e(R_{\mathfrak{p}})$. However, $\text{ht}(\mathfrak{p}) = 2$ and $s(\mathfrak{p}) = 3$.

Example 4.14. Let $R = k[[X, Y, Z, W]]/(XY - ZW)$ with $\mathfrak{p} = (0)R$. Then $\text{ht}(\mathfrak{p}) = 0 = s(\mathfrak{p})$. However, $e(R) = 2$ and $e(R_{\mathfrak{p}}) = 1$.

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