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Multiplicities and a Dimension Inequality for Unmixed Modules

Sean Sather-Wagstaff[1]

Department of Mathematics, University of Illinois, 273 Altgeld Hall, 1409 West Green Street, Urbana, IL, 61801
E-mail: ssather@math.uiuc.edu

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We prove the following result, which is motivated by the recent work of Kurano and Roberts on Serre's positivity conjecture. Assume that $(R, \mathfrak{m})$ is a local ring with finitely generated module M such that $R/\mathrm{Ann}(M)$ is quasi-unmixed and contains a field, and that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals in the support of M such that $\mathfrak{p}$ is analytically unramified, $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $e(M_\mathfrak{p}) = e(M)$. Then

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(M).$$

We also prove a similar theorem in a special case of mixed characteristic. Finally, we provide several examples to explain our assumptions as well as an example of a noncatenary local domain R with prime ideal $\mathfrak{p}$ such that $e(R_p) > e(R) = 1$. © 2001 Academic Press

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1. INTRODUCTION

Throughout this paper, all rings are commutative and Noetherian, and all modules are finitely generated and unital.

Let $(R, \mathfrak{m})$ be a local Neotherian ring of dimension d, and let M and N be finitely generated R-modules such that M has finite projective dimension and $M \otimes_R N$ is a module of finite length. Serre [20] defined the *intersection multiplicity* of M and N to be

$$\chi(M, N) = \sum_{i=0}^{d} (-1)^i \text{ length } (\mathrm{Tor}_i^R(M, N))$$

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and considered the following properties when R is regular:

1. $\dim(M) + \dim(N) \leq \dim(R)$.
2. (Nonnegativity) $\chi(M, N) \geq 0$.
3. (Vanishing) If $\dim(M) + \dim(N) < \dim(R)$, then $\chi(M, N) = 0$.
4. (Positivity) If $\dim(M) + \dim(N) = \dim(R)$, then $\chi(M, N) > 0$.

Serre was able to verify the first statement for any regular local ring and the others in the case where R is unramified. Since $\chi(M, N)$ has many of the characteristics we desire from an intersection multiplicity (for example, Bézout's Theorem holds), it was reasonable to suppose that these further properties are satisfied over an arbitrary regular local ring. The results were left unproved for ramified rings.

The vanishing conjecture was proved about 10 years ago by Gillet and Soulé [5] and independently by Roberts [17], with K-theoretic methods. Gabber proved the nonnegativity conjecture recently [2, 9, 18] by using a theorem of de Jong [4]. The Positivity Conjecture remains open in the ramified case. Kurano and Roberts have proved the following with the use of methods introduced by Gabber.

THEREOM 1.1 [10, Theorem 3.2]. *Assume that* $(R, \mathfrak{m})$ *is a regular local ring which either contains a field or is ramified. Also, assume that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals in* R *such that* $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ *and* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$. *If* $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$ *then*

$$\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1} \qquad \text{for all } n > 0, \tag{1}$$

where $\mathfrak{p}^{(n)}$ *denotes the nth symbolic power of* $\mathfrak{p}$.

As a result, they conjectured that (1) should hold for all regular local rings.

Conjecture 1.2. Assume that $(R, \mathfrak{m})$ is a regular local ring and that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals in R such that $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$. Then $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$ for all $n > 0$.

We study Conjecture 1.2, as a verification of this conjecture could introduce new tools to apply to the positivity conjecture.

For any local ring $(A, \mathfrak{n})$ with finite module M let $e(M)$ denote the Samuel multiplicity of M with respect to the ideal $\mathfrak{n}$. (For the definition of the Hilbert–Samuel multiplicity, see Definition 2.1.) It is straightforward to verify that, if R is a regular local ring with prime ideal $\mathfrak{p}$ and $0 \neq f \in \mathfrak{p}$, then $e(R_{\mathfrak{p}}/(f)) = m$ if and only if $f \in \mathfrak{p}^{(m)} \backslash \mathfrak{p}^{(m+1)}$. Thus, Conjecture 1.2 may be rephrased as the following.

Conjecture 1.2′. Assume that $(R, \mathfrak{m})$ is a regular local ring and that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals in R such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = m$. If there exists $0 \neq f \in \mathfrak{p} \cap \mathfrak{q}$ such that $e(R_{\mathfrak{p}}/(f)) = e(R/(f))$, then $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R) - 1$.

Conjecture 1.2′ motivates the following generalization.

Conjecture 1.3. Assume that $(R, \mathfrak{m})$ is a local ring with finitely generated module M such that $R/\mathrm{Ann}(M)$ is quasi-unmixed, and $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals in the support of M such that $\mathfrak{p}$ is analytically unramified, $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $e(M_{\mathfrak{p}}) = e(M)$, then

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(M).$$

In Examples 6.1–6.5 we give examples showing the necessity of the assumptions of Conjecture 1.3.

In a previous paper [19] we considered the case where $M = R$ and R is Cohen–Macaulay. In Conjecture 1.3, if $\dim(M) = \dim(R)$ and R is quasi-unmixed, then the Associativity Formula for multiplicities tells us that the condition $e(M_{\mathfrak{p}}) = e(M)$ is slightly weaker than the condition $e(R_{\mathfrak{p}}) = e(R)$. The condition $e(R_{\mathfrak{p}}) = e(R)$ depends on all the minimal primes of R, while the condition $e(M_{\mathfrak{p}}) = e(M)$ depends only on the minimal primes of R which are in the support of M. In Example 6.6 we give an example demonstrating this.

The following is the main result of this paper, in which we prove Conjecture 1.3 in the case where $R/\mathrm{Ann}(M)$ contains a field.

THEOREM 3.2. *Assume that* $(R, \mathfrak{m})$ *is a local ring and that* M *is an* R*-module such that* $R/\mathrm{Ann}(M)$ *is quasi-unmixed and contains a field. Let* $\mathfrak{p}$ *and* $\mathfrak{q}$ *be prime ideals of* R *in the support of* M *such that* $\mathfrak{p}$ *is analytically unramified,* $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$, *and* $e(M_{\mathfrak{p}}) = e(M)$. *Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(M)$.

Note that, if R is assumed to be excellent, then the condition that $\mathfrak{p}$ is analytically unramified is automatically satisfied, and $R/\mathrm{Ann}(M)$ is quasi-unmixed if and only if it is equidimensional.

Here we give a sketch of the proof of the main theorem. By passing to the quotient $R/\mathrm{Ann}(M)$, we may assume that R is quasi-unmixed and contains a field, and that $\mathrm{Supp}(M) = \mathrm{Spec}(R)$. By using the Associativity Formula for multiplicities (see Section 2), we may replace the module M with the ring R. By passing to the ring $R[X]_{\mathfrak{m}R[X]}$, we may assume that the residue field of R is infinite. The fact that $\mathfrak{p}$ is analytically unramified allows us to pass to the completion of R so that we may assume that R is complete and equidimensional and contains an infinite field K. To prove the complete case, we construct an injection $R' = K[\![X_1, \ldots, X_r]\!] \to R$

where $r = \dim(R)$, and we let $\mathfrak{p}' = \mathfrak{p} \cap R'$ and $\mathfrak{q}' = \mathfrak{q} \cap R'$. To prove the result, it suffices to show that $\mathfrak{p}' + \mathfrak{q}'$ is primary to the maximal ideal of R'. This is sufficient because R' is regular and Serre's result shows that

$$\begin{aligned}\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) &= \dim(R'/\mathfrak{p}') + \dim(R'/\mathfrak{q}') \\ &\leq \dim(R') = \dim(R).\end{aligned}$$

To prove that $\mathfrak{p}' + \mathfrak{q}'$ is primary to the maximal ideal of R', it suffices to show that $\mathfrak{p}$ is the unique prime ideal of R which contracts to $\mathfrak{p}'$ in R'. The desired uniqueness follows from our key lemma, which is essentially a corollary of a standard formula in multiplicity theory.

LEMMA 3.1. *Assume that R is an equidimensional ring containing a regular local ring $(R', \mathfrak{m}')$ such that the extension $R' \to R$ is module finite. Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{p}' = \mathfrak{p} \cap R'$. Assume that $e(R_\mathfrak{p}) = \operatorname{rank}_{R'}(R)$. Then $\mathfrak{p}$ is the unique prime ideal of R contracting to $\mathfrak{p}'$ in R' and $\kappa(\mathfrak{p}') \cong \kappa(\mathfrak{p})$.*

In addition, we use similar methods to prove the following theorem in mixed characteristic.

THEOREM 4.1. *Assume that $(R, \mathfrak{m})$ is a local ring and that M is an R-module such that $R/\operatorname{Ann}(M)$ is quasi-unmixed of mixed characteristic p and that p is part of a system of parameters of $R/\operatorname{Ann}(M)$ which generates a reduction of the maximal ideal of $R/\operatorname{Ann}(M)$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R in the support of M such that $\mathfrak{p}$ is analytically unramified, $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$, and $e(M_\mathfrak{p}) = e(M)$. Then $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(M)$.*

As noted above, Serre proved Conjecture 1.3 in the case where R is regular and $M = R$. Many attempts have been made to generalize Serre's result to the nonregular situation. It can be shown easily that, if one drops the assumption of regularity on the ring, then one must assume that the objects under investigation have additional properties which would be automatic if the ring were regular. In our conjectures and results, we assume that the module M satisfies the property $e(M_\mathfrak{p}) = e(M)$. When the ring is regular and M has positive rank, this is automatic because the localization $R_\mathfrak{p}$ is also regular, so that $e(R) = 1$ and $e(R_\mathfrak{p}) = 1$, and if $r = \operatorname{rank}_R(M)$ then $r = \operatorname{rank}_{R_\mathfrak{p}}(M_\mathfrak{p})$ and $e(M) = re(R) = r = re(R_\mathfrak{p}) = e(M_\mathfrak{p})$. In a famous conjecture, Peskine and Szpiro focus on the finiteness of projective dimensions over regular local rings.

Conjecture 1.4 [16]. Assume that $(R, \mathfrak{m})$ is a local ring and $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals in R such that $\mathfrak{p}$ has finite projective dimension and $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$. Then $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$.

We include this here because it places Conjecture 1.3 in a second context: not only is our conjecture motivated by a consequence of positivity, but it is also a generalization of Serre's dimension inequality for regular local rings. We note that, in the nonregular situation, Conjectures 1.3 and 1.4 are not comparable (cf. [19]).

In Section 2 we present definitions and background results. In Section 3 we prove the key lemma and the main theorem. In Section 4 we prove the mixed characteristic result mentioned above. In Section 5 we briefly consider Conjecture 1.2 in light of Theorems 3.2 and 4.1. In Section 6 we present several examples.

2. DEFINITIONS AND BACKGROUND RESULTS

The Hilbert–Samuel multiplicity shall play a central role in our work. For the sake of clarity we specify which multiplicity we are considering.

DEFINITION 2.1. Assume that $(R, \mathfrak{m})$ is a local ring and that M is an R-module of dimension d. Let $\mathfrak{a}$ be an ideal of R such that $\sqrt{\mathfrak{a}} = \mathfrak{m}$. For $n \gg 0$ the Hilbert function $H_{\mathfrak{a}, M}(n) = \text{length}(M/\mathfrak{a}^{n+1}M)$ is a polynomial in n of degree d with rational coefficients. If e_d is the leading coefficient of this polynomial then the *Hilbert–Samuel multiplicity* of $\mathfrak{a}$ on M is the positive integer $e_R(\mathfrak{a}, M) = d!e_d$. We will write $e(\mathfrak{a}, M)$ instead of $e_R(\mathfrak{a}, M)$ if doing so causes no confusion. We denote $e(\mathfrak{m}, M)$ by $e(M)$.

Recall that the Hilbert–Samuel multiplicity satisfies the *Associativity Formula*

$$e(M) = \sum_{\mathfrak{p}} \text{length}\,(M_{\mathfrak{p}})e(R/\mathfrak{p}),$$

where the sum is taken over all prime ideals $\mathfrak{p}$ of R such that $\dim(R/\mathfrak{p}) = \dim(M)$. Because we need only take the sum over such prime ideals which are also in the support of M, this sum is finite.

Furthermore, recall the following formula (cf. Nagata [13, (23.1)]). Assume $(R', \mathfrak{m}')$ is a local ring contained in a semilocal ring $(R, \mathfrak{m}_1, \ldots, \mathfrak{m}_n)$ such that each $\mathfrak{m}_i \cap R' = \mathfrak{m}'$. Assume also that R is a finite R'-module. Then

$$e_{R'}(\mathfrak{m}', R) = \sum_i [R/\mathfrak{m}_i : R'/\mathfrak{m}'] e_{R_{\mathfrak{m}_i}}(\mathfrak{m}' R_{\mathfrak{m}_i}, R_{\mathfrak{m}_i}), \quad (2)$$

where the sum is taken over all indices i such that $\text{ht}(\mathfrak{m}_i) = \dim(R')$.

The following theorem tells us that, under certain circumstances, the Hilbert–Samuel multiplicity is well behaved under localizations. We say that a local ring R is *analytically unramified* if its completion has no nonzero nilpotents. We say that a prime ideal $\mathfrak{p}$ in a local ring R is *analytically unramified* if the quotient $R/\mathfrak{p}$ is analytically unramified.

THEOREM 2.2 [13, (40.1)]. *Let* $\mathfrak{p}$ *be a prime ideal of a local ring* R. *If* $\mathfrak{p}$ *is analytically unramified and if* $ht(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$, *then* $e(R_{\mathfrak{p}}) \leq e(R)$.

Note that, if R is catenary and equidimensional then the condition $\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$ is automatically satisfied. (A ring R is *equidimensional* if, for every minimal prime $\mathfrak{t}$ of R, $\dim(R/\mathfrak{t}) = \dim(R)$.) Theorem 2.2 gives our motivation for the assumption "$\mathfrak{p}$ is analytically unramified" in Conjecture 1.3. If we do not assume that the Hilbert–Samuel multiplicity is well behaved with respect to localization, then there is no reason to suspect that the assumption $e(M_{\mathfrak{p}}) = e(M)$ is meaningful. Example 6.7 is an example of a local domain with prime ideal $\mathfrak{p}$ which is analytically unramified such that $\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) \neq \dim(R)$ and $e(R_{\mathfrak{p}}) > e(R)$. At this time, we do not know of an example of an equidimensional, catenary local ring R with prime ideal $\mathfrak{p}$ which is analytically ramified where either Conjecture 1.3 or the conclusion of Theorem 2.2 fails to hold.

The following result tells us that multiplicities behave well under certain flat extensions.

PROPOSITION 2.3 (Herzog [8, Lemma 2.3]). *Assume that* $R \to \widetilde{R}$ *is a flat local homomorphism of local rings* $(R, \mathfrak{m})$ *and* $(\widetilde{R}, \widetilde{\mathfrak{m}})$ *and that* $\mathfrak{m}\widetilde{R} = \widetilde{\mathfrak{m}}$. *Then* $e(\widetilde{R}) = e(R)$.

3. THE MAIN THEOREM

The key lemma for our main theorem is the following. For an integral domain A let $Q(A)$ denote the field of fractions of A. For a prime ideal $\mathfrak{p}$ of a ring R, let $\kappa(\mathfrak{p}) = Q(R/\mathfrak{p})$.

LEMMA 3.1. *Assume that* R *is an equidimensional ring containing a regular local ring* $(R', \mathfrak{m}')$ *such that the extension* $R' \to R$ *is module finite. Let* $\mathfrak{p}$ *be a prime ideal of* R *and let* $\mathfrak{p}' = \mathfrak{p} \cap R'$. *Assume that* $e(R_{\mathfrak{p}}) = \mathrm{rank}_{R'}(R)$. *Then* $\mathfrak{p}$ *is the unique prime ideal of* R *contracting to* $\mathfrak{p}'$ *in* R' *and* $\kappa(\mathfrak{p}') \cong \kappa(\mathfrak{p})$.

Proof. The fact that the extension $R' \to R$ is module finite and injective implies that R is semilocal and dominates R'. Let $r = \mathrm{rank}_{R'}(R)$ so that $r = \mathrm{rank}_{R'_{\mathfrak{p}'}}(R_{\mathfrak{p}'}) = \mathrm{rank}_{R'_{\mathfrak{p}'}}(R \otimes_{R'} R'_{\mathfrak{p}'})$. By Eq. (2) and Matsumura [12, Theorem 14.8],

$$r = re(R'_{\mathfrak{p}'}) = e(\mathfrak{p}'R'_{\mathfrak{p}'}, R_{\mathfrak{p}'}) = \sum_{\mathfrak{q}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p}')] e(\mathfrak{p}'R_{\mathfrak{q}}, R_{\mathfrak{q}}),$$

where the sum is taken over all prime ideals $\mathfrak{q}$ of R which contract to $\mathfrak{p}'$ and such that $\mathrm{ht}(\mathfrak{q}) = ht(\mathfrak{p}')$. Because the extension $R' \to R$ is finite, R' is integrally closed, and R is equidimensional, the going-up and going-down properties hold for the extension. In particular, any prime ideal $\mathfrak{q}$ of R which contracts to $\mathfrak{p}'$ automatically satisfies the condition $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}')$. Our assumption $e(R_\mathfrak{p}) = r$ implies that

$$r = \sum_{\mathfrak{q}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p}')] e(\mathfrak{p}' R_\mathfrak{q}, R_\mathfrak{q}) \geq e(\mathfrak{p}' R_\mathfrak{p}, R_\mathfrak{p}) \geq e(R_\mathfrak{p}) = r,$$

where the sum is taken over all prime ideals $\mathfrak{q}$ such that $\mathfrak{q} \cap R' = \mathfrak{p}'$. Therefore, we have equality at each step. The only way this can be true is if $\mathfrak{p}$ is the unique such prime and $[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p}')] = 1$. This is the desired conclusion. ∎

The following is our main theorem. Recall that a local ring is *quasi-unmixed* (or *formally equidimensional*) if its completion is equidimensional. This is equivalent to the ring being universally catenary and equidimensional by Ratliff [16].

THEOREM 3.2. *Assume that* $(R, \mathfrak{m})$ *is a local ring and that* M *is an* R-*module such that* $R/\mathrm{Ann}(M)$ *is quasi-unmixed and contains a field. Let* $\mathfrak{p}$ *and* $\mathfrak{q}$ *be prime ideals of* R *in the support of* M *such that* $\mathfrak{p}$ *is analytically unramified,* $\sqrt{\mathfrak{p} + \mathfrak{q}} = \mathfrak{m}$, *and* $e(M_\mathfrak{p}) = e(M)$. *Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(M)$.

As we noted above, if R is assumed to be excellent, then the condition that $\mathfrak{p}$ is analytically unramified is automatically satisfied, and $R/\mathrm{Ann}(M)$ is quasi-unmixed if and only if it is equidimensional.

Proof. Step 1. By passing to the quotient $R/\mathrm{Ann}(M)$, we reduce to the case where R is quasi-unmixed and contains a field and $\mathrm{Supp}(M) = \mathrm{Spec}(R)$. The fact that $\mathfrak{p}$ and $\mathfrak{q}$ are in the support of M implies that $\mathrm{Ann}(M) \subseteq \mathfrak{p} \cap \mathfrak{q}$. Let $R' = R/\mathrm{Ann}(M)$ with $\mathfrak{m}' = \mathfrak{m}R'$, $\mathfrak{p}' = \mathfrak{p}R'$ and $\mathfrak{q}' = \mathfrak{q}R'$. Our assumptions imply that R' is a quasi-unmixed local ring which contains a field, and $\mathrm{Supp}_{R'}(M) = \mathrm{Spec}(R')$. The ideals $\mathfrak{p}'$ and $\mathfrak{q}'$ are primes in the support of M such that $\mathfrak{p}'$ is analytically unramified and $\sqrt{\mathfrak{p}' + \mathfrak{q}'} = \mathfrak{m}'$. Also, $e_{R'}(M) = e_R(M)$ because the Hilbert polynomials of M over R and R' are the same. Similarly, $e_{R'_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) = e_{R_\mathfrak{p}}(M_\mathfrak{p})$ so that $e_{R'_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) = e_{R'}(M)$. Thus, the case where R is quasi-unmixed and contains a field and $\mathrm{Supp}(M) = \mathrm{Spec}(R)$ implies that

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R'/\mathfrak{p}') + \dim(R'/\mathfrak{q}') \leq \dim(M),$$

giving the desired result.

Step 2. We reduce to the case where $M = R$. Because we are assuming now that $\mathrm{Supp}(M) = \mathrm{Spec}(R)$ and that R is quasi-unmixed, the Associativity Formula tells us that

$$\sum_{\mathfrak{t}} \text{length }(M_{\mathfrak{t}})e(R/\mathfrak{t}) = e(M) = e(M_{\mathfrak{p}})$$
$$= \sum_{\mathfrak{t}\subseteq\mathfrak{p}} \text{length }(M_{\mathfrak{t}})e(R_{\mathfrak{p}}/\mathfrak{t}_{\mathfrak{p}}), \tag{3}$$

where the first sum is taken over all minimal primes of R and the second sum is taken over all minimal primes of R which are contained in $\mathfrak{p}$. The fact that each $R/\mathfrak{t}$ is quasi-unmixed and that $\mathfrak{p}/\mathfrak{t}$ is analytically unramified for all such $\mathfrak{t}$ contained in $\mathfrak{p}$ implies that $e(R_{\mathfrak{p}}/\mathfrak{t}_{\mathfrak{p}}) \le e(R/\mathfrak{t})$ by Theorem 2.2. Thus, Eq. (3) implies that every minimal prime $\mathfrak{t}$ of R is contained in $\mathfrak{p}$ and $e(R_{\mathfrak{p}}/\mathfrak{t}_{\mathfrak{p}}) = e(R/\mathfrak{t})$. Thus,

$$e(R) = \sum_{\mathfrak{t}} \text{length }(R_{\mathfrak{t}})e(R/\mathfrak{t}) = \sum_{\mathfrak{t}} \text{length }(R_{\mathfrak{t}})e(R_{\mathfrak{p}}/\mathfrak{t}_{\mathfrak{p}}) = e(R_{\mathfrak{p}}).$$

Of course, $\mathfrak{p}$ and $\mathfrak{q}$ are in the support of R, so that, if we know the result for $M = R$, then

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \le \dim(R) = \dim(M).$$

Step 3. By passing to the ring $R(X) = R[X]_{\mathfrak{m}R[X]}$, we reduce to the case where R has infinite residue field. The ring $R(X) = R[X]_{\mathfrak{m}R[X]}$ is quasi-unmixed with infinite residue field $R/\mathfrak{m}(X)$. Because the extension $R \to R(X)$ is flat and local, and $\mathfrak{m}$ extends to the maximal ideal of $R(X)$, Proposition 2.3 implies that $e(R(X)) = e(R)$. For any ideal I of R let $I(X) = IR(X)$. Then $\mathfrak{p}(X)$ and $\mathfrak{q}(X)$ are prime ideals of $R(X)$ such that $\sqrt{\mathfrak{p}(X) + \mathfrak{q}(X)} = \mathfrak{m}(X)$. By [13, (36.8)], $\mathfrak{p}(X)$ is analytically unramified. The extension $R_{\mathfrak{p}} \to R(X)_{\mathfrak{p}(X)}$ is faithfully flat and $\mathfrak{p}_{\mathfrak{p}}$ extends to the maximal ideal of $R(X)_{\mathfrak{p}(X)}$, so that $e(R(X)_{\mathfrak{p}(X)}) = e(R_{\mathfrak{p}})$. If we know the result for rings with infinite residue field, then

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R(X)/\mathfrak{p}(X)) + \dim(R(X)/\mathfrak{q}(X))$$
$$\le \dim(R(X)) = \dim(R).$$

Step 4. By passing to the completion $\widehat{R}$ of R, we reduce to the case where R is complete and equidimensional. Let $\hat{\mathfrak{p}}$ and $\hat{\mathfrak{q}}$ be prime ideals of $\hat{R}$ which are minimal over $\mathfrak{p}\widehat{R}$ and $\mathfrak{q}\widehat{R}$, respectively. Then $\dim(\widehat{R}/\hat{\mathfrak{p}}) = \dim(R/\mathfrak{p})$ and similarly for $\widehat{R}/\hat{\mathfrak{q}}$. The ring $\widehat{R}$ is quasi-unmixed. Also,

$$\hat{\mathfrak{m}} \supseteq \sqrt{\hat{\mathfrak{p}} + \hat{\mathfrak{q}}} \supseteq \sqrt{\mathfrak{p}\hat{R} + \mathfrak{q}\hat{R}} = \mathfrak{m}\hat{R} = \hat{\mathfrak{m}},$$

so that $\sqrt{\hat{\mathfrak{p}} + \hat{\mathfrak{q}}} = \hat{\mathfrak{m}}$. The extension $R \to \widehat{R}$ is flat and local and the extension of $\mathfrak{m}$ into $\widehat{R}$ is the maximal ideal $\hat{\mathfrak{m}}$, so that $e(\widehat{R}) = e(R)$ by

Proposition 2.3. The fact that $\mathfrak{p}$ is analytically unramified implies that $\mathfrak{p}\widehat{R}_{\hat{\mathfrak{p}}} = \hat{\mathfrak{p}}\widehat{R}_{\hat{\mathfrak{p}}}$, so that $e(\widehat{R}_{\hat{\mathfrak{p}}}) = e(R_{\mathfrak{p}})$. Thus, the complete, equidimensional case implies that

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(\hat{R}/\hat{\mathfrak{p}}) + \dim(\hat{R}/\hat{\mathfrak{q}}) \leq \dim(\hat{R}) = \dim(R).$$

Step 5. We prove the case where R is complete and equidimensional with infinite residue field, and $M = R$. By Bruns and Herzog [3, Proposition 4.6.8], there exists a system of parameters $x = x_1, \ldots, x_r$ of R which generates a reduction ideal of $\mathfrak{m}$.

Let K denote the residue field of R, which we may assume is contained in R, as R is complete and contains a field. Let R' denote the power series ring $K[[X_1, \ldots, X_r]]$. The natural map $R' \to R$ given by $X_i \mapsto x_i$ is injective and makes R into a finite R'-module. By [12, Theorem 14.8] and [3, Lemma 4.6.5],

$$\operatorname{rank}_{R'}(R) = \operatorname{rank}_{R'}(R)e(R') = e(\mathfrak{m}'R, R) = e((x)R, R) = e(R) = e(R_{\mathfrak{p}}).$$

Let $e = e(R)$, $\mathfrak{p}' = \mathfrak{p} \cap R'$, and $\mathfrak{q}' = \mathfrak{q} \cap R'$. The fact that the extension $R'/\mathfrak{p}' \to R/\mathfrak{p}$ is injective and module-finite implies that $\dim(R'/\mathfrak{p}') = \dim(R/\mathfrak{p})$, and similarly, $\dim(R'/\mathfrak{q}') = \dim(R/\mathfrak{q})$.

To show that $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(R)$, it suffices to show that $\sqrt{\mathfrak{p}' + \mathfrak{q}'} = \mathfrak{m}'$, as Serre's intersection theorem for regular local rings implies that

$$\begin{aligned}\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) &= \dim(R'/\mathfrak{p}') + \dim(R'/\mathfrak{q}') \\ &\leq \dim(R') = \dim(R),\end{aligned}$$

as desired.

The fact that $e(R_{\mathfrak{p}}) = e(R) = \operatorname{rank}_{R'}(R)$ with the previous lemma implies that $\mathfrak{p}$ is the unique prime ideal of R contracting to $\mathfrak{p}'$ in R'. Let $\mathfrak{s}'$ be a prime ideal of R' containing $\mathfrak{p}' + \mathfrak{q}'$. It suffices to show that $\mathfrak{s}' = \mathfrak{m}'$. By the going-up property, there is a prime ideal $\mathfrak{s}$ of R containing $\mathfrak{q}$ such that $\mathfrak{s} \cap R' = \mathfrak{s}'$. By the going-down property, there is a prime ideal $\mathfrak{p}_1$ of R contained in $\mathfrak{s}$ such that $\mathfrak{p}_1 \cap R' = \mathfrak{p}'$. By our uniqueness statement, $\mathfrak{p}_1 = \mathfrak{p}$, so that $\mathfrak{s}$ contains $\mathfrak{p} + \mathfrak{q}$. Therefore, $\mathfrak{s} = \mathfrak{m}$ and $\mathfrak{s}' = \mathfrak{s} \cap R' = \mathfrak{m} \cap R' = \mathfrak{m}'$, as desired. This completes the proof. ∎

4. A THEOREM IN MIXED CHARACTERISTIC

The following result in mixed characteristic is slightly more restrictive than Theorem 3.2, but is rather interesting. For brevity, we say that a system of parameters $x_1, \ldots, x_d$ of a local ring $(R, \mathfrak{m})$ is a *reductive system of* *parameters* if $(\mathbf{x})R$ is a reduction of $\mathfrak{m}$. If the residue field of R is infinite and the sequence $y_1, \ldots, y_r$ generates a reduction of $\mathfrak{m}$, then Northcott and Rees [15, Theorem 4.2] tell us that the ideal $(\mathbf{y})R$ is a minimal reduction of $\mathfrak{m}$ if and only if $r = \dim(R)$. That is, a system of parameters is reductive if and only if it generates a minimal reduction of $\mathfrak{m}$.

THEOREM 4.1. *Assume that $(R, \mathfrak{m})$ is a local ring and that M is an R-module such that $R/\mathrm{Ann}(M)$ is quasi-unmixed of mixed characteristic p and that p is part of a reductive system of parameters of $R/\mathrm{Ann}(M)$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R in the support of M such that $\mathfrak{p}$ is analytically unramified, $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$, and $e(M_{\mathfrak{p}}) = e(M)$. Then* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \leq \dim(M)$.

Again, we note that if R is excellent, then the assumption that $\mathfrak{p}$ is analytically unramified is automatically satisfied and that $R/\mathrm{Ann}(M)$ is quasi-unmixed if and only if it is equidimensional.

Proof. We follow the same steps as in the proof of Theorem 3.2. Steps 1–4 are independent of the assumption on the characteristic of the ring. To verify that we may assume that R is complete, equidimensional, and mixed characteristic p with infinite residue field and that p is part of a reductive system of parameters of R, it suffices to show that this final property passes through each of the steps. Steps 1 and 2 are trivial. For Steps 3 and 4, we note that each extensions $R \to R(X)$ and $R \to \widehat{R}$ are flat, local extensions such that the extension of the maximal ideal of R into the extension ring is the maximal ideal of the extension ring. It is straightforward to show that, in this situation, a reductive system of parameters $\mathbf{x}$ of R extends to a reductive system of parameters of the extension ring. Therefore we may pass to the completion.

Step 5. Let $p = x_1, x_2, \ldots, x_d$ be a reductive system of parameters of R. The fact that p is part of a system of parameters for R implies that p is contained in no minimal prime of R. In particular, R has characteristic 0. The fact that R is complete then implies that R has a coefficient ring (V, pV) which is a complete discrete valuation ring contained in R. Let $R' = V[[X_2, \ldots, X_d]]$, which is a regular local ring of dimension $d = \dim(R)$. By the proof of [12, Theorem 29.4(iii)], the map $R' \to R$ given by $X_i \mapsto x_i$ is injective and makes R into a finite R'-module. The fact that $x_1, \ldots, x_d$ generate a reduction of $\mathfrak{m}$ implies, as in the proof of Theorem 3.2, that $\mathrm{rank}_{R'}(R) = e(R) = e(R_{\mathfrak{p}})$, so that the proof is now identical to that of Theorem 3.2. ∎

In Example 6.8 we show that the construction of Step 5 fails if the assumption "p is part of a reductive system of parameters of $R/\mathrm{Ann}(M)$" is dropped.

5. THE CONJECTURE OF KURANO AND ROBERTS

At this point, it seems wise to consider the status of Conjecture 1.2 in light of Theorems 3.2 and 4.1. Conjecture 1.2 holds for regular local rings containing a field by Theorem 3.2. Of course, because the positivity conjecture holds for regular local rings containing a field, this also follows from Theorem 1.1. In the mixed characteristic, unramified case, Theorem 4.1 does not completely resolve this conjecture, but it shows us exactly where to focus our attention.

Assume that $(R, \mathfrak{m})$ is an unramified regular local ring of mixed characteristic p with prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. We want to show that $\mathfrak{p}^{(n)} \cap \mathfrak{q} \subseteq \mathfrak{m}^{n+1}$. Using standard methods, we may assume that R is complete with infinite residue field. (Note that, for the question of Kurano and Roberts, we do not need to assume that $\mathfrak{p}$ is analytically unramified to make this reduction.) Suppose that $f \in \mathfrak{p}^{(n)} \cap \mathfrak{q}$ and $f \notin \mathfrak{m}^{n+1}$. Using the Associativity Formula, we may assume that f is irreducible. Let $R_1 = R/fR$, $\mathfrak{m}_1 = \mathfrak{m}R_1$, and so on. The data $R_1, \mathfrak{m}_1, \mathfrak{p}_1, \mathfrak{q}_1, M_1 = R_1$ then give a counterexample to Conjecture 1.3 where the ring is in fact a complete domain. By Theorems 3.2 and 4.1, we therefore know that R_1 does not contain a field and the residual characteristic p is not part of a reductive system of parameters of R_1. An appropriate choice of variables then shows that we can write R_1 in the form

$$R_1 = V[\![X_1, \dots, X_d]\!]/(p^n + a_1 p^{n-1} + \cdots + a_n),$$

where V is a complete p-ring, $a_i \in (X_1, \dots, X_d)^i$, and $a_n \in \mathfrak{m}^{n+1}$. Therefore, with an appropriate choice of variables for R, we know that the only form f can have is $f = p^n + a_1 p^{n-1} + \cdots + a_n$.

6. EXAMPLES

The following example demonstrates that, in Conjecture 1.3, the ring $R/\mathrm{Ann}(M)$ must be equidimensional.

EXAMPLE 6.1. Let $R = M = k[\![X, Y, Z]\!]/(XY, XZ) = k[\![x, y, z]\!]$ and let $\mathfrak{p} = (x)R$ and $\mathfrak{q} = (y, z)R$. Then R is excellent, $\mathfrak{p}$ and $\mathfrak{q}$ are in the support of M, $\mathfrak{p}$ is analytically unramified, $e(M_\mathfrak{p}) = 1 = e(M)$, and $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. However, R is not equidimensional and

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 3 > 2 = \dim(M).$$

The following example demonstrates that, in Conjecture 1.3, the condition $e(M_\mathfrak{p}) = e(M)$ is necessary.

Example 6.2. Let $R = M = k[[X, Y]]/(XY) = k[[x, y]]$ and let $\mathfrak{p} = (x)R$ and $\mathfrak{q} = (y)R$. Then R is excellent and equidimensional, $\mathfrak{p}$ and $\mathfrak{q}$ are in the support of M, $\mathfrak{p}$ is analytically unramified, and $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. However, $e(M_{\mathfrak{p}}) = 1 < 2 = e(M)$ and

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 2 > 1 = \dim(M).$$

The following example demonstrates that, in Conjecture 1.3, the condition $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$ is necessary.

Example 6.3. Let $R = M = k[[X]]$ and let $\mathfrak{p} = \mathfrak{q} = (0)R$. Then R is excellent and equidimensional, $\mathfrak{p}$ and $\mathfrak{q}$ are in the support of M, $\mathfrak{p}$ is analytically unramified, and $e(M_{\mathfrak{p}}) = 1 = e(M)$. However, $\sqrt{\mathfrak{p}+\mathfrak{q}} \neq \mathfrak{m}$ and

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = 2 > 1 = \dim(R).$$

The following example demonstrates that, in Conjecture 1.3, the ideal $\mathfrak{q}$ must be in the support of M. (Note that the question makes no sense if $\mathfrak{p}$ is not in the support of M, since then $e(M_{\mathfrak{p}}) = 0 < e(M)$.)

Example 6.4. Let

$$R = k[[X, Y, Z, W]]/(XY, YZ, ZW, WX) = k[[x, y, z, w]]$$

and let $M = R/(x, z) \cong k[[Y, W]]$. Let $\mathfrak{p} = (x, z)$ or $\mathfrak{p} = (x, y, z)$ so that $e(M) = e(M_{\mathfrak{p}}) = 1$ and $\dim(R/\mathfrak{p}) \geq 1$. Let $\mathfrak{q} = (y, w)$ so that $M_{\mathfrak{q}} = 0$ and $\dim(R/\mathfrak{q}) = 2$. Then R is excellent, $R/\mathrm{Ann}(M)$ is a domain, $\mathfrak{p} \in \mathrm{Supp}(M)$, $\mathfrak{p}$ is analytically unramified, and $\sqrt{\mathfrak{p}+\mathfrak{q}} = \mathfrak{m}$. However, $\mathfrak{q} \notin \mathrm{Supp}(M)$ and

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) \geq 3 > 2 = \dim(M).$$

The following example demonstrates that, in Conjecture 1.3, the quotient ring $R/\mathrm{Ann}(M)$ must be catenary.

Example 6.5 [13, Appendix (E.2)]. Let K be a field and let x be an indeterminate. Let r be a positive integer and $z_1, \ldots, z_r \in K[[x]]$ be power series which are algebraically independent over $K(x)$. Write $z_i = \sum_j a_{ij}x^j$, and for $j > 0$ let

$$z_{ij} = \frac{z_i - \sum_{k<j} a_{ik}x^k}{x^{j-1}} = \sum_{k \geq j} a_{ik}x^{k-j+1}.$$

Let m be a positive integer and let $y_1, \ldots, y_m$ be algebraically independent elements (indeterminates) over $K[x, z_1, \ldots, z_r]$. Let $R_0 = K[x, \{z_{ij}\}]$, and for $i = 1, \ldots, m$ let $R_i = R_0[y_1, \ldots, y_i]$.

The ideal xR_0 is maximal in R_0, because $z_{ij} = xz_{ij+1} + a_{ij}x \in xR_0$. As Nagata notes, $(R_0)_{xR_0}$ is a discrete valuation ring. Let $\mathfrak{m}_i = (x, y_1, \ldots, y_i)R_i$ and $\mathfrak{n}_i = (x-1, z_1, \ldots, z_r, y_1, \ldots, y_i)R_i$ for $i = 0, \ldots, m$. The rings $V_i = (R_i)_{\mathfrak{m}_i}$ are regular local rings of dimension $i+1$ and the rings $W_i = (R_i)_{\mathfrak{n}_i}$

are regular local rings of dimension $r+i+1$. Let $S_i = R_i \backslash (\mathfrak{m}_i \cup \mathfrak{n}_i)$ and let $A_i = (R_i)_{S_i}$. The maximal ideals of A_i are $\mathfrak{m}_i A_i$ and $\mathfrak{n}_i A_i$, and $(A_i)_{\mathfrak{m}_i A_i} = V_i$ and $(A_i)_{\mathfrak{n}_i A_i} = W_i$. It follows [13, Appendix (E1.2)] that each A_i is Noetherian and $A_i/\mathfrak{m}_i A_i = A_i/\mathfrak{n}_i A_i = K$.

Let $A = A_m$ and let J denote the Jacobson radical of A. Let $R = K + J$, which is a subring of A. In fact, by [13, Appendix (E2.1)] R is a local ring with maximal ideal J, the residue field of R is K, and A is a finite R-module. We claim that R is the desired example.

First, we show that, for every prime ideal $\mathfrak{p}$ of R, $e(R_\mathfrak{p}) = 1$. To show this, it suffices to show the following: (i) $e(R) = 1$ and (ii) for every nonmaximal prime ideal $\mathfrak{p}$ of R, there exists a unique prime ideal P of A such that $P \cap R = \mathfrak{p}$ and that $R_\mathfrak{p} = A_P$. This is sufficient because A_P is a regular local ring. For (i) we note that Nagata's computation shows that

$$e(R) = e((A_m)_{\mathfrak{n}_m A_m}) = e(W_m) = 1.$$

For (ii), it suffices to verify the following.

Claim. For every nonzero element λ of the R-module A/R, $\operatorname{Ann}_R(\lambda) = J$.

Before we prove the claim, we show how it implies (ii). Let $\mathfrak{p}$ be a nonmaximal prime ideal of R and let P be a prime ideal of A contracting to $\mathfrak{p}$ in R. (Note that the fact that the extension $R \to A$ is integral implies that such a prime P always exists and is not maximal.) To verify (ii) it suffices to show that $(A/R)_\mathfrak{p} = (A/R) \otimes_R R_\mathfrak{p} = 0$, because then we will have an isomorphism $R_\mathfrak{p} \cong A_\mathfrak{p}$ so that $A_\mathfrak{p}$ is a local ring; the integrality of the extension $R \to A$ implies that there are no containments between prime ideals contracting to $\mathfrak{p}$ in R so there is a unique such prime in A. Furthermore, it is straightforward to show that $A_P \cong A_\mathfrak{p}$. By the claim, for every nonzero element $\lambda \in A/R$, $\operatorname{Ann}_R(\lambda) = J$, and this implies that every such λ is annihilated by an element $s \in J \backslash \mathfrak{p}$. If $\lambda_1, \ldots, \lambda_u \in A/R$ is a generating set of A/R as an R-module, then $\lambda_1/1, \ldots, \lambda_u/1 \in (A/R)_\mathfrak{p}$ is a generating set of $(A/R)_\mathfrak{p}$ over $R_\mathfrak{p}$. The fact that every λ_i is annihilated by an element $s_i \in J \backslash \mathfrak{p}$ implies that $\lambda_i/1 = s_i \lambda_i / s_i = 0$ so that $(A/R)_\mathfrak{p} = 0$ as desired.

Now, we prove the claim. If $t \in A \backslash R$, let $\bar{t}$ denote the class of t in the quotient A/R. The fact that $\bar{t} \neq 0$ implies that $\operatorname{Ann}_R(\bar{t}) \subseteq J$. However, $Jt \subseteq J \subset R$, which implies that $J\bar{t} = 0$, so that $J \subseteq \operatorname{Ann}(\bar{t}) \subseteq J$, giving the desired equality.

As an ideal of A, J is generated by the elements $x(x-1), \{y_i\}, \{z_j\}$. As an ideal of R,

$$J = (x(x-1), \{y_i\}, \{z_j\}, \{xy_i\}, \{xz_j\}, \{(x-1)y_i\}, \{(x-1)z_j\})R.$$

Let $P = (x)A$ and $Q = (x-1)A$, which are prime ideals in A, and let $\mathfrak{p} = P \cap R$ and $\mathfrak{q} = Q \cap R$. Then

$$\mathfrak{p} = (x(x-1), xy_1, \ldots, xy_m, xz_1, \ldots, xz_r)R$$

and

$$\mathfrak{q} = (x(x-1), (x-1)y_1, \ldots, (x-1)y_m, (x-1)z_1, \ldots, (x-1)z_r)R,$$

so that $\mathfrak{p} + \mathfrak{q} = J$. The condition $e(R_\mathfrak{p}) = e(R)$ is automatic. Furthermore, $\mathfrak{p}$ is analytically unramified, as follows. We have an inclusion of rings $R/\mathfrak{p} \hookrightarrow A/P = A/xA$ such that the extension is module finite. Also, A/xA is a regular local ring and is therefore analytically unramified. The maximal ideal of $R/\mathfrak{p}$ extends to the maximal ideal of A/xA so that the completion of A/xA contains the completion of $R/\mathfrak{p}$ (cf. Atiyah and MacDonald [1, Theorem 10.11 and Corollary 10.3]). Therefore, $\widehat{R/\mathfrak{p}}$ is a domain.

Finally,

$$\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(A/P) + \dim(A/Q) = m + (r+m),$$

which is strictly greater than $\dim(R) = r + m + 1$ if we assume that $m > 1$.

I do not know of an example where $R/\mathrm{Ann}(M)$ is a local, catenary, equidimensional ring which is not universally catenary and where Conjecture 1.3 does not hold.

The following example demonstrates that, for a given ring R and module M, the equality $e(M_\mathfrak{p}) = e(M)$ is weaker than the equality $e(R_\mathfrak{p}) = e(R)$, even when R is complete and equidimensional and $\dim(M) = \dim(R)$.

EXAMPLE 6.6. Let

$$R = k[\![X, Y, Z, W]\!]/(X, Y) \cap (Z, W) = k[\![x, y, z, w]\!].$$

Let $M = R/(x, y) \cong k[\![Z, W]\!]$. Then, $e(R) = 2$, and, for every nonmaximal prime ideal $\mathfrak{p}$ of R, $e(R_\mathfrak{p}) = 1$. However, for every prime $\mathfrak{p}$ in the support of M, $e(M_\mathfrak{p}) = 1 = e(M)$.

The following example demonstrates that, if $\mathfrak{p}$ is a prime ideal in a local domain which is analytically unramified and does not satisfy the condition $\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$, then the conclusion of Theorem 2.2 does not hold.

EXAMPLE 6.7. We continue with the notation of Example 6.5. Consider the polynomial $f = y_m^s + x^t(x-1) \in A_{m-1}[y_m]$ where $s \geq t > 1$. The fact that A_{m-1} is a regular semilocal domain implies that A_{m-1} is a unique factorization domain [13, (28.8)]. The ideal x generates a prime ideal in A_{m-1}, and it follows from Eisenstein's criterion that f is irreducible in the polynomial ring $A_{m-1}[y_m]$. Therefore, f generates a prime ideal in $A_{m-1}[y_m]$. Because A_m is a localization of $A_{m-1}[y_m]$ and f is a nonzero nonunit in A_m, we see that the ideal fA_m is prime. Let $A = A_m/fA_m$, which is a semilocal domain with maximal ideals $\mathfrak{m}_m A$ and $\mathfrak{n}_m A$.

Let J denote the Jacobson radical of A and let $R = K + J$, which is a subring of A. As in Example 6.5, R is a local ring with maximal ideal J, the residue field of R is K, and A is a finite R-module. We claim that R is the desired example.

First, we verify that $e(R) = 1$. This is straightforward, as Nagata's computation shows that

$$e(R) = e((A_m)_{\mathfrak{n}_m A_m}/(f)) = e(W_m/(f)).$$

The fact that $x - 1$ is a minimal generator of the maximal ideal of W_m and that x is a unit in W_m implies that $f = y_m^s + x^t(x-1)$ is a minimal generator of the maximal ideal of W_m. Because W_m is a regular local ring, it follows that $e(R) = 1$.

To verify that there is a prime ideal $\mathfrak{p}$ of R such that $e(R_\mathfrak{p}) > 1$, it suffices to verify the following.

Claim. If $\mathfrak{p}$ is a nonmaximal prime ideal of R, then there exists a unique prime ideal P of A contracting to $\mathfrak{p}$ in R and $A_P = R_\mathfrak{p}$. The verification of this claim is identical to that in Example 6.5, so we omit it here.

We show how the claim implies the desired result. Let $P = (x, y_m)A$. Then P is a prime ideal of A because x is a prime element of A_{m-1} and therefore the ideal $(x, y_m)A_m$ is a prime ideal of A_m containing f. Let $\mathfrak{p} = P \cap R$. Then $\mathfrak{p}$ is not maximal because the extension $R \to A$ is integral and P is not maximal. The verification of the fact that $\mathfrak{p}$ is analytically unramified is similar to that in Example 6.5. (By Theorem 2.2 it follows that $\operatorname{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) \neq \dim(R)$.) As noted, $R_\mathfrak{p} \cong A_P$, and it follows that

$$e(R_\mathfrak{p}) = e(A_P) = e(V_m/fV_m).$$

The fact that x is a minimal generator of the maximal ideal of V_m and that $x - 1$ is a unit in V_m implies that $f \in (\mathfrak{m}_m V_m)^t \setminus (\mathfrak{m}_m V_m)^{t+1}$. The fact that V_m is a regular local ring implies that $e(R_\mathfrak{p}) = t > 1$, as desired.

I do not know of a local, catenary, equidimensional ring which is not universally catenary where this inequality of multiplicities does not hold. A number of examples have been constructed (e.g., [13, Appendix (E.2)]; Heinzer et al. [6, Example (4.5)], Nishimura [14, Sections 2, 4, and 6]) where the hypotheses of Theorem 2.2 do not hold. However, in each example, the conclusion of Theorem 2.2 does hold.

The following example demonstrates that if R is a complete domain of mixed characteristic p such that p is not part of a reductive system of parameters of R, then there does not exist in general a regular local ring $R' \subseteq R/\mathrm{Ann}(M)$ such that the extension is finite and $e(R) = \mathrm{rank}'_R(R)$. In particular, in Theorem 4.1, if the assumption "p is part of a reductive system of parameters of $R/\mathrm{Ann}(M)$" is dropped, then the construction of Step 5 of the proof does not work.

EXAMPLE 6.8. Let p be a prime number and let (V, pV, k) a complete p-ring. Let $A = V[[X, Y]]$, $f = pX + Y^3$, and $R = A/(f)$. It is straightforward to verify that f is irreducible in A, so that R is a complete domain of dimension 2 and mixed characteristic p. Let $\mathfrak{m}$ and $\mathfrak{n}$ denote the maximal ideals of R and A, respectively.

The element p is not part of a reductive system of parameters of R, as follows. Suppose the contrary. Because $\dim(R) = 2$ there would exist an element $z \in R$ such that p, z is a reductive system of parameters of R. Let $\mathfrak{q} = (p, z)R$. By [15, Theorem 4.2], $\mathfrak{q}$ is a minimal reduction of $\mathfrak{m}$. By Herrmann et al. [7, Proposition 10.17], the fact that p, z generate a minimal reduction of $\mathfrak{m}$ implies that the initial forms of p and z in the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ form a homogeneous system of parameters for $\mathrm{gr}_{\mathfrak{m}}(R)$. The fact that A is regular and $R = A/(f)$ implies that $\mathrm{gr}_{\mathfrak{m}}(R)$ is the quotient of $\mathrm{gr}_{\mathfrak{n}}(A)$ by the initial form of f. More specifically, let $\mathbf{P}, \mathbf{X}, \mathbf{Y}$ denote the initial forms of p, X, Y in $\mathfrak{n}/\mathfrak{n}^2$, respectively. Then $\mathbf{P}, \mathbf{X}, \mathbf{Y}$ are indeterminates in $\mathrm{gr}_{\mathfrak{n}}(A)$, and $\mathrm{gr}_{\mathfrak{n}}(A)$ is the polynomial ring $k[\mathbf{P}, \mathbf{X}, \mathbf{Y}]$. Then $\mathrm{gr}_{\mathfrak{m}}(R)$ is a quotient of $\mathrm{gr}_{\mathfrak{n}}(A)$:

$$\mathrm{gr}_{\mathfrak{m}}(R) = k[\mathbf{P}, \mathbf{X}, \mathbf{Y}]/(\mathbf{PX}).$$

The initial form of p in $\mathrm{gr}_{\mathfrak{m}}(R)$ is the image of $\mathbf{P}$ which is contained in a minimal prime ideal of $\mathrm{gr}_{\mathfrak{m}}(R)$. Therefore, this initial form cannot be part of a homogeneous system of parameters of $\mathrm{gr}_{\mathfrak{m}}(R)$, giving the desired contradiction.

Now, we show that there does not exist a complete regular local ring R' contained in R such that the extension $R' \to R$ is module-finite and $e(R) = \mathrm{rank}_{R'}(R)$. This will prove that we have the desired example. Suppose that such a ring R' existed. Because R is a domain of mixed characteristic p, R does not contain a field. Therefore R' does not contain a field. Because R is unramified, R' is also unramified because otherwise, $p \in (\mathfrak{m}')^2 \subseteq \mathfrak{m}^2$, which contradicts the fact that R is unramified. Therefore, R' is of the form $W[[Z]]$, where (W, pW) is a complete p-ring and Z is analytically independent over W. By assumption, $e(\mathfrak{m}, R) = e(R) = \mathrm{rank}_{R'}(R) = e(\mathfrak{m}'R, R)$. By the theorem of Rees (cf. [7, Theorem 19.5]) this implies that $\mathfrak{m}'R$ is a reduction of $\mathfrak{m}$. Because $\mathfrak{m}'R = (p, Z)R$ we see that this implies that p is part of a reductive system of parameters of R, a contradiction.

We also note that it is straightforward to verify that no change in variables will remedy the noted behavior of this example.

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